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Topological and Algebraic Structures

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Abstracts

Erich Peter Klement, Stephen E. Rodabaugh
Editors

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Propositional Theories, Frames, and Fuzzy Algebras

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Entities like the T-fuzzy subalgebras and congruences of an algebra (in the sense of Universal Algebra) will be described as the T-valued models of suitable propositional theories, thus linking them to frames (= complete Heyting algebras) and providing a systematic approach to the study of the partially ordered sets arising in this context. It will be shown how this approach leads to suggestive new proofs of familiar facts as well as a variety of new results.

Pure States

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The concept of a pure state of a C*-algebra A is equivalent to that of an irreducible representation of the C*-algebra. Within the quantale Max A that is the spectrum of A , the notion of pure state may be abstracted to play an important role. In turn, this leads to links with orthocomplemented sup-lattices through the concept of Hilbert quantale. In this talk, we explore the interaction between the roles played by pure states within these contexts.

Discrete Triangular Norms

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1 T-norms and additive generators

Triangular norms (t-norms) have been introduced in the sixties by Schweizer and Sklar [7] as commutative, associative and increasing $[0,1]^2 \to [0,1]$ mappings with neutral element 1. Continuous t-norms have been completely characterized as ordinal sums with continuous Archimedean summands [5, 7]. Recall that continuous Archimedean t-norms are characterized by the diagonal inequality $(\forall x \in]0,1[)(\mathcal{T}(x,x) < x)$.

Continuous Archimedean t-norms are representable by means of additive generators [5]: a t-norm $\mathcal{T}$ is continuous and Archimedean if and only if there exists a continuous, strictly decreasing $[0,1] \to [0,\infty]$ mapping f with $f(1) = 0$ (called an additive generator of $\mathcal{T}$) such that $\mathcal{T}(x,y) = f^{-1}(\min(f(0), f(x) + f(y)))$, for any $(x,y) \in [0,1]^2$. Note that an additive generator f of a continuous Archimedean t-norm $\mathcal{T}$ is unique up to a positive multiplicative constant.

Any continuous Archimedean t-norm $\mathcal{T}$ is either a strict t-norm (i.e. continuous and strictly increasing on $]0,1]^2$) or a nilpotent (i.e. non-strict) t-norm. Strict t-norms are characterized by unbounded additive generators ($f(0) = +\infty$) and are isomorphic to the product t-norm $T_P(x,y) = xy$. On the other hand, nilpotent t-norms are characterized by bounded additive generators ($f(0) < +\infty$) and are isomorphic to the Łukasiewicz t-norm $T_L(x,y) = \max(0, x+y-1)$.

2 Discrete t-norms

Practical applications supported by computer implementations are often based on arguments taken from a finite scale, i.e. a finite subchain $\{x_1, \ldots, x_n\}$ of $[0,1]$, where $x_1 = 0 < x_2 < \ldots < x_n = 1$. Note that any other scale of length n can be transformed into a scale of the above type. Since there is no problem with introducing the concept of a t-norm on an arbitrary bounded poset [3] (for an in-depth study, in particular on product lattices, see [2]), we can introduce t-norms on a finite chain as well.

Definition 1. Consider a finite chain $C_n = \{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$, with $x_1 < x_2 < \ldots < x_n$. A $C^2 \to C$ mapping $\mathcal{D}$ is called a discrete t-norm (on C) if it is commutative, associative, increasing and has x_n as neutral element, i.e. $(\forall i \in \{1, \ldots, n\})(\mathcal{D}(x_i, x_n) = x_i)$.

All algebraic notions (the Archimedean property, strict monotonicity, nilpotent elements, ...) introduced for t-norms on $[0,1]$ can be introduced for discrete t-norms in a straightfor-

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ward way. The particular structure of a finite chain, however, leads to additional observations [2]: the Archimedean property is equivalent to the diagonal inequality, there exists no strictly increasing discrete t-norm, etc.

The number of discrete t-norms on a finite chain C_n is known only for $n \leq 14$ [1] (for $n = 14$ there are 382.549.464 discrete t-norms). An important subclass is the class of smooth discrete t-norms characterized by Mayor and Torrens as a counterpart of continuous t-norms on $[0, 1]$ [6]. In fact, the continuity of a t-norm $\mathcal{T}$ on $[0, 1]$ is equivalent with

$$(\forall(x, y) \in [0, 1]^2)(x \leq y \Leftrightarrow (\exists z \in [0, 1])(x = \mathcal{T}(y, z))).$$

However, for a discrete t-norm $\mathcal{D}$ on a finite chain C_n, the above property is equivalent with the following: for any $(i, j) \in \{2, \ldots, n\}^2$ it holds that if $\mathcal{D}(x_i, x_j) = x_r$, then $\mathcal{D}(x_{i-1}, x_j) = x_p$ and $\mathcal{D}(x_i, x_{j-1}) = x_q$ with $r - 1 \leq p, q \leq r$. This property is called the smoothness property by Godo and Sierra [4]; it can be seen as some kind of Lipschitz condition. The class of smooth discrete t-norms on a finite chain C_n has been characterized completely by Mayor and Torrens [6]. Firstly, there exists a unique smooth Archimedean discrete t-norm D_L on it, defined by $D_L(x_i, x_j) = x_{\max(1,i+j-n)}$. Secondly, for any given subset I of $\{x_2, \ldots, x_{n-1}\}$, there exists a unique smooth discrete t-norm that has I as set of non-trivial idempotent elements. As a consequence, smooth discrete t-norms show an ordinal sum structure similar to that of continuous t-norms on $[0, 1]$. It then also follows that there exist 2^{n-2} smooth discrete t-norms on C_n.

3 Discrete versus discretized

An interesting problem with possible practical consequences is that of the extension of a discrete t-norm $\mathcal{D}$ on a finite subchain $C_n = \{x_1, \ldots, x_n\}$ of $[0, 1]$, where $x_1 = 0 < x_2 < \ldots < x_n = 1$, to a t-norm $\mathcal{T}$ on $[0, 1]$, i.e. to a t-norm $\mathcal{T}$ on $[0, 1]$ such that $\mathcal{T}|_{C^2} = \mathcal{D}$. The following results hold:

(i) A right-continuous extension of $\mathcal{D}$ is always possible.

(ii) A continuous extension of $\mathcal{D}$ is not always possible. In fact, there exist Archimedean discrete t-norms without continuous extension. Hence, Archimedean discrete t-norms are not necessarily representable by means of a continuous additive generator.

(iii) If $\mathcal{D}$ is smooth, then it can always be extended to a continuous t-norm $\mathcal{T}$ that has the same idempotent elements as $\mathcal{D}$.

(iv) There exist non-smooth discrete t-norms with a continuous extension. This is the case for the weakest discrete t-norm D_W (defined by $D_W(x_i, x_j) = 0$ whenever $\max(x_i, x_j) < 1$).

Obviously, the problem of characterizing all discrete t-norms admitting a continuous extension is still open!

On the other hand, we may wonder when a discretization of a t-norm on $[0, 1]$ yields a discrete t-norm, i.e. given a t-norm $\mathcal{T}$ on $[0, 1]$, for which finite subchains C of $[0, 1]$ is $\mathcal{D} = \mathcal{T}|_{C^2}$ a discrete t-norm. The main problem is, of course, that the discretization should yield an operation which is internal on the selected subchain. The following results hold:

(i) For $\mathcal{T} = T_M$ (the minimum operator), any finite subchain C is suitable and $\mathcal{D} = D_M$ is also the minimum operator.

(ii) For a strict t-norm $\mathcal{T}$, only the trivial scale $C = \{0, 1\}$ is acceptable and then, of course, $\mathcal{D}$ is nothing else but the Boolean conjunction.

(iii) For a nilpotent t-norm $\mathcal{T}$ with additive generator f, it holds that $\mathcal{D}$ is a discrete t-norm if and only if $f(C) = \{f(x) \mid x \in C\}$ is relatively closed under addition, i.e. for any $(x, y) \in C^2$ there either exists $z \in C$ such that $f(x)+f(y) = f(z)$ or $f(x)+f(y) > f(0)$.

(iv) Finally, we consider the case of a general continuous t-norm $\mathcal{T}$ with ordinal sum representation $(\langle a_k, b_k, \mathcal{T}_k\rangle)_{k\in K}$. In this case, C can contain any of the idempotent elements of $\mathcal{T}$ and $C_k = C\cap\,]a_k, b_k[$ can be non-empty if and only if $\mathcal{T}_k$ is nilpotent. In that case, if f_k is an additive generator of $\mathcal{T}_k$, for any $k \in K$, a necessary and sufficient condition is again that $f_k(C_k)$ is relatively closed under addition.

References

[1] B. De Baets, *Database entry A030453*, Sloane's On-Line Encyclopedia of Integer Sequences, (URL: http://www.research.att.com/˜njas/sequences/index.html).

[2] B. De Baets and R. Mesiar, *Triangular norms on product lattices*, Fuzzy Sets and Systems, to appear.

[3] G. De Cooman and E. Kerre, *Order norms on bounded partially ordered sets*, J. Fuzzy Math. **2** (1994), 281–310.

[4] L. Godo and C. Sierra, *A new approach to connective generation in the framework of expert systems using fuzzy logic*, Proc. Eighteenth Internat. Symposium on Multiple-Valued Logic (Palma de Mallorca, Spain), IEEE Computer Society Press, 1988, pp. 157–162.

[5] C. Ling, *Representation of associative functions*, Publ. Math. Debrecen **12** (1965), 189–212.

[6] G. Mayor and J. Torrens, *On a class of operators for expert systems*, Internat. J. Intell. Sys. **8** (1993), 771–778.

[7] B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, Elsevier, 1983.

Non-Continuous Generated T-Norms

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Let us recall some definitions.

Definition 1. A function $T : [0,1]^2 \to [0,1]$ which $\forall x, y, z \in [0,1]$ fulfills:

$T(T(x,y),z) = T(x,T(y,z))$	(associativity)
$x \leq z \Rightarrow T(x,y) \leq T(z,y)$	(monotonicity)
$T(x,y) = T(y,x)$	(commutativity)
$T(x,1) = x$	(boundary condition)

is called a *triangular norm* (a *t-norm* for short).

Definition 2. Let $f : [0,1] \to [0,\infty]$ be a non-increasing function. Then the function $f^{(-1)} : [0,\infty] \to [0,1]$ defined by

$$f^{(-1)}(y) = \sup\{x \in [0,1] \mid f(x) > y\}$$

is called the *pseudo-inverse of function* f.

Definition 3. Let $f : [0,1] \to [0,\infty]$ be a strictly decreasing function, $f(1) = 0$ and let the function $T : [0,1]^2 \to [0,1]$ be given by formula

$$T(x,y) = f^{(-1)}(f(x) + f(y)) \quad \forall x, y \in [0,1] \tag{1}$$

where $f^{(-1)}$ is the pseudo-inverse of the function f. Then the function f is called a *conjunctive additive generator of the function* T.

Definition 4. Let $f : [0,1] \to [0,\infty]$ be a non-increasing function. The range of the function f is *relatively closed under the addition* if and only if for all $x, y \in [0,1]$ we have $f(x) + f(y) \in Ran(f)$ or $f(x) + f(y) \geq \lim_{x \to 0^+} f(x)$.

Theorem 1 was introduced by Klement, Mesiar and Pap in [4].

Theorem 5. *Let $f : [0,1] \to [0,\infty]$ be a conjunctive additive generator of T with Ran(f) relatively closed under the addition. Then the function T is a t-norm.*

The requirement 'Ran(f) is relatively closed under the addition' is sufficient but not necessary condition of associativity of corresponding generated function T. We can formulate questions: What is the characterization of additive generators of t-norms whose range is not relatively closed under the addition? What are the sufficient (or necessary) conditions of the associativity of T defined by (1)?

Next Theorem 2 gives one of the necessary conditions of the associativity of T. First we have to introduce some notation:

$$D_f(0,1) = \{a \in (0,1) \mid f(a_-) > f(a_+)\}$$

$$L_-(f) = \{v \in R^+ \mid \exists t \in (0,1], v = f(t_-)\}$$

$$H_-(f) = \{v \in R^+ \mid \exists a \in D_f(0,1), v = f(a_-) - f(a)\}.$$

Theorem 6. *Let $f : [0,1] \to [0,\infty]$ be a conjunctive additive generator of T. If T is a t-norm, then $H_-(f) \cap L_-(f) = \emptyset$.*

From this theorem we can obtain that if $f : [0,1] \to [0,\infty]$ is a left-continuous conjunctive additive generator of the function T and T is a t-norm then f is continuous function on $(0,1]$ (and then T is a continuous Archimedean t-norm).

Now we will characterize the set I_T of all non-trivial idempotent elements of generated function T. Denote

$$D_0 = D_0(f) = \{a \in (0,1) \mid a \in D_f \text{ and } 2f(a) \leq f(a_-)\}.$$

$$I_T = \{x \in (0,1) \mid T(x,x) = x\} \quad \text{(idempotent elements)}$$

for $M \subset [0,\infty]$,

$$C(M) = \{t \in [0,\infty] \mid \exists \{x_n\}_{n\in N} \subset M - \{t\} \text{ and } t = \lim_{n\to\infty} x_n\}.$$

Theorem 7. *Let f be some additive generator of t-norm T. Then $I_T = D_0$.*

Set I_T has following properties: $I_T \subset D_f$, I_T is countable set and $C(I_T) \subset \{0,1\}$.

Next Theorem 4 gives one of the sufficient conditions ensuring the associativity of T. Let $D_f = D_0$ and $\exists c \in D_f$ such that $c = \max D_f$.

Denote

$$\bar{a} = \max\{D_f \cap [0,a)\} \quad \forall a \in D_f \cup 1$$

(note that $\max \emptyset = 0$). We know that $\bar{a} < a$.

Theorem 8. *Let f be a conjunctive additive generator of T and $D_f = D_0$. If there exists $c = \max D_f$ and for all $a \in D_f$ it holds $f(\bar{a}_+) - f(a_-) \leq f(c_+)$, then T is a t-norm.*

There are examples of generated t-norms with infinite set of non-trivial idempotents (these t-norms are not Archimedean).

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References

[1] Abel N. H., *Untersuchungen der Funktionen zweier unabhängigen veränderlichen Größen x und y wie f(x,y), welche die Eigenschaft haben, daß f(z,f(x,y)) eine symmetrische Funktionen von x,y und z ist,* J. Reine Angew. Math., Vol 1, 1826, 11-15.

[2] Ling C. M., *Representation of associative functions,* Publ. Math., Debrecen 12, 1965, 189-212.

[3] Klement E. P., Mesiar R. and Pap E., *Additive generators of t-norms which are not necessarily continuous,* EUFIT'96, Vol. 1, Aachen, 1996, 70-73.

[4] Klement E. P., Mesiar R. and Pap E., *Triangular norms,* (monograph in preparation).

[5] Schweizer B. and Sklar A., *Espaces Métriques Aléatoires,* Comptes Rendus Acad. Sci. 247, Paris, 1958.

[6] Schweizer B. and Sklar A., *Probabilistic metrics spaces,* North-Holland, New York, 1983.

[7] Viceník P., *A note on generators of t-norms,* Busefal 75, 1998.

[8] Viceník P., *Additive generators of non-continuous triangular norms,* (in preparation).

Powers of T-Norms

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The rth power of a continuous t-norm is defined for positive real numbers r, generalizing the notion of the 2nd power discussed in a paper of Mesiar and Navara [4] on diagonals of continuous triangular norms. Our definition is based on an example presented in [2] that called upon powers of strict t-norms. We generalize the results of Mesiar and Navara somewhat, and make the point that, in general, use of automorphisms of the ordered unit interval as generators enables one to take advantage of the underlying algebra, as this group provides a natural mathematical framework in which to sort out what is going on and to describe the results. In this spirit, we point out that the representation theorems for Archimedean t-norms can be phrased as follows.

Theorem. *A t-norm $\circ$ is strict if and only if it is isomorphic to multiplication on the unit interval—that is, given a strict t-norm $\circ$, there is an automorphism f of $I = ([0,1], \leq)$ satisfying $f(x \circ y) = f(x) f(y)$ for all $x, y \in [0,1]$. Two automorphisms f and g of I give isomorphisms for the same t-norm if and only if there is a positive real number s such that $f(x) = (g(x))^s$ for all $x \in [0,1]$.*

The automorphism f is called a (multiplicative) **generator** for $\circ$, since $\circ$ is obtained from multiplication by the formula $x \circ y = f^{-1}(f(x) f(y))$.

Theorem. *A t-norm $\circ$ is nilpotent if and only if it is isomorphic to the Łukasiewicz t-norm $x \bullet y = (x + y - 1) \vee 0$, that is, there is an isomorphism $f : (\mathbb{I}, \circ) \to (\mathbb{I}, \bullet)$ satisfying $f(x \circ y) = (f(x) + f(y) - 1) \vee 0$ for all $x, y \in [0,1]$. The isomorphism f is unique.*

We call the isomorphism f the **L-generator** for $\circ$, since $\circ$ is obtained from the Łukasiewicz t-norm by the formula $x \circ y = f^{-1}((f(x) + f(y) - 1) \vee 0)$. Note that f is, in particular, an automorphism of $\mathbb{I}$.

For positive integers n, the nth power of $\circ$ is $x^{[n]} = x \circ x \circ \cdots \circ x$ (n times). The 2nd power of $\circ$ is $x^{[2]} = x \circ x$, commonly known as the **diagonal** of the t-norm $\circ$. The notion of nth power naturally extends to rth powers for positive real numbers r. The theory readily provides a mechanism for representing these rth powers for strict t-norms as the functions $x^{[r]} = f^{-1} r f(x)$, where f is any isomorphism of $\circ$ with multiplication and r represents the ordinary rth power $r(x) = x^r$, and for nilpotent t-norms as the functions $x^{[r]} = f^{-1}((r \cdot f(x) - r + 1) \vee 0)$, where f is the L-generator of $\circ$ and $r \cdot f(x)$ denotes the ordinary product of r and $f(x)$. The functions $x^{[r]}$ are independent of the choice of isomorphism and satisfy the usual properties of powers.

Mesiar and Navara proved in [4] that any automorphism δ of $\mathbb{I}$ satisfying $\delta(x) < x$ for all $x \in (0,1)$ is the diagonal of a strict t-norm, that is, for any such δ there exists an $f \in \mathrm{Aut}(\mathbb{I})$ such that $\delta(x) = f^{-1}(f(x) f(x))$. We generalize this characterization to arbitrary rth powers as follows.

Theorem. *Let $r \in R^+$. A function $\delta : [0,1] \to [0,1]$ is the rth power for some strict t-norm $\circ$ if and only if $\delta \in \mathrm{Aut}(\mathbb{I})$ and one of the following holds:*

1. $r > 1$ *and* $\delta(x) < x$ *for all* $x \in (0,1)$.

2. $r = 1$ *and* $\delta(x) = x$ *for all* $x \in (0,1)$.

3. $r < 1$ *and* $\delta(x) > x$ *for all* $x \in (0, 1)$.

Two automorphisms $f, g \in \mathrm{Aut}(\mathbb{I})$ generate strict t-norms having the same rth power if and only if fg^{-1} is in the centralizer of r in $\mathrm{Aut}(\mathbb{I})$. The question of uniqueness of the strict t-norm is thus reduced to the problem of describing the centralizer of r in $\mathrm{Aut}(\mathbb{I})$. This is done in terms of automorphisms of subintervals of the unit interval.

We prove theorems analogous to the above for nilpotent t-norms. The characterization of rth powers of continuous t-norms can be pieced together from the characterizations for strict and nilpotent t-norms.

References

[1] M. Gehrke, C. Walker, and E. Walker, "DeMorgan systems on the unit interval," *Int. Jour. Intelligent Systems*, **11**, 733-750, 1996.

[2] M. Gehrke, C. Walker, and E. Walker, "A mathematical setting for fuzzy logics," *Int. Jour. of Uncertainty, Fuzziness and Knowledge-Based Systems*, **5**(3), 223-238, 1997.

[3] M. Gehrke, C. Walker, and E. Walker, "A note on negations and nilpotent t-norms," preprint.

[4] R. Mesiar and M. Navara, "Diagonals of continuous triangular norms," preprint.

Semiring-Valued Measures and Integrals

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In modeling fuzzy systems and decision making there are different types of integrals as Choquet, Sugeno, Weber, Maslov, Sugeno-Murofushi, etc. (see [1], [2], [5]). On other side there are general integrals with values in abstract structures (see [6]) and multi-valued integrals.

We shall present an integral, which will cover many of important integrals, and which takes values in an abstract semiring endowed with some special metric. Namely, it is known, that the topology of a uniform semigroup $(X, \oplus)$ with a neutral element $\mathbf{0}$ can be characterized by a family of pseudo-metrics $\{d_i\}_{i \in I}$ which satisfy the inequality

$$d_i(x \oplus x', y \oplus y') \leq d_i(x, y) + d_i(x', y') \tag{1}$$

for $i \in I, x, x', y, y' \in X$ (see [3]).

We consider a semiring $(P, \oplus, \odot)$ with a compatible partial ordering $\leq$ which is endowed with a metric d compatible with $\leq$ and which if $\oplus$ is idempotent satisfies the condition

$$d(x \oplus x', y \oplus y') \leq \max(d(x, y), d(x', y'))$$

otherwise d satisfies the condition (1).

Let Σ be a σ-algebra. We consider $\oplus$-measure $m : \Sigma \to P$, i.e., $m(\emptyset) = 0$; for every $A, B \in \Sigma$ such that $A \cap B = \emptyset$ we have $m(A \cup B) = m(A) \oplus m(B)$, and for every sequence $\{A_n\}$ ¿from Σ and $A \in \Sigma$ such that $A_n \uparrow A$ we have $m(A_n) \uparrow m(A)$. We introduce an integral with respect to $\oplus$-measure using ε-net construction (see [5]). If $\oplus$ is idempotent extension of $\oplus$-measure is not unique (see [4]). This has implications on the corresponding integrals.
We present some applications in optimization, and nonlinear partial differential equations.

References

[1] V.P. Maslov, S.N. Samborskij, *Idempotent Analysis*, Advances in Soviet Mathematics, AMS, Providence, 1992.

[2] R. Mesiar, Possibility measures, integration, fuzzy possibility measures, *Fuzzy Sets and Systems* 92 (1997), 191-196.

[3] E.Pap, *Functional Analysis (Sequential convergences)* (in Serbian, English summary), Novi Sad, 1982.

[4] E. Pap, Extension of $\oplus$-decomposable measures, *Atti Sem. Mat. Fis. Univ. Modena* 41 (1993), 109-120.

[5] E. Pap, *Null-Additive Set Functions*, Kluwer, Dordrecht, 1995.

[6] M. Sion, *Theory of Semigroup Valued Measures* , Lecture Notes in Mathematics 355, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

Varieties Generated by Strict de Morgan Systems on the Unit Interval

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Much research in fuzzy logic and fuzzy sets concentrates on the study of various algebraic systems based on the unit interval such as de Morgan systems since such algebraic systems give rise to corresponding operations on fuzzy sets and connectives of fuzzy logic. As described in our paper [1], two algebras with the same arity operations give rise to the same fuzzy logic – or equivalently, yield the same equational properties of fuzzy sets – if and only if the two algebras generate the same variety in the sense of universal algebra [According to Birkhoff's fundamental theorem, a variety may either be seen as a class of algebras closed under homomorphic images, subalgebras, and products, or as the class of all algebras satisfying some set of equations].

In our paper [2], we advocate an abstract algebraic point of view in the study of de Morgan systems. Rather than talking of generators for norms and negations, we considered these as isomorphisms between algebraic systems. Thus we determined exactly which Archimedean de Morgan systems are isomorphic. It is clear that if two algebraic systems are isomorphic then they satisfy the same equational properties and thus generate the same variety. The converse is however far ¿from true. That is, non-isomorphic algebras can generate the same variety. In fact, all varieties other than ones consisting solely of one element algebras have many generators. For a more pertinent example, the systems $(\mathbb{I}, \circ)$ and $(\mathbb{I}, \triangle)$ where $\mathbb{I}$ is the unit interval as a bounded lattice and $\circ$ is any strict t-norm and $\triangle$ is any nilpotent t-norm generate the same variety. In contrast, we will show that the variety generated by one strict de Morgan system is contained in another such if and only if the de Morgan systems are isomorphic. In conjunction with our previous characterization of the isomorphism classes of strict de Morgan systems [2], we thus get a complete account of the distinct varieties generated by a strict de Morgan system since there is one for each isomorphism class. This yields uncountably many distinct fuzzy logics.

Other questions we consider are: existence of a finite basis for such varieties, finite generation, subvariety structure and generalizations to varieties generated by other de Morgan systems. The existence of a finite basis, that is, a finite set of equations that determines the variety – such as there is for the variety generated by the isomorphism class of Boolean systems, also known as the variety of MV-algebras – essentially means that we have a logic in the classical sense: that is, a finitary syntactic deduction system corresponding to the logic in question. It should be noted that it follows from a cardinality consideration that all but at most countably many of the varieties generated by de Morgan systems are not finitely based. Finite generation, that is, a variety being generated by a single finite algebra – such as is the case for classical fuzzy logic (given by lattice meet) [1] – has as consequence that the corresponding logical equivalence relation is decidable. This is also extremely rare: we show that none of the varieties generated by strict de Morgan systems are finitely generated. The subvarieties of a variety correspond to the extensions of the corresponding logic. Finally, we discuss generalizations of these results.

References

[1] A Mathematical Setting for Fuzzy Propositional Logic, M. Gehrke, C. Walker, and E. Walker, *International Journal of Uncertainty, Fuzziness, and Knowledge-based Systems*, **5**, no. 3 (1997), pp 223-238.

[2] de Morgan Systems on the Unit Interval, M. Gehrke, C. Walker, and E. Walker, *International Journal of Intelligent Systems*, **11**(1996), pp. 733-750.

Fuzzy Functions: A Fuzzy Extension of the Category *Set* and Some Related Categories

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Preliminaries. Let $L = (L, \leq, \wedge, \vee, *)$ be a completely distributive GL-monoid (cf e.g. [1], [2]). It is well known that every GL-monoid is residuated, i.e. there exists a further binary operation - implication " $\longmapsto$ " such that $\alpha * \beta \leq \gamma \iff \alpha \leq \beta \longmapsto \gamma \quad \forall \alpha, \beta, \gamma \in L$. We set $\alpha^2 = \alpha * \alpha$ and further by induction: $\alpha^n = \alpha^{n-1} * \alpha$

Modifying slightly the terminology of U.Höhle (cf e.g. [2]) by an *L-valued set* we call a pair (X, E) where X is a set and E is an *L-valued equality*, i.e. a mapping $E : X \times X \to L$ such that

(1eq) $E(x, x) = 1$;

(2eq) $E(x, y) = E(y, x)$;

(3eq) $E(x, y) * E(y, z) \leq E(x, z)$.

An L-valued set (X, E) is called *separated* if

(4eq) $E(x, y) = 1 \iff x = y$

By $L - SET(L)$ we denote the category whose objects are triples (X, E, A) where (X, E) is an L-valued set and A is its *extensional L-subset* (i.e. a mapping $A : X \to L$ such that $\sup_x A(x) * E(x, y) \leq A(y), \forall y \in X$) and morphisms from (X, E_X, A) to (Y, E_Y, B) are mappings $f : X \to Y$ which preserve equalities (i.e. $E_X(x_1, x_2) \leq E_Y(fx_1, fx_2)$) and "respect L-subsets", i.e. $A \leq B \circ f$. Let $L - SET'(L)$ stand for the full subcategory of the category $L - SET(L)$ determined by separated L-valued sets.

To recall the concept of an *$L-$fuzzy category* [4, 5], consider an ordinary (classical) category $\mathcal{C}$ and let $\omega : Ob(\mathcal{C}) \to L$ and $\mu : Mor(\mathcal{C}) \to L$ be $L-$fuzzy subclasses of its objects and morphisms respectively. Now, an *$L-$fuzzy category* can be defined as a triple $(\mathcal{C}, \omega, \mu)$ satisfying the following axioms ([5], cf [4] in case $* = \wedge$):

1^0 $\mu(f) \leq \omega(X) \wedge \omega(Y) \quad \forall X, Y \in Ob(\mathcal{C})$ and $\forall f \in Mor(X, Y)$;

2^0 $\mu(g \circ f) \geq \mu(f) * \mu(g)$ whenever the composition $g \circ f$ is defined;

3^0 $\mu(e_X) = \omega(X)$ where $e_X : X \to X$ is the identity morphism.

Our aim is, starting from the category $L - SET(L)$, to define a fuzzy category $L - \mathcal{F}SET(L)$ having the same class of objects as $L - SET(L)$ but an essentially wider class of "potential" morphisms.

Definition of the fuzzy category $L - \mathcal{F}SET(L)$**.**
Let $L - FSET(L)$ denote the (ordinary) category having the same objects as $L - SET(L)$ and whose morphisms, called (potential) *fuzzy functions*, from (X, E_X, A) to (Y, E_Y, B) are L-relations $F : X \times Y \to L$ (cf e.g. [3]) such that

(1ff) $\sup_x A(x) * F(x,y) \leq B(y) \quad \forall y \in Y$;

(2ff) $F(x,y) * E_X(x,x') * E_Y(y,y') \leq F(x',y') \quad \forall x,x' \in X, \forall y,y' \in Y$;

(3ff) $F(x,y) * F(x',y') * E_X(x,x') \leq E_Y(y,y') \quad \forall x,x' \in X, \forall y,y' \in Y$.

Given two fuzzy functions $F : (X, E_X, A) \to (Y, E_Y, B)$ and $G : (Y, E_Y, B) \to (Z, E_Z, C)$ we define their composition $G \circ F : (X, E_X, A) \to (Z, E_Z, C)$ by setting $(G \circ F)(x,z) = \bigvee_{y \in Y}(F(x,y) * G(y,z))$; the identity morphism is defined by the corresponding L-valued equality: $E : (X, E, A) \to (X, E, A)$.

Further, we define an L-subclass μ of the class of all morphisms of $L - FSET(L)$ by setting
$$\mu(F) = \inf_x \sup_y F(x,y).$$
In case $\mu(F) \geq \alpha$ we refer to F as a *fuzzy α-function* or a *fuzzy α-morphism*
Noticing that $\mu(G \circ F) \geq \mu(G) * \mu(F)$ (by complete distributivity of L) and $\mu(E) = 1$ we conclude that a *fuzzy category* $L - \mathcal{F}SET(L)$= $(L - FSET(L), 1_{Ob(L-FSET(L))}, \mu)$ is thus obtained.

Some (fuzzy) subcategories of the fuzzy category $L - \mathcal{F}SET(L)$**.**
For a fixed α let $L - \mathcal{F}_\alpha SET(L)$ consist of all objects of $L - \mathcal{F}SET(L)$ and its fuzzy α-morphisms. In case α is idempotent, $L - \mathcal{F}_\alpha SET(L)$ is a usual (crisp) category. In particular, it is a crisp category in case $\alpha = 1$.
If L_1, L_2, $L_3 \subset L$, then by $L_1 - \mathcal{F}SET(L_2, L_3)$ we denote the (fuzzy) subcategory of $L - \mathcal{F}SET(L)$, whose objects (X, E, A) satisfy the conditions $A(X) \subset L_1$ and $E(X \times X) \subset L_2$, and whose morphisms satisfy the condition $F(X \times Y) \subset L_3$. By specifying the sets L_1, L_2 and L_3 some known and new (fuzzy) categories related to L-sets can be characterized as (fuzzy) subcategories of $L_1 - \mathcal{F}SET(L_2, L_3)$-type or of $L_1 - \mathcal{F}SET'(L_2, L_3)$-type.

Images and preimages of L**-sets under fuzzy functions.**
Given a fuzzy function $F : (X, E_X) \to (Y, E_Y)$ and L-subsets $A : X \to L$ and $B : Y \to L$ of X and Y respectively, we define a fuzzy set $F(A) : Y \to L$ (the image of A under F) by the equality $F(A)(y) = \sup_x F(x,y) \odot A(x)$ and a fuzzy set $F(B) : X \to L$ (the preimage of B under F) by the equality $F(B)(x) = \sup_y F(x,y) \odot B(y)$; here $\odot$ can be chosen either as $\wedge$ or as $*$.
Proposition 1 [*Basic properties of images and preimages of L-sets under fuzzy functions*]

1 $F(\bigvee_{i \in \mathcal{I}}(A_i) = \bigvee_{i \in \mathcal{I}} F(A_i) \qquad \forall \{A_i : i \in I\} \subset L^X$;

2 $F(A_1 \bigwedge A_2) \leq F(A_1) \bigwedge F(A_2) \qquad \forall A_1, A_2 \in L^X$;

3 $(\bigwedge_{i \in \mathcal{I}} F(B_i))^5 \leq F(\bigwedge_{i \in \mathcal{I}}(B_i)) \leq \bigwedge_{i \in \mathcal{I}} F(B_i) \qquad \forall \{B_i : i \in I\} \subset L^X$;

$3_\wedge$ $(\bigwedge_{i \in \mathcal{I}} F(B_i))^3 \leq F(\bigwedge_{i \in \mathcal{I}}(B_i)) \leq \bigwedge_{i \in I} F(B_i) \qquad \forall \{B_i : i \in I\} \subset L^X$ – in case $\odot = \wedge$;

$3_\wedge^\wedge$ $\bigwedge_{i \in \mathcal{I}} F(B_i) = F(\bigwedge_{i \in \mathcal{I}}(B_i)) \qquad \forall \{B_i : i \in I\} \subset L^X$ – in case $\odot = * = \wedge$

4 $(\bigvee_{i \in \mathcal{I}} F(B_i))^5 \leq F(\bigvee_{i \in \mathcal{I}}(B_i)) \leq \bigvee_{i \in \mathcal{I}} F(B_i) \qquad \forall \{B_i : i \in \mathcal{I}\} \subset L^X$;

$4_\wedge$ $(\bigvee_{i \in \mathcal{I}} F(B_i))^3 \leq F(\bigvee_{i \in \mathcal{I}}(B_i)) \leq \bigvee_{i \in \mathcal{I}} F(B_i) \qquad \forall \{B_i : i \in \mathcal{I}\} \subset L^X$ – in case $\odot = \wedge$;

$4^{\wedge}_{\wedge}$ $(\bigvee_{i\in\mathcal{I}} F(B_i)) = F(\bigvee_{i\in\mathcal{I}}(B_i)) \quad \forall\{B_i : i\in\mathcal{I}\}\subset L^X$ – in case $\odot = * = \wedge$;

5_* $F(F(B)) \leq B$ if $\odot = *$.

Injectivity, surjectivity and bijectivity of fuzzy functions.
A fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$ is called *injective*, if
(inj) $F(x, y) * F(x', y') * E_Y(y, y') \leq E_X(x, x')$;
a fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$ is called *α-surjective* if
$(sur1^\alpha)$ $\inf_y \sup_x F(x, y) \geq \alpha$ and
$(sur2)$ $F(A) = B$.
In case F is injective and α-surjective, it is called *α-bijective.*
A fuzzy function $F : (X, E_X, A) \to (Y, E_Y, B)$ defines a fuzzy *relation* $F^{-1} : (Y, E_Y, B) \to (X, E_X, A)$ by setting $F^{-1}(y, x) = F(x, y) \quad \forall x \in X, \forall y \in Y$.
Proposition 2 [*Basic properties of injections, α-surjections and α-bijections*]

1 F^{-1} is a fuzzy function iff F is injective;

2 F is α-bijective iff F^{-1} is α-bijective.

3 If F is injective then $(F(A_1) \bigwedge F(A_2))^5 \leq F(A_1 \wedge A_2) \leq F(A_1) \bigwedge F(A_2) \quad \forall A_1, A_2 \in L^X$.
In particular:

$3_\wedge$ $(F(A_1) \bigwedge F(A_2))^3 \leq F(A_1 \wedge A_2) \leq F(A_1) \bigwedge F(A_2) \quad \forall A_1, A_2 \in L^X$ – in case $\odot = \wedge$;

$3^{\wedge}_{\wedge}$ $F(A_1 \wedge A_2) = F(A_1) \bigwedge F(A_2) \quad \forall A_1, A_2 \in L^X$ – in case $\odot = \wedge = *$;

4 If F is α-surjective, then $F(F(B)) \geq B \quad \forall B \in L^Y$; and hence, in particular, $F(F(B)) = B$ in case $\odot = *$.

Some categorical properties of the fuzzy category $L - \mathcal{F}SET(L)$ will be discussed. In particular, it will be shown that
Products and coproducts in the fuzzy category $L - \mathcal{F}SET(L)$ are defined, respectively, by the products and coproducts in the category $L - SET(L)$.

On the basis of $L - \mathcal{F}SET(L)$ some fuzzy categories related to topology and algebra can be naturally defined. Here are two examples:
Category $\mathcal{F}TOP(L)$. Let $(X, E_X) = (X, E_X, 1_X)$ and let $\tau_X \subset L^X$ be the (Chang-Goguen) L-topology on X. A fuzzy function $F : (X, E_X, \tau_X) \to (Y, E_Y, \tau_Y)$ is called *continuous* if $F(V) \in \tau_X$ for all $V \in \tau_Y$. L-topological spaces and continuous fuzzy mappings between them form the category $\mathcal{F}TOP(L)$.
Category $L - \mathcal{F}Gr(L)$. Let X be a group,
E_X be an L-valued equality on X such that $E_X(x \cdot y,\ x' \cdot y') \geq E_X(x, x') * E_X(y, y')$ for all $x, x', y, y' \in X$ and $G_X : X \to L$ be an L-subgroup of X (see e.g. [3]). A fuzzy function $F : (X, E_X, G_X) \to (Y, E_Y, G_Y)$ is called a *fuzzy homomorphism* if $F(x \cdot x',\ y \cdot y') \geq F(x, y) * F(x', y')$ for all $x, x', \in X,\ y, y' \in Y$
Some properties of the categories $\mathcal{F}TOP(L)$ and $L - \mathcal{F}Gr(L)$ will be discussed.

References

[1] U.Höhle, Commutative, residuated l-monoids, In: *Non-classical Logics and Their Applications to Fuzzy Subsets: A Handbook of the Mathematical Foundations of Fuzzy Subsets*, E.P. Klement and U. Höhle eds., Kluwer Acad. Publ., 1994, 53-106.

[2] U.Höhle, M-valued sets and sheaves over integral commutative *cl*-monoids, In: *Applications of Category Theory to Fuzzy Sets.*, S.E. Rodabaugh, E.P. Klement and U. Höhle eds., Kluwer Acad. Publ., 1992, pp. 33-72.

[3] C.V. Negoita, D.A. Ralescu, Application of Fuzzy Sets to System Analysis, *John Wiley & Sons*, New York, 1975.

[4] A. Šostak, On a concept of a fuzzy category, In: *14th Linz Seminar on Fuzzy Set Theory: Non-classical Logics and Applications.* Linz, Austria, 1992, pp. 62-66.

[5] A. Šostak, Fuzzy categories versus categories of fuzzily structured sets: Elements of the theory of fuzzy categories, In: *Mathematik-Arbeitspapiere*, Universität Bremen, vol **48** (1997), pp. 407-437.

Groups and Families of Triple Systems

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Let A be the automorphism group of $([0,1],\leq)$, the unit interval with its usual order structure. That is, A is the set of one-to-one order preserving maps of $[0,1]$ onto itself, and the group operation is composition of maps. Elements f of A give strict t-norms via $x \bigtriangleup y = f^{-1}(f(x)f(y))$. Some well known families of t-norms come from subgroups of A. For example, consider the one parameter family $\{\frac{xy}{a+(1-a)(x+y-xy)} : a > 0\}$ of t-norms. A generator of $\frac{xy}{a+(1-a)(x+y-xy)}$ is the automorphism $\frac{x}{x+a(1-x)}$, and the set $\{\frac{x}{x+a(1-x)} : a > 0\}$ is a subgroup G. of A. It is an example of a one parameter subgroup of A. It is a particularly transparent example: composition in G corresponds to multiplication of the parameters involved. Thus, if $f_a(x) = \frac{x}{x+a(1-x)}$, then $f_a f_b = f_{ab}$. So this group is isomorphic to the multiplicative group of positive real numbers. Clearly, one parameter subgroups of A give one parameter families of t-norms.

Some well known families of t-norms do not fit this mold, the Frank family

$$\left\{\log_a\left(1+\frac{(a^x-1)(a^y-1)}{a-1}\right) : a > 0, a \neq 1\right\}$$

with generators $\left\{\frac{a^x-1)}{a-1} : a > 0, a \neq 1\right\}$ being one such.

One goal here is to examine the well known families of t-norms from this viewpoint, and to construct new families by finding one and two parameter subgroups of A. But instead of limiting ourselves to subgroups of A, we consider subgroups G of the group M of all automorphisms and antiautomorphisms of $([0,1],\leq)$. If such a group contains a negation η, then for any automorphism $f \in G$, f and η give a triple system with t-norm generated by f, negation given by η, and t-conorm cogenerated by $f\eta$, all of these elements being in G. So such a group gives rise to a family of triple systems. Here is an example. Let $G = \left\{e^{-(-\ln x)^r} : r \neq 0\right\}$. The automorphisms in G are the elements with $r > 0$, the antiautomorphisms with $r < 0$, and only $r = -1$ gives a negation. The family of t-norms is $\left\{e^{-((-\ln x)^r+(-\ln y)^r)^{\frac{1}{r}}} : r > 0\right\}$. So for $r > 0$, we get the triple system with t-norm $e^{-((-\ln x)^r+(-\ln y)^r)^{\frac{1}{r}}}$, negation $e^{\frac{1}{\ln x}}$, and t-conorm $e^{-((-\ln x)^r+(-\ln y)^r)^{\frac{1}{r}}}$. For $r = 1$, this becomes the triple system with t-norm xy, negation $e^{\frac{1}{\ln x}}$, and t-conorm $e^{\frac{\ln x \ln y}{\ln xy}}$. We will present other examples, and some pertinent group theoretic facts. This research is in its infancy.

The Structure of Girard Monoids on $[0, 1]$

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4 Motivation

Triangular norms play basic role in several disciplines of mathematics, e.g., in fuzzy sets theory and its applications. Left-continuity of a t-norm is a frequently quoted property in the literature. The role of left-continuous t-norms having strong induced negations (that is, Girard monoids on $[0, 1]$) are even more relevant. They are applied e.g. in the field of non-classical logics.

Continuous t-norms are known as ordinal sums of continuous Archimedean t-norms (see e.g. [10]) hence their structure is well understood. But despite of the importance of the topic, there isn't any result in the literature concerning the structure left-continuous t-norms. Moreover, only three basic examples of left-continuous t-norms having strong induced negations are known so far: One is the (continuous) nilpotent class, its representative is the Łukasiewicz t-norm given by

$$T(x, y) = \max(x + y - 1, 0). \tag{1}$$

An other is the family of nilpotent minimum [1], its representative is

$$T(x, y) := \begin{cases} 0 & \text{if } y \leq 1 - x \\ \min(x, y) & \text{otherwise} \end{cases}. \tag{2}$$

The third is the family of nilpotent ordinal sums [4] (in narrow sense). This family (in wide sense) contains the two previous ones. A representative is given by

$$T(x, y) = \begin{cases} 0 & \text{if } x \leq 1 - y \\ \frac{1}{3} + x + y - 1 & \text{if } \frac{1}{3} \leq x, y \leq \frac{2}{3} \text{ and } x > 1 - y \\ \min(x, y) & \text{otherwise} \end{cases}. \tag{3}$$

In order to fill in the gap between the particular importance of left-continuous t-norms and the pure knowledge concerning them we completely describe here the structure of left-continuous t-norms having strong induced negations.

In case of continuous t-norms this description is the following: The indecomposable class is the class of continuous Archimedean t-norms. Decomposition and construction is answered by the well-known ordinal sum theory. So for the description of left-continuous t-norms having strong induced negations we need to know the following three things: What is **indecomposability**, how to **decompose** into indecomposable units and finally, how to **construct** left-continuous t-norms having strong induced negations. All these questions are answered in this talk.

After the basic definitions in Section 5 for left-continuous t-norms two properties are introduced in Section 6: The *rotation invariancy property* and the *self quasi-inverse property.* These properties hold for a t-norm T if and only if T is a left-continuous triangular norm having strong induced negation; whence any of these properties characterizes the class of left-continuous triangular norms having strong induced negations. The names and the geometrical meaning of these properties are investigated, explained and examples are given. For more on this topic we refer to [3].

To make the understanding of the constructions in Sections 8 and 9 easier we define 'in medias res' indecomposability and present the decomposition theorem in Section 7. The interested reader can find the details in [9].

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Inspired by results of Section 6 a new method is introduced in Section 8 which from any left-continuous triangular norm which has no 0-divisors produces a left-continuous but not continuous triangular norm having strong induced negation. The method is called *rotation*. The name is motivated by a geometrical feature of the resulted t-norm, namely, its graph is produced via the 'rotation' of the graph of the starting t-norm. For more on this construction see [7]. For the deeper understanding of the motivation of rotation construction we refer the reader to [3].

Then in Section 9 we introduce the second method which produces left-continuous t-norms which have strong induced negations from a pair of certain connectives. The construction is called *rotation-annihilation*. For more on this contruction we refer to [8].

An infinite number of new families of left-continuous triangular norms having strong induced negations can be generated with these constructions, which provides a tremendously wide spectrum of choice for e.g. logical and set theoretical connectives in non-classical logic and in fuzzy theory; thus fairly enlarging the set of the above described three families which are until now the only known examples.

If first we decompose and than we construct from the decomposed units we get back the starting connective. If we construct first and than we can decompose the constructed t-norm into the starting connective(s). Therefore the presented decomposition and construction are inverse operations of each other. In this way the structure of left-continuous triangular norms having strong induced negations is completely described.

Finally, we remark that results of the present talk are generalized in the setting of partially-ordered semigroups in [5] and [6].

5 Basic Definitions

We need the following definitions in order to make the formulation and the reading of the results easier.

Definition A A triangular norm (t-norm for short) on $[a, b] \subset \mathbb{R}$ is a function $T : [a, b]^2 \to [a, b]$ such that for all $x, y, z \in [a, b]$ the following four axioms (T1)-(T4) are satisfied:

(*T1*) *Symmetry*	$T(x, y) = T(y, x)$	
(*T2*) *Associativity*	$T(x, T(y, z)) = T(T(x, y), z)$	
(*T3*) *Monotonicity*	$T(x, y) \leq T(x, z)$	whenever $y \leq z$
(*T4*) *Boundary condition*	$T(x, b) = x$	
(*T5*) *Boundary condition*	$T(x, a) = a$	
(*T6*) *Range condition*	$T(x, y) \leq \min(x, y)$.	

It is immediate to see that (T3) and (T4) imply (T5), and that (T1), (T3) and (T4) imply (T6). Clearly, a t-norm on $[0, 1]$ means simply t-norm in the usual sense.

Now we introduce a new class of two-place functions. This class will play a key-role in the sequel:

Definition 1. A triangular subnorm (t-subnorm for short) on $[a, b] \subset \mathbb{R}$ is a function $T : [0, 1]^2 \to [0, 1]$ such that for all $x, y, z \in [0, 1]$ axioms (T1), (T2), (T3) and (T6) are satisfied. A t-subnorm on $[0, 1]$ is called simply a t-subnorm.

Clearly any t-norm is a t-subnorm.

We say that T has 0-divisors if there is $x, y \in]a, b]$ such that $T(x, y) = a$. A t-norm (resp. t-subnorm) is said to be *continuous* or *left-continuous* if it is continuous or left-continuous as a two-place function. Further, a t-norm (resp. t-subnorm) on $[a, b]$ is clearly not else but a linear transformation of a t-norm (resp. t-subnorm):

Definition B Let $[a, b] \subset \mathbb{R}$. For any function $T : [0, 1] \times [0, 1] \to [0, 1]$, call the function $T_{[a,b]}$: $[a, b] \times [a, b] \to [a, b]$ defined by

$$T_{[a,b]}(x, y) = a + T\left(\frac{x-a}{b-a}, \frac{y-a}{b-a}\right)$$

the linear transformation of T into $[a,b]$. For $T_{[a,b]} : [a,b] \times [a,b] \to [a,b]$ call the function $T : [0,1] \times [0,1] \to [0,1]$ defined by

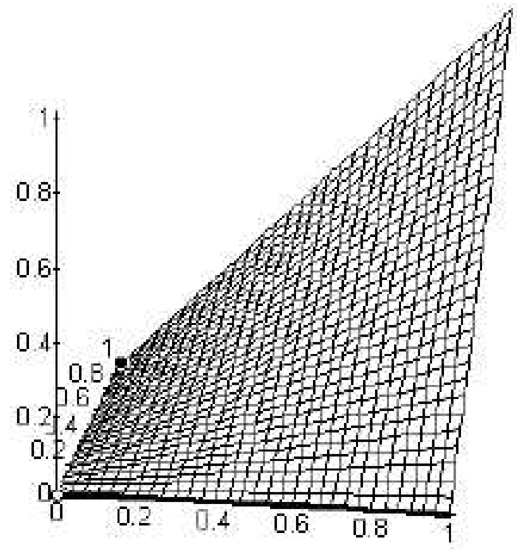
$$T(a,b) = T_{[a,b]}\left((a + x(b-a)), (a + y(b-a))\right)$$

the linear transformation of $T_{[a,b]}$ into $[0,1]$. Observe that linear transformation preserves left-continuity and 0-divisors.

Definition C A negation N on $[a,b] \subset \mathbb{R}$ is a non-increasing function on $[a,b]$ with boundary conditions $N(a) = b$ and $N(b) = a$. Such a negation is called strong if $N(N(x)) = x$ holds for all $x \in [a,b]$. We remark, that a strong negation is automatically a continuous function.

Definition D Let T be a left-continuous t-subnorm on $[a,b] \subset \mathbb{R}$. The residuated implication I_T generated by T is given by $I_T(x,y) = \sup\{t \in [a,b] \mid T(x,t) \leq y\}$. The negation induced by T is a negation on $[a,b]$ it is denoted by N_T and given by $N_T(x) = I_T(x,a)$. We say that a t-norm T *has strong induced negation* if N_T is a strong negation on $[a,b]$.

In Figure 1 we present the three-dimensional plots of the product t-norm given by $T(x,y) = x \cdot y$ and an ordinal sum with one Lukasiewicz summand. One can see easily the induced negation of them in the plane XY. It is basically the 'limit line' between the 0 and the positive part of the graphs.



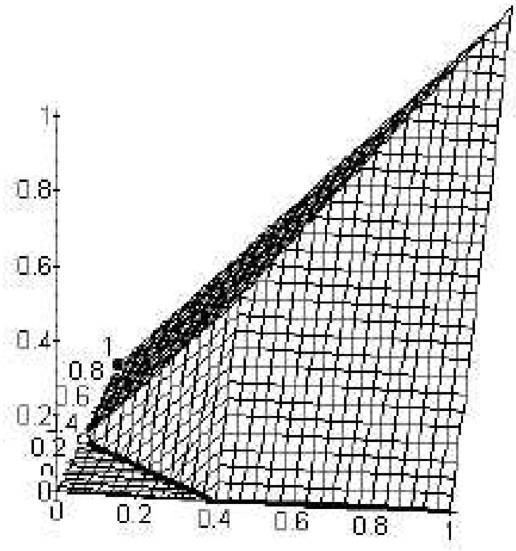


Figure 1: How the induced negation can be seen on the graph of the t-norm

We will need the following definition ([4]).

Definition 2. Let N be a strong negation (on $[0,1]$) and t is unique fixed point. Let $d \in [t, 1[$. Then $N_d : [0,1] \to [0,1]$ defined by

$$N_d(x) = \frac{N(x \cdot [d - N(d)] + N(d)) - N(d)}{d - N(d)}$$

is a strong negation (on $[0,1]$). Call N_d the zoomed d-negation of N.

6 Geometrical Properties

We introduce now two properties of left-continuous t-norms:

Definition 3. Let $[a,b] \in [0,1]$, T be a left-continuous t-norm on $[a,b]$ and N be a strong negation on $[a,b]$. We say that T admits the rotation invariancy property with respect to N (or T is rotation invariant with respect to N) if for all $x,y,z \in [a,b]$ we have

$$T(x,y) \le z \;\Leftrightarrow\; T(y,N(z)) \le N(x).$$

We say that T admits the self quasi-inverse property (with respect to N) if for all $x,y,z \in [a,b]$ we have

$$I_T(x,y) = z \;\Leftrightarrow\; T(x,N(y)) = N(z).$$

Theorem 4. *Let N be a strong negation on $[a,b] \subset [0,1]$. A left-continuous t-norm on $[a,b]$ is rotation invariant (resp. has the self quasi-inverse property) w.r.t. N if and only if its induced negation is equal to N.*

Hence any of the above two properties in characteristic for the class of left-continuous t-norms having strong induced negations.

The following theorem explains the geometrical content of the rotation invarinacy property.

Theorem 5. *Rotation invariancy property for T means exactly that the part of the space $[0,1] \times [0,1] \times [0,1]$ which is strictly below the graph of T remains invariant under an order 3 transformation. This transformation is indeed a rotation of $[0,1]^3$ (with angle $\frac{2\pi}{3}$ around the axe which is based on the points $(0,0,1)$ and $(1,1,0)$) when $N(x) = 1-x$.*

In Figure 2 the first coloumn presents the three-dimensional plots of the t-norms given by (1), (2) and (3). Since their induced negations equal $1-x$ one can realise easily the geometrical meaning of the rotation invariancy property.

The following theorem explains the geometrical content of the self quasi-inverse property.

Theorem 6. *The self quasi-inverse property for T w.r.t. $1-x$ means exactly that the graph of any partial mapping $T(x,\cdot) : [0,1] \to [0,1]$ has the following geometrical property. First extend the discontinuities of $T(x,\cdot)$ with vertical line segments. Then the obtained graph is invariant under the reflection at the second median (given by $y = 1-x$).*

In Figure 2 the second coloumn presents plots on the partial mappings $T\left(x,\frac{1}{2}\right)$ of the t-norms given by (1), (2) and (3).

7 Decomposition

Definition 7. Let T be a left-continuos t-subnorm having strong induced negation N and t be its unique fixed point. Define the set of decomposition points of T by

$$D = \{x \in [t,1[\mid T(y,z) = z \text{ for all } y \in]t,x] \; z \in]x,1]\}.$$

Call T indecomposable if $D = \emptyset$, decomposable if $D \ne \emptyset$ and totally decomposable if $t \in D$.

Theorem 8. (Total decomposition) *Let T be a totally decomposable a left-continuous t-norm having strong induced negation N. Define a binary operator T_3 on $[t,1]$ by*

$$T_3(x,y) = \begin{cases} T(x,y) & \text{if } x,y > t \\ t & \text{otherwise} \end{cases}.$$

Let T_1 be the linear transformation of T_3 into $[0,1]$. Then T_1 is a left-continuous t-norm without 0-divisors.

In Figure 3 the total decomposition of the t-norm, which is given by (2) is presented.

Theorem 9. (Decomposition) *Let T be decomposable a left-continuous t-norm having strong induced negation N. Suppose that $d \in D$ is a decomposition point of T which is different to the fixed point of N and let N_d be the zoomed d-negation of N. Define a binary operator T_3 on $[d,1]$ by*

$$T_3(x,y) = \begin{cases} T(x,y) & \text{if } x,y > d \\ d & \text{otherwise} \end{cases},$$

and another binary operation T_4 on $[N(d), d]$ by

$$T_4(x,y) = \begin{cases} N(d) & \text{if } x \le N(y) \\ T(x,y) & \text{otherwise} \end{cases}.$$

Let T_1 (resp. T_2) be the linear transformation of T_3 (resp. T_4) into $[0,1]$. Then

i. T_1 is a left-continuous t-norm.

ii. If T_1 has no 0-divisors then T_2 is a left-continuous t-subnorm fulfilling the rotation invariancy property w.r.t. N_d.

If T_1 has 0-divisors then T_2 is a left-continuous t-norm having (strong) induced negation equal to N_d.

Theorem 10. (Maximal decomposition) *Let T be decomposable a left-continuous t-norm having strong induced negation, D be the set of its decomposition points. Then $\inf D \in D$, that is, the infimum of the decomposition points is a decomposition point.*

By decomposing with the least decomposition point we obtain the 'biggest possible' T_1 and a 'smallest possible' indecomposable T_2.

In Figure 4 a possible decomposition of the t-norm given by (3) is presented. The chosen decomposition point is $\frac{2}{3}$, T_1 is the minimum t-norm, T_2 is the Łukasiewicz t-norm. This decomposition is at the same time the maximal decomposition, since $\frac{2}{3}$ is the least decomposition point of this t-norm.

8 Rotation Construction

In this section we introduce a new method which produces left-continuous (but not continuous) t-norms which have strong induced negations from any left-continuous t-norm T_1 which has no 0-divisors. This construction is the 'inverse' of the total decomposition theorem in the sense as it is explained at the end of Section 4. First we need a definition.

Definition 11. Let N be a strong negation, t its unique fixed point and T_1 be a left-continuous t-norm having no 0-divisors. Let T_3 be the linear transformation of T_1 into $[t,1]$. Let $I^+ =]t,1]$, $I^- = [0,t]$ and define $T_{\mathbf{J}} : [0,1] \times [0,1] \to [0,1]$ by

$$T_{\mathbf{J}}(x,y) = \begin{cases} T_3(x,y) & \text{if } x,y \in I^+ \\ N\left(I_{T_3}(x, N(y))\right) & \text{if } x \in I^+ \text{ and } y \in I^- \\ N\left(I_{T_3}(y, N(x))\right) & \text{if } x \in I^- \text{ and } y \in I^+ \\ 0 & \text{if } x,y \in I^- \end{cases}.$$

Further, define

$$I_{T_{\mathbf{J}}}(x,y) = \begin{cases} I_{T_3}(x,y) & \text{if } x,y \in I^+ \\ N\left(T_3(x, N(y))\right) & \text{if } x \in I^+ \text{ and } y \in I^- \\ 1 & \text{if } x \in I^- \text{ and } y \in I^+ \\ I_{T_3}(N(y), N(x)) & \text{if } x,y \in I^- \end{cases}. \qquad (4)$$

Call $T_{\mathbf{J}}$ the *N–rotation* of T_1. If $N(x) = 1-x$ (the standard negation) then call $T_{\mathbf{J}}$ simply the *rotation* of T_1.

Theorem 12. (Rotation construction) *Let N be a strong negation, t its unique fixed point and T_1 be a left-continuous t-norm having no 0-divisors. Let $T_{\mathbf{J}}$ be the N–rotation of T_1. Then $T_{\mathbf{J}}$ is a left-continuous t-norm, its induced negation is N and the residuated implication generated by $T_{\mathbf{J}}$ is $I_{T_{\mathbf{J}}}$ given by (4).*

Remark 13. Let T_1 be a left-continuous t-norm without 0-divisors, N be a strong negation. Observe that the fixed point t of N is a decomposition point of $T_{\mathbf{J}}$ (the N–rotation of T_1) and the total decomposition of $T_{\mathbf{J}}$ with t gives back T_1.

On the other hand, suppose T is a totally decomposable left-continuous t-norm having strong induced negation N. If we decompose T with the fixed point of N (with Theorem 8) then the N–rotation of the obtained 0-divisor free left-continuous t-norm gives back T.

In Figure 5 the rotation of the minimum t-norm and the rotation of the product t-norm can be seen. Observe that the nilpotent minimum t-norm (see (2)) is not else but the rotation of the minimum t-norm.

9 Rotation-Annihilation Construction

In this section we introduce the second method which produces left-continuous t-norms which have strong induced negations from a pair of certain connectives as it is given in the following definition. This construction is the 'inverse' of the decomposition theorem in the sense as it is explained at the end of Section 4.

Definition 14. Let N be a strong negation, t its unique fixed point, $d \in]t,1[$ and N_d be the zoomed d-negation of N. Let T_1 be a left-continuous t-norm.

1. If T_1 has no 0-divisors then let T_2 be a t-subnorm which admits the rotation invariancy property with respect to N_d. Further, let $I^+ =]d,1]$, $I^- = [0,d[$ and $I^0 = [N(d), d]$.
2. If T_1 has 0-divisors then let T_2 be a left-continuos t-norm having strong induced negation equal to N_d. Further, let $I^+ = [d,1]$, $I^- = [0,d]$ and $I^0 =]N(d), d[$.

Let T_3 be the linear transformation of T_1 into $[d,1]$, T_4 be the linear transformation of T_2 into $[N(d), d]$ and

$T_5 : [N(d), d] \times [N(d), d] \to [0,1]$ be the annihilation of T_4 given by

$$T_5(x,y) = \begin{cases} 0 & \text{if } x \le N(y) \\ xT_4y & \text{otherwise} \end{cases} .$$

Define $T_{\mathbf{J}} : [0,1] \times [0,1] \to [0,1]$ by

$$T_{\mathbf{J}}(x,y) = \begin{cases} T_3(x,y) & \text{if } x,y \in I^+ \\ N(I_{T_3}(x, N(y))) & \text{if } x \in I^+,\ y \in I^- \\ N(I_{T_3}(y, N(x))) & \text{if } x \in I^-,\ y \in I^+ \\ 0 & \text{if } x,y \in I^- \\ T_5(x,y) & \text{if } x,y \in I^0 \\ y & \text{if } x \in I^+ \text{ and } y \in I^0 \\ x & \text{if } x \in I^0 \text{ and } y \in I^+ \\ 0 & \text{if } x \in I^- \text{ and } y \in I^0 \\ 0 & \text{if } x \in I^0 \text{ and } y \in I^- \end{cases} .$$

Further, define $I_{T_{\mathbf{J}}} : [0,1] \times [0,1] \to [0,1]$ by

$$I_{T_{\mathbf{J}}}(x,y) = \begin{cases} I_{T_3}(x,y) & \text{if } x,y \in I^+ \\ N(T_3(x, N(y))) & \text{if } x \in I^+,\ y \in I^- \\ 1 & \text{if } x \in I^-,\ y \in I^+ \\ I_{T_3}(N(y), N(x)) & \text{if } x,y \in I^- \\ I_{T_4}(x,y) & \text{if } x,y \in I^0 \\ y & \text{if } x \in I^+ \text{ and } y \in I^0 \\ N(x) & \text{if } x \in I^0 \text{ and } y \in I^- \\ 1 & \text{if } x \in I^- \text{ and } y \in I^0 \\ 1 & \text{if } x \in I^0 \text{ and } y \in I^+ \end{cases} . \tag{5}$$

Call $T_{\mathbf{J}}$ the *N-d–rotation-annihilation* of T_1 and T_2. If $N(x) = 1 - x$ (the standard negation) then call $T_{\mathbf{J}}$ simply the *d–rotation-annihilation* of T_1 and T_2.

Theorem 15. (Rotation-annihilation construction) *Let N be a strong negation, t its unique fixed point, $d \in]t,1]$ and T_1 be a left-continuous t-norm. Take T_2 depending on the 0-divisors of T_1, as it is taken in Definition 14 and let $T_{\mathbf{J}}$ be the N-d–rotation-annihihation of T_1 and T_2. Then $T_{\mathbf{J}}$ is a left-continuous t-norm, its induced negation is N and the residuated implication generated by $T_{\mathbf{J}}$ is given by (5).*

Remark 16. The analogue of Remark 13 holds true for this case.

In Figure 6 the rotation-annihilation of T_1 and T_2 is presented, where T_1 is an ordinal sum defined by a Łukasiewicz t-norm and a product t-norm and T_2 is the rotation of the product.

To show an example for a rotation invariant t-subnorm (which is not a t-norm), consider the following example:

$$T(x,y) = \max(x + y - 1 - \varepsilon, 0), \tag{6}$$

where ε is in $[0,1]$. Observe that the case $\varepsilon = 0$ gives back the Łukasiewicz t-norm, while $\varepsilon = 1$ defines the drastic t-subnorm $T : [0,1] \times [0,1] \to [0,1]$, $T(x,y) = 0$ (which is not the so called drastic t-norm!)

In Figure 7 the rotation-annihilation of T_1 and T_2 is presented, where T_1 is the same as in the previous example and T_2 is the t-subnorm given by (6) with $\varepsilon = \frac{1}{2}$.

10 Further Examples

Even when we start only with continuous t-norms having no 0-divisors we can obtain by rotation an infinite number of new families of left-continuous but not continous t-norms having strong induced negations. (On a family we understand a t-norm together with its φ-transformations.)

Another way to construct such t-norms is to rotate an ordinal sum of left-continuous (or/and continuous) t-norms. The only thing we have to pay attention is that this ordinal sum should have no 0-divisors. The left-continuous summands may be generated by a previous rotation or rotation-annihilation. Iteration of this idea leads to quite 'egzotic and beautiful' t-norms.

The role of T_2 in Theorem 15 can be played by e.g. any left-continuous t-norm having strong induced negation (see Theorem 4).

For the brief illustration of the wide spectrum of left-continuous t-norms having strong induced negations and for the illustration of the power of our construction methods we give an 'exotic' example via the corresponding 3-dimensional plot in Figure 8:

References

[1] J.C. Fodor, Contrapositive symmetry of fuzzy implications, *Fuzzy Sets and Systems*, 69 (1995) 141–156.

[2] J.Y. Girard, Linear logic, *Theor. Comp. Sci.*, 50 (1987), 1–102.

[3] S. Jenei, Geometry of left-continuous triangular norms having strong induced negations, *submitted*

[4] S. Jenei, New family of triangular norms via contrapositive symmetrization of residuated implications, *Fuzzy Sets and Systems*, (to appear)

[5] S. Jenei, Rotation construction of commutative, residuated, 0-closed, partially-ordered semigroups, *submitted*

[6] S. Jenei, Rotation-annihilation construction and decomposition of commutative, residuated, 0-closed, partially-ordered semigroups, *submitted*

[7] S. Jenei, The structure of left-continuous t-norms with strong induced negations. (I) Rotation construction, *submitted*

[8] S. Jenei, The structure of left-continuous t-norms with strong induced negations. (II) Rotation construction with center summand annihilation, *submitted*

[9] S. Jenei, The structure of left-continuous t-norms with strong induced negations. (III) Decomposition, *submitted*

[10] C-H. Ling, Representation of associative functions, *Publ. Math. Debrecen*, 12 (1965) 189–212.

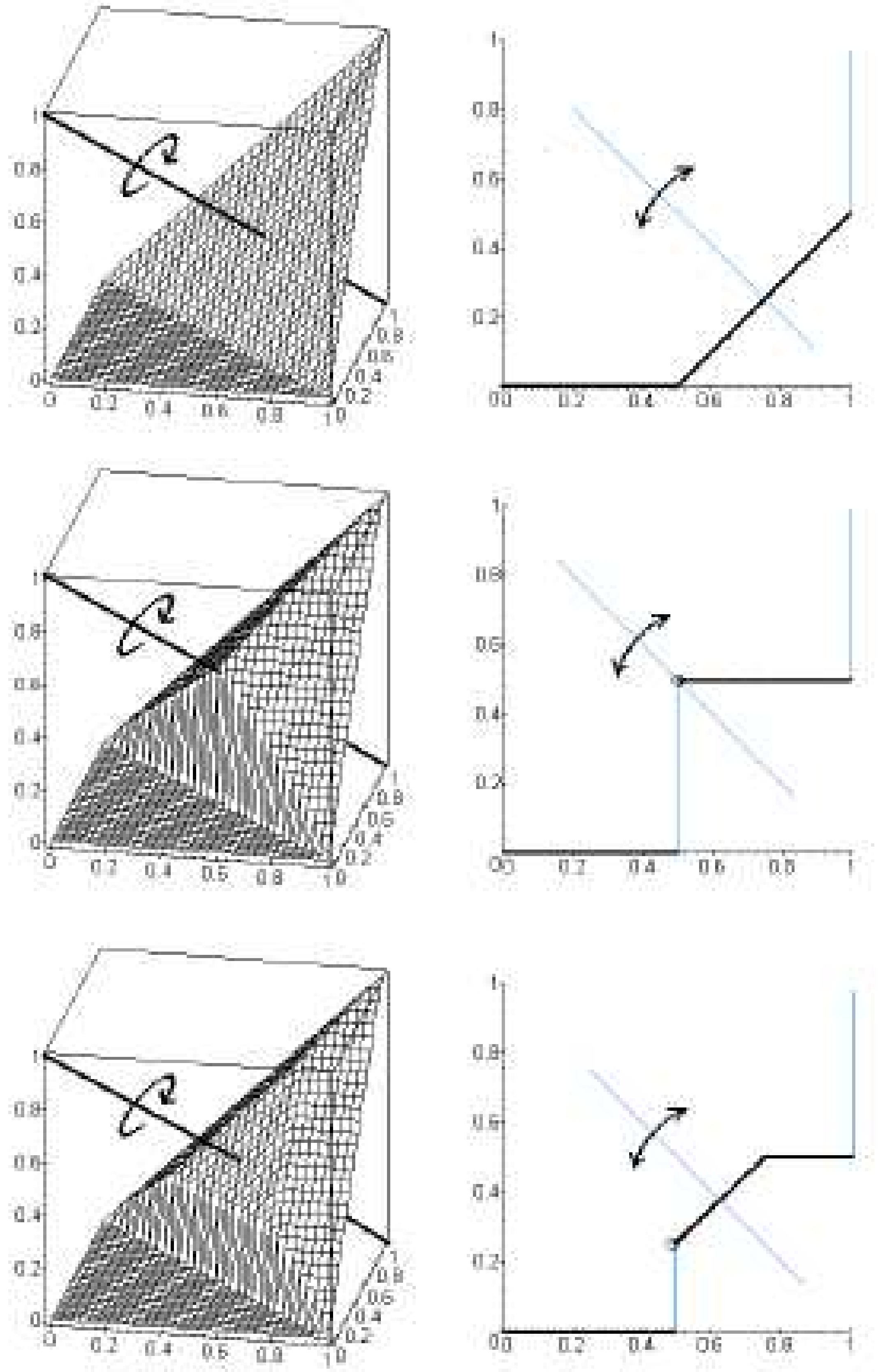


Figure 2: Geometrical interpretation of the rotation invariancy property and the self quasi-inverse property

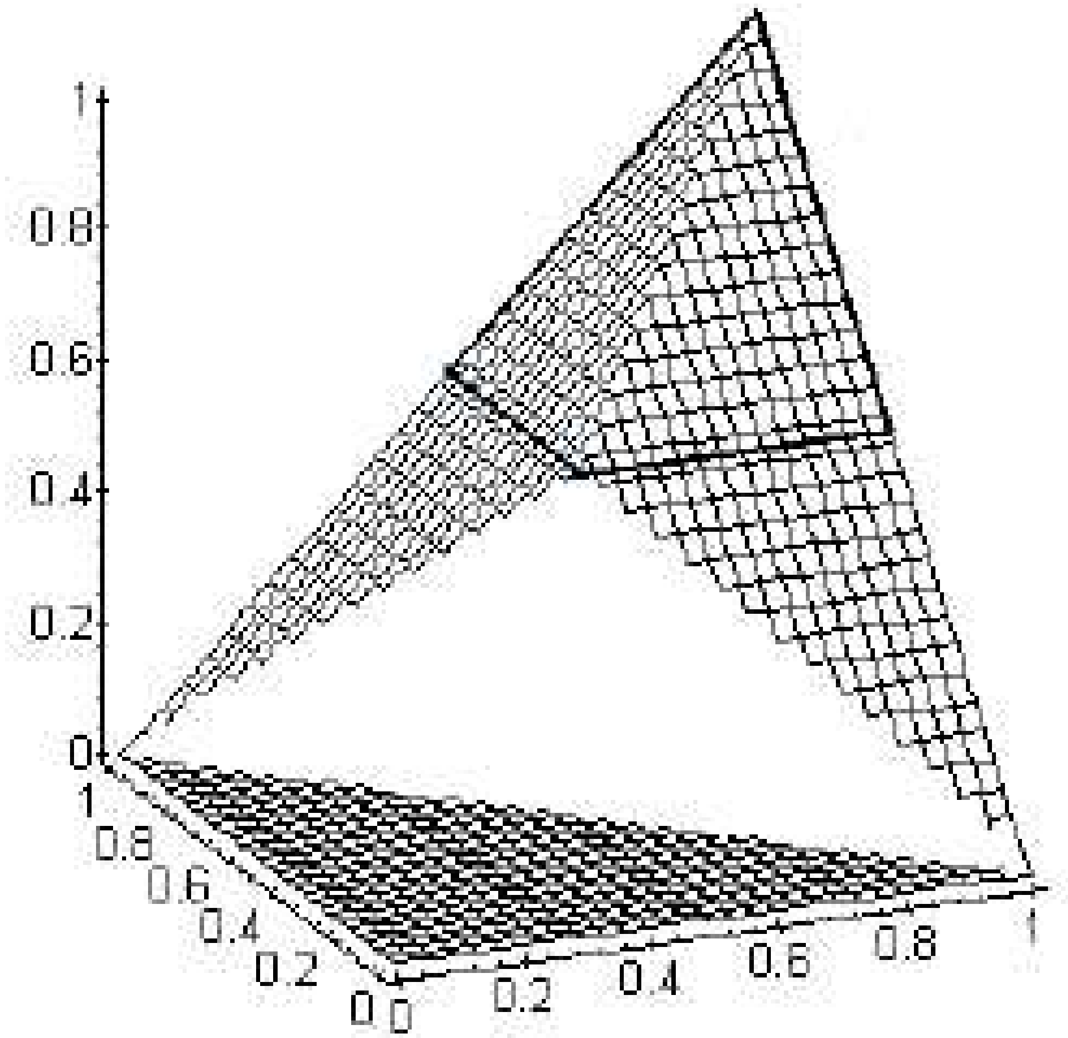


Figure 3: Total decomposition of the Nilpotent Minimum t-norm

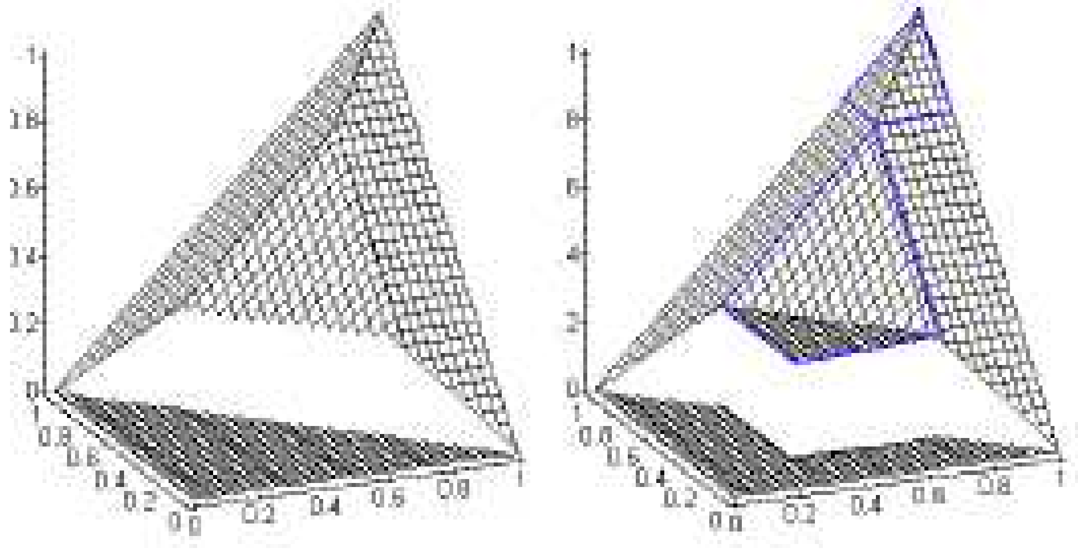


Figure 4: A possible decomposition of the Nilpotent Ordinal Sum t-norm

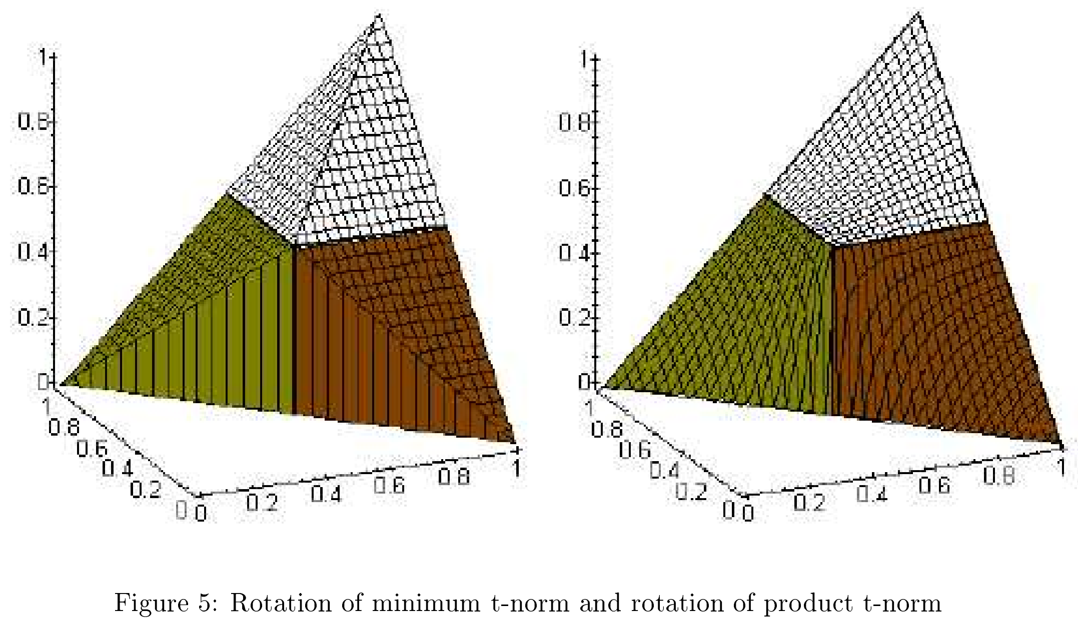


Figure 5: Rotation of minimum t-norm and rotation of product t-norm

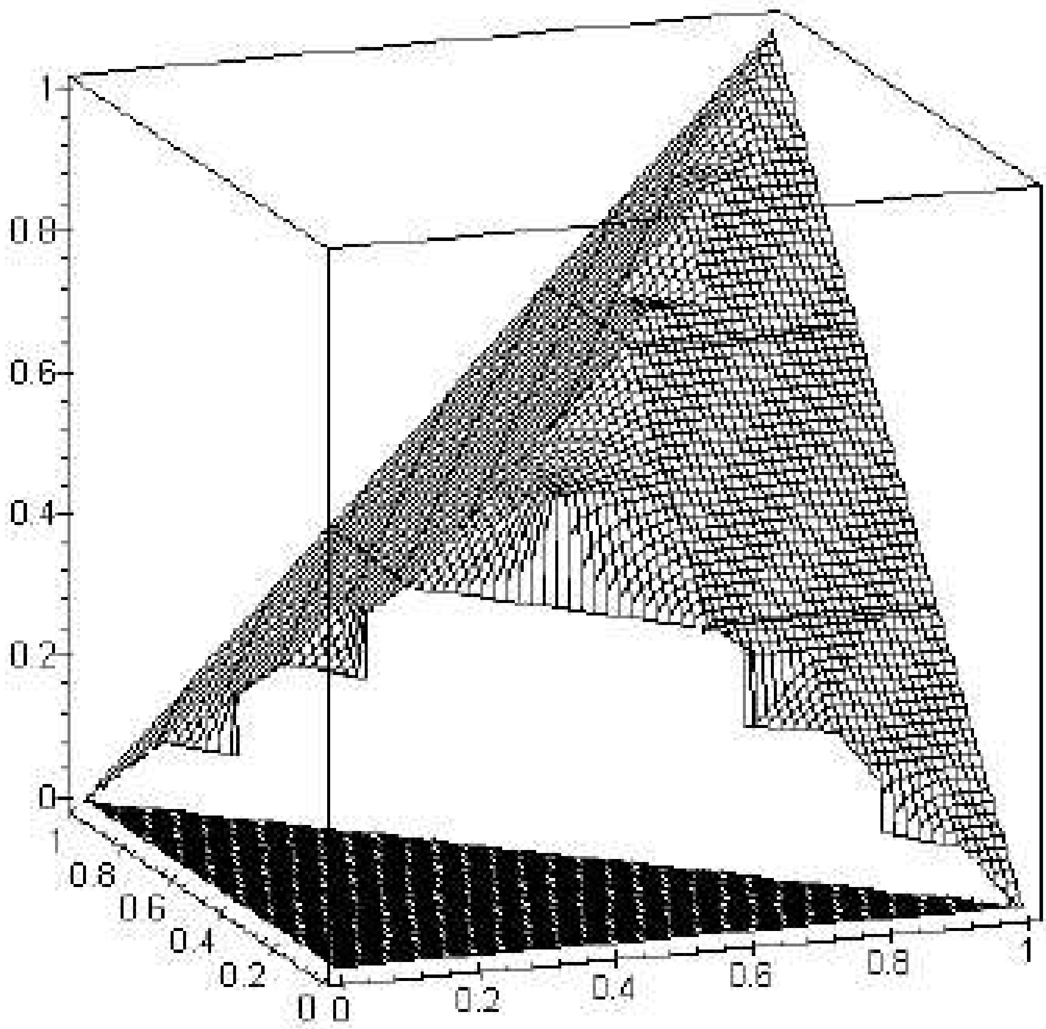


Figure 6: A t-norm generated by the rotation-annihilation construction

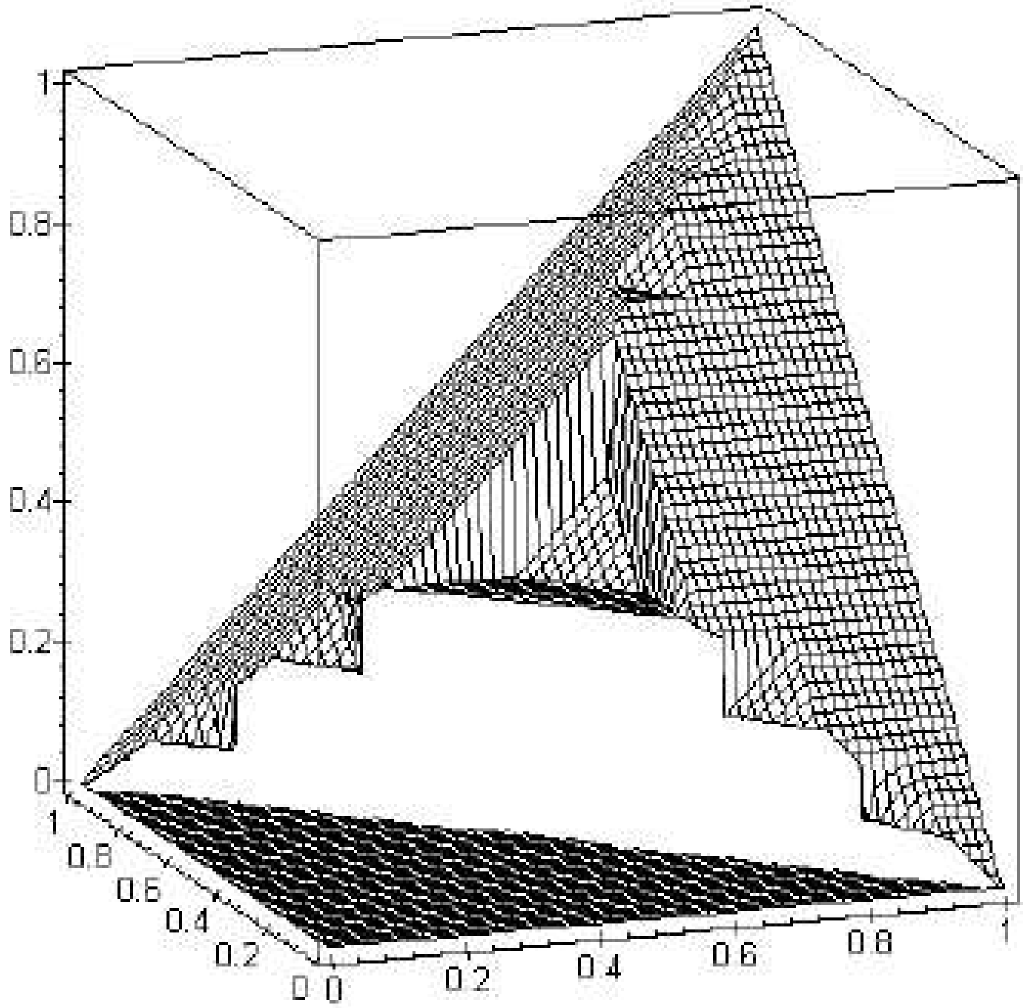


Figure 7: A second t-norm generated by the rotation-annihilation construction

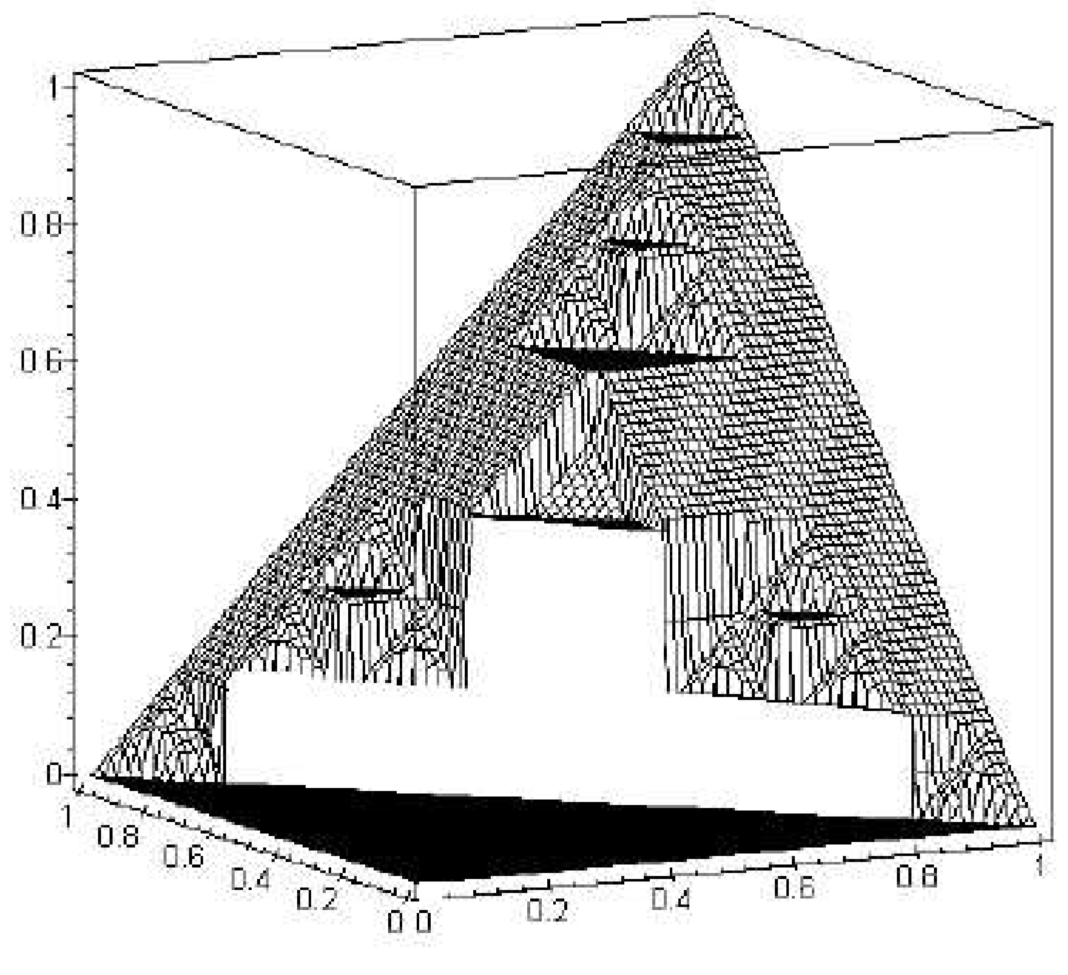


Figure 8: A t-norm generated by the combination of rotation, rotation-annihilation and ordinal sums

Variable Basis Fuzzy Topology and Compactifications: A Status Report

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This report summarizes categorical foundations for topology and fuzzy topology in which the basis of a space—the lattice of membership values—is allowed to change from one object to another *within the same category* (the basis of a space being distinguished from the basis of the topology of a space). With such foundations, all the following questions can be answered in the affirmative:

1. Are there categories for variable-basis topology or variable-basis fuzzy topology which are topological over their ground (or base) categories?
2. Are there categories for variable-basis topology or variable-basis fuzzy topology which cohere or unite all known canonical examples of point-set lattice-theoretic (or **poslat**) spaces, even when they are based on different lattices of membership values? For example, is there a *single category* containing all the fuzzy real lines and unit intervals

   $$\{\mathbb{R}(L), \mathbb{I}(L) : L \in |\mathbf{DQML}|\}$$

   and in which fuzzy real lines with different underlying bases may be "compared" by "homeomorphisms" or "non-homeomorphic continuous morphisms" (where **DQML** is the category of deMorgan quasi-monoidal lattices defined below)?
3. Are there categories for variable-basis topology or variable-basis fuzzy topology which cohere or unite known, important fixed-basis categories for topology or fuzzy topology as subcategories within a *single category*?
4. Are there categories for variable-basis topology or variable-basis fuzzy topology which make no essential use of algebraic notions such as associativity, commutivity, and idempotency of the traditional meet operation?
5. Are there categories for variable-basis topology or variable-basis fuzzy topology in which there exist variable-basis compactification reflectors, and are there informative relationships between such compactifications, traditional compact Hausdorff spaces, and "canonical" lattice-valued spaces such as the fuzzy unit intervals?

Restated, the topological theory summarized in this report may be thought of as satisfying four important **boundary conditions:**

- the **topological** condition addressed in (1) above,
- the **unification/coherence** conditions addressed in (2,3) above,
- the **non-algebraic** condition addressed in (4) above, and
- the **applicability** condition addressed in (5).

Unexpected bonuses arising from this program of research will also be highlighted: for example, the richness of morphisms in variable-basis topology, the close link between variable fuzzy topology being topological and the Adjoint Functor Theorem, a resolution of the separation axiom question, and the creation of a new class of fuzzy real lines and unit intervals, the former having jointly-continuous arithmetic operations for all such real lines having semiframe bases. An outline of this report is as follows:

Lattice-Theoretic Bases
Motivating Examples of Fixed-Basis Objects
Variable-Basis Ground Categories

References

[1] U. Höhle, A. Šostak, *Axiomatic foundations of fixed-basis topology*, in U. Höhle, S. E. Rodabaugh, eds., *Mathematics of Fuzzy Sets*, 1999, volume in *The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers (Dordrecht).

[2] S. E. Rodabaugh, *Categorical foundations of variab-basis topology,* in U. Höhle, S. E. Rodabaugh, eds., *Mathematics of Fuzzy Sets*, 1999, volume in *The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers (Dordrecht).

[3] S. E. Rodabaugh, *Separation axioms: representation theorems, compactness, and compactifications,* in U. Höhle, S. E. Rodabaugh, eds., *Mathematics of Fuzzy Sets*, 1999, volume in *The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers (Dordrecht).

Monadic Foundations of Topology

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The aim of this paper is to investigate the problem to which extent *topological spaces* are *algebras* — i.e. objects provided with a certain set of n-ary operations. It is interesting to see that a partially ordered monad $\mathbf{T}$ over a given monoidal closed category $\mathcal{K}$ (cf. [5, 6]) offers an appropriate axiomatic foundation. In this context a topological space can be defined as an *object* X of the associated Kleisli category $\mathcal{K}_{\mathbf{T}}$ provided with a unary operation $\mathcal{U}$ satisfying the important axiom (cf. [3])

$$\mathcal{U} \quad \leq \quad \eta_X$$

where η_X denotes the identity of $A \in |\mathcal{K}_{\mathbf{T}}|$. Hence topology is *not* necessarily based on *set theory* !

In order to illustrate this situation we mention the following special cases: Let $\mathcal{K}$ always denote the category **SET** of ordinary sets.

1. Let $\mathbf{T}$ be the *semi-filter monad.* Then topological spaces are generalized topological spaces of type $\mathfrak{V}$. Here the symbol $\mathfrak{V}$ refers to the french word "voisinage" (cf. [1, 2]) . In particular , if $\mathcal{U}$ is idempotent, then $(X,\mathcal{U})$ is a generalized topological space of type $\mathfrak{V}_\alpha$ (cf. [1, 7]).
2. Let $\mathbf{T}$ be the *filter monad.* Then topological spaces are generalized topological spaces of type $\mathfrak{V}_D$ (cf. [1, 2]). In particular, if $\mathcal{U}$ is idempotent, then $(X,\mathcal{U})$ is a generalized topological space of type $\mathfrak{V}_{D\alpha}$ — i.e. a topological space satisfying the usual Kuratowski axioms (cf. [1]).
3. Let L be a complete lattice and $\mathbf{T}_L$ be the *L-filter monad.* Further let $(A,\mathcal{U})$ be a topological space. If $\mathcal{U}$ is idempotent, then $(A,\mathcal{U})$ is an L-topological space (cf. [4]).

References

[1] A. Appert and Ky Fan, *Espaces topologiques intermédiaires*, Actualités Scientifiques et Industrielles **1121** (Paris, Herman 1951).

[2] M. Fréchet, *Les espaces abstraits et leur théorie considerée comme introduction à l'analyse générale* (Paris, Gauthier-Villars 1928) (New edition 1951).

[3] U. Höhle, *Categorical axioms of neighborhood systems*, Quaestiones Mathematicae **20** (1997), 373–414.

[4] U. Höhle and A.P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, in: Mathematics of Fuzzy Sets, eds U. Höhle and S.E. Rodabaugh, Handbook Series, Volume 3 (Boston, Kluwer Academic Publishers 1999).

[5] E.G. Manes, *Algebraic Theories* (Springer-Verlag, Berlin, New York 1976).

[6] H. Schubert, *Categories* (Springer-Verlag, Berlin, New York 1972).

[7] W. Sierpiński, *Intorduction to General Topology* (University of Toronto Press, Toronto 1934). (English translation of the Polish edition: Zarys teorji mnogości. Część druga: Topologja ogólna (Warszawa 1928)).

Powerset Operator Based Approach to M-Fuzzy Topological Spaces on L-Fuzzy Sets

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The purpose of this talk is to exibit an approach to fuzzy topology regarded as a suitable M-fuzzy set on a powerset associated to any L-fuzzy set.

The ground category $L - \mathcal{A}FSet$ is the well known catagory (see [2,3]) whose objects are L-fuzzy sets and whose morphisms from $Z \in L^T$ to $Y \in L^X$ are L-fuzzy functions i.e. maps $f : T \to X$ s. t. $Y \circ f \geq Z$.

This category is the domain of fuzzy powerset functors

$$\to_L : L - \mathcal{A}FSet \to Set \quad \text{and} \quad \leftarrow_L : L - \mathcal{A}FSet \to CLat^{op}$$

which associate to an L-fuzzy set $Y \in L^X$ the complete lattice $[O, Y] = \{A \in L^X : A \leq Y\}$, and to an L-fuzzy function $f : Z \to Y, Z \in L^T, Y \in L^X$, the forward powerset operator $f^{\to} : [O, Z] \to [O, Y]$ defined by $f^{\to}(A)(x) = \vee\{A(t) : f(t) = x\}$ and the opposite of the backward powerset operator $f^{\leftarrow} : [O, Y] \to [O, Z] s.t. f^{\leftarrow}(B) = B \circ f \wedge Z$, respectively (see [1,3]).

Then M-fuzzy topological spaces on L-fuzzy sets are defined as pairs (Y, τ), Y any L-fuzzy set on some set X, τ any function from $[O, Y]$ to M that satisfy the well known conditions

1)$\tau(O) = \tau(Y) = 1$

2)$A, A' \in [O, Y] \Rightarrow \tau(A \wedge A') \geq \tau(A) \wedge \tau(A')$

3)$A^j \in [O, Y] \forall j \in J \Rightarrow \tau(\vee A^j : j \in J) \geq \wedge\{A^j : j \in J\}$

so extending similar definitions given in [4,5].

An L-fuzzy function from $Z \in L^T$ to $Y \in L^X$ is M-fuzzy continuous with respect to M-fuzzy topologies σ in Z and τ in Y iff the related backward powerset operator is an L-fuzzy function from τ to σ

$$f^{\leftarrow} : \tau \to \sigma$$

Which means $\sigma \circ f^{\to} \geq \tau$.

M-fuzzy topological spaces on L-fuzzy sets are objects and M-fuzzy continuous L-fuzzy functions are morphisms of a category

$$(L, M) - \mathcal{A}FTop$$

.

By the obvious ordering, the set of all M-fuzzy topologies on a fixed L-fuzzy set Y can be given the structure of a complete lattice, but this require L to be a frame and M to be a completely distributive complete lattice.

Given a morphism $f : Z \to Y$ in $L - \mathcal{A}FSet$ (L a frame), a completely distributive complete lattice M and M-fuzzy topologies σ in Z and τ in Y, by means of the powerset operators of $f^{\leftarrow} : M^{[O,Y]} \to M^{[O,Z]}$ the initial M-fuzzy topology of τ and the final M-fuzzy topology of σ by f can be constructed, as they are respectively

$$(f^{\leftarrow})^{\to}(\tau) \quad \text{and} \quad (f^{\leftarrow})^{\leftarrow}(\sigma).$$

This allow to characterize M-fuzzy topological L-subspaces and M-fuzzy topological L-product spaces in $(L, M) - \mathcal{A}FTop$.

We remark that the use of powerset operators of (suitable) maps between (suitable) complete lattices, namely in the above context $f^{\leftarrow}$, suggests and allows a possible pointless approach to M-fuzzy topology as a suitable M-fuzzy set on a complete lattice, so overcoming the problem of changing the base L in the above description.

Aknowledgement. Most of the result in this note were obtained during a visit of the author at the Mathematical Department of Rhodes University in Grahamstown.

References [1] De Mitri C. and Guido C., *Some remarks on fuzzy powerset operators*, (submitted).

[2] Goguen J.A., *Categories of V-sets*, Bull. Amer. Math. Soc. 75 (1969), 622-624.
[3] Guido C., *The subspace problem in the traditional point-set context of fuzzy topology*, Quaestiones Mathematicae 20 (1997), 351-372.
[4] Kubiak T., On fuzzy topologies, Ph.D. Thesis, A. Mickiewisz Univ. Poznan, 1985.
[5] Šostak A.P., *On a fuzzy topological structure*, Suppl. Rend. Circ. Matem. Palermo, Sr II, 11 (1985), 89-103.

Some Meeting Points of Fuzzy and Point-Free Topology

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Although some concrete facts will be mentioned, the main aim of the talk is to open discussion on some contrasts and some common features of the fuzzy and point-free approaches to topology.

Metaphorically: While fuzzy topology is concerned with the fuzziness of open sets, point-free topology, in particular the enriched one (nearness, uniform, metric locales) is concerned with fuzzy, that is, blurred points and fuzzy (blurred) maps. The basic notions of point-free nearness, etc., will be recalled and the mentioned point of view explained.

Further topics to be discussed:

- the information lost when viewing a fuzzy space as a frame,
- fuzzification of a point-free space (is there a motivation for studying various degrees of openness of crisp subsets?),
- Banaschewski interpretation of completeness in the point-free context (T-valued Cauchy points vs. T-valued points),
- imitating point-free techniques in studying enriched fuzzy spaces (e.g.: fuzzy metric spaces).

Fuzzy Reals: Some Topological Results Surveyed, Brouwer Fixed Point Theorems, and a Few Open Questions

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There is no doubt that the L-reals $\mathbb{R}(L)$ and, in particular, the unit L-interval $I(L)$ are among the most important and canonical examples of L-topological structures (at least in one of those various disciplines which all claim to be named *fuzzy topology*).

In this talk, we first give a brief account os some of the topological results involving L-reals, separation axioms, and compactness.

We then put emphasis on how the functor ι_L modifies the L-unit interval and on the properties of the *natural topology* it produces. This natural topology on $I(L)$ can either be described as a certain quotient topology of a subspace of the product $L^{[0,1]}$ or just as the interval topology of the lattice $I(L)$.

The new material involves the natural topology of $I(L)$. For a suitable L, we shall present an analogue of the Brouwer fixed point theorem for Hilbert L-cubes (= products of countably many copies of $I(L)$ with its Hutton L-topology). The proof depends on a fixed point theorem od D. Papert Strauss. Actually, as proved recently by D. Zhang and the author, each Tychonoff L-cube has the fixed point property.

Some related open questions will also be recalled.

Each lattice L must have an order-reversing involution; minimal assumption: completeness; maximal assumption: complete distributivity plus a countable basis.

Fuzzy Compactness via Categorical Closure Operators

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Let FTS denote the category of fuzzy topological spaces (in the sense of Lowen) and fuzzy continuous maps. The categorical compactness notions (as defined in [3]) corresponding to three categorical closure operators on FTS identified by the author and G. Castellini in [1] are characterized.

Two of these closure operators are weakly hereditary and idempotent, and give rise to alpha-compactness and alpha*-compactness [4] respectively. The third closure operator, although weakly hereditary, is not idempotent. The corresponding compactness notion in FTS can be characterized in terms of the compactness of certain closure spaces, as defined by Cech in [2].

References

[1] I.W. Alderton and G. Castellini, Closure Operators and fuzzy connectedness. To appear in Fuzzy sets and Systems.

[2] E. Cech, Topological Spaces. John Wiley & Sons, 1966

[3] M.M. Clementino, E. Giuli and W. Tholen, Topology in a category: compactness. Portugaliae Mathematicae, 53(4), 1996, 397 - 433.

[4] T.E. Gantner, R.C. Steinlage and R.H. Warren, Compactness in fuzzy topological spaces. J. Math. Anal. Appl., 62 ,1978, 547 - 562.

Families of Separated Fuzzy Sets

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In [1], an α-Hausdorff space is defined to be a fuzzy topological space (X, τ), where $\forall\, x, y \in X$ s.t. $x \neq y$, $\exists\, u, v \in \tau$ s.t. $u(x) > \alpha$, $v(y) > \alpha$, and $v \wedge y = 0$. In other words, the fuzzy topology separates the crisp points to a degree $> \alpha$, since the crisp points belong separately to the disjoint sets u and v to a degree $> \alpha$. This paper explores an "extension" of the notion of α-separation of crisp points which suggests itself from the work of S. Watson [2]. In this paper, Watson calls a discrete family of points D of a topological space X **separated** if there is a disjoint family $\{U_d : d \in D\}$ of open sets in X s.t. $(\forall\, d \in D)\ d \in U_d$; and given any discrete family of points, then, it is possible to form $\mathcal{I}$, the family of all separated subsets of D. Watson also defines the following property, which is crucial to our arguments below: a family of separated subsets $\mathcal{I} \in P(\kappa)$ (κ a cardinal—see [2]) is said to be **diagonally-closed** if, whenever $A \subset \kappa$, $\{E_j : j \in \omega\}$ is an increasing family of finite subsets of $\mathcal{I}$ and $\{B_j : j \in \omega\}$ is a partition of A s.t.

$$(\forall\, j \in \omega)\ \ (\forall\, b \in B_j)\ \ (\forall \gamma \in A)\ \ (\exists\, C \in E_j)\ \ \{\beta, \gamma\} \subset C$$

then $A \in \mathcal{I}$. Now suppose, given a fuzzy Hausdorff topological space (X, τ), we take the family $\mathcal{I}$ of all subsets of $P(\kappa)$ for which there exists a disjoint family of fuzzy sets $\{U_d : d \in D\}$ in τ s.t.

$$(\forall d \in D)\ \ U_d(d) > \alpha$$

(we shall refer to such families as **α-F-separated**). Is it true (as is proved for the crisp case in [2], Theorem 3) that for such an α-F-separated family of sets $\mathcal{I}$, it must be the case that $\mathcal{I}$ is diagonally-closed and contains all sets of size 2?

A number of issues arise before this question can be addressed. It is important to note that diagonal closure is a set-theoretic property of (a subset of) the crisp elements which interacts with (helps to characterize) the topology over these elements (i.e., diagonal closure is a non-topological property of a cardinal which accrues as a result of the topological "standing" of the cardinal). We must wonder, therefore, whether it is possible to fuzzify such theorems as Theorem 3 of [2] in any meaningful way, since the exact value of α in an α-F-separation would seem to be irrelevant. This raises issues related to properties of fuzzy (Hausdorff) topologies which have been known for many years but whose implications remain essentially unexplored, *viz.*, the fact that many fuzzy topologies can have the same topological modification and the fact that fuzzy topologies can be "highly non-topological" [3]. In any case, it would appear to be difficult to make any reasonable claims about the connection between particular fuzzy Hausdorff topological spaces and diagonally-closed subsets of the domain of their topological modifications; as a result, diagonal closure would not appear to be uniquely extendible (in the sense of [3]) to fuzzy topology. Issues involving local and global separation of points [4] also arise in this connection and are explored in the paper.

The concept required to provide effective extendibility of diagonal closure to fuzzy topological spaces would appear to be sobriety (good summaries, references, and applications in [5], [6]). Take any fuzzy topological space (X, F, I) (I an A_3B_1 lattice [6]), along with ptF, the collection of all frame maps from F to I. If (X, F, I) is (fuzzy) sober, then ptF should inherit the separability of X in its topology $\{\Phi(u) : u \in F\}$. We show that this is indeed the case, i.e., that the correct generalization of separation and diagonal closure is to be found in ptF mediated through bijective (sober) maps from X to ptF. For the crisp case, such "property transference" (from X to ptT) is clear; for the fuzzy case, we show that sobriety carries the relevant properties from the (crisp) element set to ptF. This, of course, lends credence to and derives support from the notion of frame fuzzy points and their connection to Hausdorfness [7], [8]. With sobriety, then, it is possible to prove a fuzzy analogue of Theorem 3 in [2], which is (in short) that if $\mathcal{J}$ is the family of all α-F-separated subsets of X (wrt (X, F, I) for some α and (X, F, I) is fuzzy Hausdorff, then

$$\{\Psi_j \ (j \in \mathcal{J}) : j \to ptF\}$$

is diagonally-closed and includes all sets of size 2 (wrt $\Phi(u)$). The method of proof of this theorem also sheds some light on the remark by Kotzé ([6], p. 263) that the relationship between Hausdorff and sobriety is not as simple as in the classical case.

References

[1] S. E. Rodabaugh, The Hausdorff separation axiom for fuzzy topological spaces, *Topology and its Applications* **11**(1980) 319-334.

[2] S. Watson, Families of separated sets, *Topology and its Applications* **75**(1997) 1-11.

[3] R. Lowen, Compact Hausdorff fuzzy topological spaces are topological, *Topology and its Applications* **12**(1981) 65-74.

[4] P. Wuyts and R. Lowen, On local and global measures of separation in fuzzy topological spaces, *Fuzzy Sets and Systems* **19**(1986) 51-80.

[5] S. E. Rodabaugh, Categorical frameworks for Stone representation theories, in S. E. Rodabaugh, U. Höhle, E. P. Klement, eds., *Applications of Category Theory to Fuzzy Subsets*, Kluwer Academic Publishers (Dordrecht, 1992) 177-231.

[6] W. Kotzé, Lattice morphisms, sobriety, and Urysohn lemmas, in S. E. Rodabaugh, U. Höhle, E. P. Klement, eds., *Applications of Category Theory to Fuzzy Subsets*, Kluwer Academic Publishers (Dordrecht, 1992) 258-274.

[7] M. W. Warner, Frame-fuzzy points and membership, *Fuzzy Sets and Systems* **42**(1991) 453-456.

[8] M. W. Warner and R. G. McLean, On compact Hausdorff L-fuzzy spaces, *Fuzzy Sets and Systems* **56**(1993) 103-110.

Some Results on L-Regular and Completely L-Regular Spaces

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Some properties of regular L-topological spaces will be presented. These are related to the Lowen functors ι_L and ω_L, where L a continuous lattice with its Scott topology. In particular maximal L-regular topologies are characterized as those which are topologically generated from maximal regular crisp topologies.

With appropiate definition of H-lindelöfness (á la Hutton compactness) one has the following: L-regular and H-Lindelöf are L-normal.

The behaviour of the lim-inf convergence with respect to the complete L-regularity is shown and a link between lim-inf convergence and the functors ι_L and ω_L (for L a continuous lattice), is established.

Compactness in Function Spaces in Fuzzy Topology

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To obtain theorems on compactness for fuzzy subsets in function spaces in the category FCS of Lowen's fuzzy convergence spaces, we first define the fuzzy convergence of pointwise convergence where compactness is easily established via a Tychonoff theorem and then secondly introduce a notion of even continuous fuzzy subsets on which pointwise convergence and the important notion of continuous convergence coincide. Both fuzzy convergences and even continuity extend well known notions from the theory of classical convergence spaces and moreover can be characterized in a natural way also in the category FNS of Lowen's fuzzy neighborhood spaces.

On the Need of Many-Valued Topologies

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The aim of this paper is to underline that *many-valued topologies* (cf. [6]) have non-trivial applications to *probability theory and statistics.*

In 1921 M. Fréchet [3] proved that there does not exist any generalized distance *dist* on the space L^0 of all almost everywhere defined measurable functions such that convergence in the sense of *dist* is equivalent to almost everywhere pointwise convergence. It is not difficult to show that this result can be strengthen as follows: There does not exist any binary, generalized topology $(\mathbb{U}_p)_{p \in L^0}$ of type $\mathfrak{V}$ (cf. p. 172—174 in [4], [1]) on L^0 such that the *convergence of sequences* in the sense of $(\mathbb{U}_p)_{p \in L^0}$ is equivalent to *almost everywhere pointwise convergence.* In this paper we show that there exists a Boolean valued topology τ on L^0 such that almost everywhere convergence is equivalent to convergence of sequences in the sense of τ. As an application of this result we give a purely topological characterization of l inear, stochastic processes with continuous trajectories (cf. [2, 5]).

Further let X be an ordinary (i.e. binary) compact topological space. There exists a $[0,1]$-valued topology τ_X on X such that the space $\mathcal{R}^1(X)$ of all Radon probability measures on X is the *compactification* of X w.r.t. τ_X — i.e. there exists a compact $[0,1]$-topology on the space $\mathcal{R}^1(X)$ such that X is a dense subspace of $\mathcal{R}^1(X)$. Since any ordinary compact subspace is an ordinary closed subset, it is clear that such a result is impossible in the case of ordinary (i.e. binary) topologies.

References

[1] A. Appert and Ky Fan, *Espaces topologiques intermédiaires*, Actualités Scientifiques et Industrielles **1121** (Paris, Herman 1951).

[2] A. Badrikian, *Séminaire sur les fonctions aléatories linéaires et measure cylindriques*, Lecture Notes in Mathematics **139** (Berlin, Heidelberg, New York, Springer-Verlag 1970).

[3] M. Fréchet, *Sur divers modes de convergence d'une suite de fonctions d'une variable*, Bull. Calcutta Math. Soc. **11** (1921), 187–206.

[4] M. Fréchet, *Les espaces abstraits et leur théorie considerée comme introduction à l'analyse générale* (Paris, Gauthier-Villars 1928) (New edition 1951).

[5] U. Höhle, *Almost everywhere convergence and Boolean–valued topologies*, Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II **29** (1992), 215–227.

[6] U. Höhle and A.P. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, in: Mathematics of Fuzzy Sets, eds U. Höhle and S.E. Rodabaugh, Handbook Series, Volume 3 (Boston, Kluwer Academic Publishers 1999).

On Monadic Convergence Structures

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A basic notion for monadic convergence structures is that of *partially ordered monad* $\Phi = (\varphi, \leq, \eta, \mu)$, where (φ, η, μ) is a monad over SET, each set φX is equipped with a partial ordering and some conditions on these partial orderings are required.

In the classical case of filter convergence structures, Φ is the *partially ordered filter monad* $(\mathsf{F}, \leq, \eta, \mu)$, where F is the filter functor, which assigns to each set X the set $\mathsf{F}X$ of all filters on X. $\leq$ indicates that the sets $\mathsf{F}X$ are equipped with the finer relations of filters.

An important more general case is that of the *partially ordered L-fuzzy filter monad* $(\mathcal{F}_L, \leq, \eta, \mu)$, where L is any non-degenerate complete Heyting algebra. For each set X, $\mathcal{F}_L X$ consists of all L-fuzzy filters on X, that is, of all mappings $\mathcal{M} : L^X \to L$ such that $\mathcal{M}(\bar{\alpha}) \leq \alpha$, $\mathcal{M}(\bar{1}) = 1$ and $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $\alpha \in L$ and $f, g \in L^X$, where $\bar{\alpha}$ is the constant mapping of X into L with value α.

There are more important examples of partially ordered monads. A method of creating such examples consists in assigning each partially ordered monad Φ its *homogeneous partially ordered submonad* defined by the subfunctor φ' of φ, where $\varphi' X = \{\mathcal{M} \in \varphi X \mid \mathcal{M} \leq \bigvee_{x \in X} \eta_X(x)\}$.

In the following let Φ be any partially ordered monad over SET. A *Φ-convergence structure* on a set X is a subset T of $\varphi X \times X$ such that, writing $\mathcal{M} \to x$ instead of $(\mathcal{M}, x) \in T$, we have (1) $\eta_X(x) \to x$ for all $x \in X$, (2) $\mathcal{M} \to x, \mathcal{N} \leq \mathcal{M} \implies \mathcal{N} \to x$, and (3) $\mathcal{M} \to x \implies \mathcal{M} \vee \eta_X(x) \to x$. Special Φ-convergence structures are the *Φ-limit structures*, the *Φ-pretopologies* and the *Φ-topologies*. Φ-pretopologies can be given as the mappings $p : X \to \varphi X$ with $\eta_X \leq p$, and Φ-topologies as the Φ-pretopology p for which $\mu_X \circ \varphi p \circ p = p$.

Let T be a Φ-convergence structure and let $\mathcal{M} \in \varphi X$. $\mathsf{nb}\mathcal{M} = (\mu_X \circ \varphi p)(\mathcal{M})$, with p the finest Φ-pretopology coarser than T, is the *neighborhood* of $\mathcal{M}$. Let t_1 and t_2 be the first and second projection of T into φX and X, respectively. Then the image $\varphi t_2(\mathcal{L})$ of the coarsest $\mathcal{L} \in \varphi T$ with $(\mu_X \circ \varphi t_1)(\mathcal{L}) = \mathcal{M}$, is the *closure* of $\mathcal{M}$, denoted $\mathsf{cl}\mathcal{M}$. For the Φ-limit structures further suitable notions can be defined, for instance the axioms T_0, T_1, T_2, *regularity* and *normality* and the notions *compactness* and *local compactness*. In the filter case we obtain the related classical notions.

By means of Φ and a partial ordered set K, a partially ordered monad $(\psi, \leq, \eta', \mu')$, denoted $K\Phi$, is defined as follows. For each set X, ψX is the set of all families $(\mathcal{M}_\alpha)_{\alpha \in K}$ of elements of φX. $\mathcal{M} \leq \mathcal{N}$ means $\mathcal{M}_\alpha \leq \mathcal{N}_\alpha$ for all $\alpha \in K$. $\eta'_X(x)_\alpha = \eta_X(x)$ for all $x \in X$ and $\alpha \in K$, and $\mu'_X(\mathcal{L})_\alpha = \mu_X(\varphi X \pi_\alpha(\mathcal{L}_\alpha))$ for all $\mathcal{L} \in \psi\psi X$ and $\alpha \in K$, where π_α is the mapping $\mathcal{M} \mapsto \mathcal{M}_\alpha$.

A *K-graded convergence structure* with respect to Φ is a family $(T_\alpha)_{\alpha \in K}$ of Φ-convergence structures. It can be canonically identified with a $K\Phi$-convergence structure. In the following let K be a complete lattice. If the family $(T_\alpha)_{\alpha \in K}$ is isotone, then for each $\mathcal{N} \in \varphi X$, $\bigwedge_{\mathsf{nb}_\alpha \mathcal{N} = \mathcal{N}} \alpha$ and $\bigwedge_{\mathsf{cl}_\alpha \mathcal{N} = \mathcal{N}} \alpha$ are the *degrees of openness* and of *closedness* of $\mathcal{N}$, respectively, where $\mathsf{nb}_\alpha \mathcal{N}$ and $\mathsf{cl}_\alpha \mathcal{N}$ are meant with respect to T_α. If this family is antitone, then for each $\mathcal{N} \in \varphi X$, $\bigvee_{(\mathcal{N}, x) \in T_\alpha} \alpha$ is the *degree of convergence* of $\mathcal{N}$ to x. In case Φ is the partially ordered filter monad, examples of isotone and antitone K-graded convergence structures are the limit towers and the probabilistic convergence structures, respectively.

If Φ is the partially ordered L-fuzzy filter monad $(\mathcal{F}_L, \leq, \eta, \mu)$, then the Φ-topologies are the *L-fuzzy topologies* and special isotone $K\Phi$-topologies are the *L, K-fuzzy topologies*, which respectively can be given as special subsets of L^X and special mappings of L^X into K. If changing here from Φ to the related homogeneous partially ordered submonad, then instead of the L-fuzzy topologies we obtain the *stratified* L-fuzzy topologies.

More informations and the related references will be given in the talk.

Fully Fuzzy Topology

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Fuzzy topology has used a definition which has a fuzzy topological space consisting of a crisp set of fuzzy subsets of a crisp set. The notion of a topological space object in **Set(L)** provides a generalization of this concept to that of a fuzzy set of fuzzy subsets of a fuzzy set. This paper provides connections with more mainstream fuzzy topology.

Definition 1. A topological space object in **Set(L)** is an object (A,α) equipped with an unbalanced subobject $T = (PA,\tau) \overset{\iota}{\rightarrowtail} P(A,\alpha)$ such that

1. The unbalanced subobjects $(A,0)$ and (A,α) both have τ with value 1.
2. T is closed under pairwise $*$: this says there is a lifting

$$\begin{array}{ccc} T\times T & \overset{*}{\to} & T \\ \downarrow & & \downarrow \\ P(A,\alpha)\times P(A,\alpha)) & \overset{*}{\to} & P(A,\alpha) \end{array}$$

3. T is closed under arbitrary internal unions: this says there is a lifting

$$\begin{array}{ccc} PT & \overset{\bigoplus}{\to} & T \\ \exists_\iota \downarrow & & \downarrow \\ P(P(A,\alpha)) & \overset{\bigoplus}{\to} & P(A,\alpha) \end{array} .$$

   where $\bigoplus : P(P(A,\alpha)) \to P(A,\alpha)$ is the adjoint of the smallest characteristic function of $\exists_{\pi_2}((\epsilon_{P(A,\alpha)} \otimes (A,\alpha)) * (P(A,\alpha) \otimes \epsilon_{(A,\alpha)}))$

Definition 2. A morphism $f : (A,\alpha) \to (B,\beta)$ is continuous with respect to the topologies T_A and T_B if there is a lifting

$$\begin{array}{ccc} T_B & \to & T_A \\ \downarrow & & \downarrow \\ PB & \overset{f^{-1}}{\to} & PA \end{array} .$$

The category **FFTop(L)** is the category of fully fuzzy topological space objects in **Set(L)** with continuous morphisms.

Theorem 3. *The category of fully fuzzy topological spaces is topological over* **Set(L)**. *That is, given any U-structured source $f_i : (A,\alpha) \to U((B_i,\beta_i),\tau_{B_i})$ there is a unique smallest structure of a fully fuzzy topology on (A,α) making all of the f_i continuous.*

Two theorems explain the relationship FFTS and topological spaces and FFTS and fuzzy topological spaces:

Theorem 4. *If (PA,τ) is a fully fuzzy topology on $(A,\top)$ then each of the sets $T_h = \{A' \subseteq A | \tau(\chi_{A'}) \geq h\}$ is a topology on A. Furthermore; if $h \leq h'$ then $T_{h'} \subseteq T_h$.*

The approach used here is different from the α-cuts approach of fuzzy topology in that instead of asking for fuzzy sets which are (crisp) members of the topology and considering the sets of elements with membership at least α we look at crisp subsets with membership at least α in the fully fuzzy topology.

Theorem 5. *A fully fuzzy topological space structure on a crisp set $(A,\top)$ with only crisp members is a fuzzy topology in the sense of Lowen if $* = \min$.*

If you do not use min for $*$ then a fuzzy topological space need not give a subobject closed under $*$. If the fuzzy topological space is not fully stratified (the Lowen constants condition), then the resulting subobject of $P(A)$ will not be closed under internal unions.

After defining an interior operator for a FFTS we consider non-emptiness conditions and how they relate to definition of neighborhoods and quasineighborhoods.

A Unified Approach to the Concept of L-Uniform Spaces

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A theory of L–uniform spaces is presented which covers U. Höhle's approach to L-valued uniformities (cf. [3]) and also Lowen's approach (viewed as generalised uniformities) (cf. [6] and [1]).

Let $(L, \leq, \otimes, *)$ be an enriched *cqm*-lattice (cf. [5] Subsection 1.2) such that $(L, \leq, \otimes)$ is a *cl*-quasi-monoid (cf. [5] Section 1) and $(L, \leq, *)$ is a commutative, strictly two-sided quantale (cf. [5] Subsection 1.1).

Definition. (L–**uniform space**) *Let X be a nonempty set and $\mathfrak{U}$ be a stratified L–filter (cf. Definition 6.1.4. in [5]) on $X \times X$. $\mathfrak{U}$ is called an L–*uniformity *on X iff it satisfies the following axioms:*

(V2) $\mathfrak{U}(d) \leq \bigwedge_{x \in X} d(x, x)$ *for all* $d \in L^{X \times X}$.

(V3) $\mathfrak{U}(d) \leq \mathfrak{U}(d^{-1})$ *where* $d^{-1}(x, y) = d(y, x)$.

(V4) $\mathfrak{U}(d) \leq \bigvee_{e_1, e_2 \in L^{X \times X}} (\mathfrak{U}(e_1) * \mathfrak{U}(e_2)) * \left(\bigwedge_{x,y \in X} ((e_1 \circ e_2) \to d)(x, y) \right)$
(where $(e_1 \circ e_2)(x, y) = \bigvee_{z \in X} e_1(x, z) * e_2(z, y)$*).*

*We call $(X, \mathfrak{U})$ an L–*uniform space.

Let $(X, \mathfrak{U}_1)$ and $(X, \mathfrak{U}_2)$ be L–uniform spaces and $\varphi : X \mapsto Y$ be a mapping. Then φ is said to be L–uniformly continuous if

$$\forall e \in L^{Y \times Y} \qquad \mathfrak{U}_1\left((\varphi \times \varphi)^{\leftarrow}(e)\right) \geq \mathfrak{U}_2(e).$$

*Obviously L–uniform spaces and L–uniformly continuous mappings form a category L–***UNIF** *over* **SET.**

In particular, in case $(L, \leq, *)$ is a complete MV–algebra with square roots and $\otimes = \circledast$ the monoidal mean operator (cf. [5] Remark 1.2.6), given a $\top$–filter $\mathbf{F}$ (cf. [5] Remark 6.2.3) we define the characteristic value of $\mathbf{F}$ with respect to $f \in L^X$ as $c^f(\mathbf{F}) = \bigwedge \{\alpha \in L : (\alpha \to \bot) \to f \in \mathbf{F}\}$ and use it to prove that any probabilistic uniformity $\mathbf{V}$ (cf. [3]) induces an L–uniformity $\mathfrak{U}$. Moreover, the L–neighbourhood systems induced by $\mathbf{V}$ and $\mathfrak{U}$ are the same. Consequently the category of probabilistic uniform spaces is a full subcategory of L–**UNIF**.

On the other hand, in case $\otimes = * = \wedge$, axiom $(V4)$ can be rewritten in the following form:

$(V4')$ $\mathfrak{U}(d) \leq \bigvee_{e \circ e \leq d} \mathfrak{U}(e)$.

Then we prove that there exists an isomorphism between the category of generalised uniformities L–**GUNIF** (cf. [1], [2]), or according to the terminology used in [4] L–uniformities of ordinary subsets, and the category of strongly stratified L–uniformities L–**SSUNIF** (cf. [2]). Moreover, if $\mathfrak{U}$ is the strongly stratified L–uniformity induced by the generalised uniformity $\mathfrak{u}$, then the L–neighbourhood systems induced by $\mathfrak{u}$ and $\mathfrak{U}$ are the same.

Taking into account that in the case $L = I$ and $\otimes = * = \wedge$ generalised uniformities and Lowen's uniformities are equivalent concepts (cf. [2]) this proves that the category of Lowen uniform spaces is isomorphic to the full subcategory I–**SSUNIF** of strongly stratified I–uniform spaces.

References

[1] M. H. Burton, M. A. De Prada Vicente and J. Gutiérrez García, *Generalised uniform spaces*, J. Fuzzy Math. **4(2)** (1996), 363-380.

[2] J. Gutiérrez García, I. Mardones Pérez and M. H. Burton, *The relationship between various filter notions on a GL–monoid*, J. Math. Anal. Appl. (To appear).

[3] U. Höhle, *Probabilistic uniformization of fuzzy topologies*, Fuzzy Sets and Systems **18** (1978), 311-332.

[4] U. Höhle, *Characterization of L–topologies by L–valued neighborhoods*, In: Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (Eds. U. Höhle and S. E. Rodabaugh), 389-432; Handbook Series Volume 3 Kluwer Academic Publishers, Dordrecht, Boston 1998/99).

[5] U. Höhle and A. Šostak, *Axiomatic foundations of fixed-basis fuzzy topology*, In: Mathematics of Fuzzy Sets: Logic, Topology and Measure Theory (Eds. U. Höhle and S. E. Rodabaugh), 123-272; Handbook Series Volume 3 Kluwer Academic Publishers, Dordrecht, Boston 1998/99).

[6] R. Lowen, *Fuzzy uniform spaces*, J. Math. Anal. Appl.**82** (1981), 370-385.

Covering/Hutton Fuzzy Uniformities: Preliminary Remarks

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It is well known that in classical uniformities, there are both the entourage approach due to Weill (and developed by Bourbaki) and the covering approach due to Tukey (and developed by Isbell [2] and others). As pointed out by this author [4], the latter has its counterpart in fuzzy sets in the Hutton uniformities [1], [3].

It is this writer's opinion that the Hutton approach is elegant, describes the important examples of the fuzzy real lines and unit intervals, plays a fundamental role in separation and metrization (particularly as they apply to the real lines and unit intervals), and is strongly lattice-theoretic. An early attempt to put this approach in a variable-basis context was that of [4].

The Hutton approach has, however, been lacking in development. This may be due in part to workers being unaware of the covering approach *ab initio* in the classical setting. It might also be due to the unsettled state of lattice-theoretic bases *vis-a-vis* uniformities (in either approach) in a fuzzy setting. In the covering approach, there has up to now been the requirement of complete distributivity because of the dependence on Raney's Lemma (see[1], [3], [4]). Among other things, we find such dependence to be unnecessary.

This short report explores the lattice-theoretic restrictions in the covering or Hutton approach. The question of restrictions divides into the following:

1. What lattice-theoretic restrictions are needed to have covering quasi-uniformities?
2. What lattice-theoretic restrictions are needed to have covering uniformities?
3. What lattice-theoretic restrictions on morphisms are needed for covering uniform continuity in the fixed-basis case?
4. What lattice-theoretic restrictions on morphisms are needed for covering uniform continuity in the variable-basis case?

References

[1] B. Hutton, *Uniformities on fuzzy topological spaces*, J. Math. Anal. Appl. 58 (1997) 559–571.

[2] J. R. Isbell, *Uniform Spaces*, Mathematics Surveys Monographs, Volume 12, 1964, American Mathematical Society, Providence, R.I.

[3] W. Kotzé, *Uniform spaces*, in U. Höhle, S. E. Rodabaugh, eds., *Mathematics of Fuzzy Sets*, 1999, volume in *The Handbooks of Fuzzy Sets Series*, Kluwer Academic Publishers (Dordrecht).

[4] S. E. Rodabaugh, *A theory of fuzzy uniformities with applications to the fuzzy real lines*, J. Math. Anal. Appl. 129(No.1)(1988) 37–70.