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23rd Linz Seminar on Fuzzy Set Theory

Abstracts

Analytical Methods and Fuzzy Sets

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Phil Diamond, Erich Peter Klement
Editors

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ANALYTICAL METHODS AND FUZZY SETS

ABSTRACTS

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Since their inception in 1979, the Linz Seminars on Fuzzy Sets have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work. LINZ2002 will be the 23^{rd} seminar carrying on this tradition.

LINZ2002 will deal with Fuzzy Analysis and its Applications. This very broad area, roughly speaking, involves looking at classical analysis when elements of vagueness and imprecision are present. It thus describes both fuzzification of established methods when important applications occur, and placing fuzzy structures as well-defined objects in classical functional analysis. The organizers hope that the talks will provide a comprehensive mathematical framework both for pure techniques and practical application of analytical methods utilizing fuzzy sets.

Phil Diamond
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Monotonicity in vague environments

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Zadeh's famous extension principle [8, 9, 10], as a general methodology for extending crisp concepts to fuzzy sets, has served as the basis for the inception of new disciplines like fuzzy analysis, fuzzy algebra, fuzzy topology, and several others. Most importantly, this fundamental principle allows to extend crisp mappings to fuzzy sets. Another well-known application—which is particularly important in fuzzy decision analysis and fuzzy control [4, 6]—is the possibility to define ordering relations for fuzzy sets.

This contribution is devoted to links between these two fields: we study in which way the monotonicity of a mapping is preserved by its extension to fuzzy sets. However, we do not restrict to the well-known methodology of extending crisp orderings to fuzzy sets, but we start from the more general case that the domain under consideration is equipped with a fuzzy ordering [1, 2] induced by some non-trivial fuzzy concept of indistinguishability. Even in such a case, it is possible to define orderings of fuzzy sets in a way similar to the extension principle [1, 3].

First, we consider the classical case of orderings of fuzzy sets defined from crisp orderings by means of the extension principle. We show that the monotonicity of a mapping directly transfers to its extension. Furthermore, the same holds for the componentwise monotonicity of n-ary operations. Next, we consider the general case that the universe is equipped with a fuzzy ordering induced by a fuzzy equivalence relation E. It is proved that the monotonicity of a mapping φ is preserved by its extension if and only if φ is extensional with respect to E [5, 7]. However, it turns out that an analogous correspondence does not necessarily hold for n-ary operations.

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A proximal point method for optimization in the space of fuzzy vectors

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A contemporary snapshot of fuzzy operational equations

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1 Introduction

Early studies of fuzzy differential equations observed that there was a fundamentally different behaviour from the crisp case: the solutions spread out as time evolved [14]. This seems to be an artifact of the Hukuhara derivative and Aumann integral of multivalued calculus, on which the fuzzy derivative is based. Ultimately, the effect stems from the fact that Minkowski addition of sets is not, in general, invertible.

Example 1. For simplicity, consider a fuzzy DE with interval values,

$$X'(t) = -AX(t), \quad X(0) = X^0 = [a,b], \tag{1}$$

where $a > 0,\ A = [1-\varepsilon, 1+\varepsilon]$, $0 < \varepsilon < 1$ and X' is the Hukuhara derivative. Put $X(t) = [\underline{X}(t), \overline{X}(t)]$ and (1) becomes the system of ordinary DEs

$$\begin{bmatrix} \underline{X}' \\ \overline{X}' \end{bmatrix} = \begin{bmatrix} 0 & -1-\varepsilon \\ -1+\varepsilon & 0 \end{bmatrix}, \quad \begin{bmatrix} \underline{X}(0) \\ \overline{X}(0) \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}.$$

Writing $\delta = \sqrt{1-\varepsilon^2}$, the system has a solution

$$X(t) = \begin{bmatrix} \underline{X}(t) \\ \overline{X}(t) \end{bmatrix} = c_1 \begin{bmatrix} 1+\varepsilon \\ \delta \end{bmatrix} e^{\delta t} + c_2 \begin{bmatrix} 1+\varepsilon \\ -\delta \end{bmatrix} e^{-\delta t},$$

where the constants $c_1,\ c_2$ depend on $a,\ b$. Clearly, for almost all $a,\ b$, the solution $X(t)$ expands without bound as t increases, no matter how small is ε (although, if *both* $b \to a+$ and $\varepsilon \to 0+$, then $c_1 \to 0$). This behaviour contrasts with the crisp equation of exponential decay $z'(t) = -z(t),\ z(0) = z^0$. Even a small incertitude in the parameters leads to vastly different and counterintuitive behaviour. Much the same happens with fuzzy integral equations and would appear highly likely in any form of fuzzy equation which involves the Hukuhara fuzzy derivative and Aumann fuzzy integral.

In a seminal paper, Hüllermeier suggested how this unsatisfactory obstacle to modelling uncertainty could be overcome [11]. His insight was to *interpret* the fuzzy DE $X'(t) = F(t, X(t)),\ X(0) = X^0$ as a *family of differential inclusions*

$$x'(t) \in F_\beta(t, x(t)), \quad x(0) = x^0 \in X^0_\beta, \quad 0 \le \beta \le 1,$$

where F_β, X^0_β are β-level sets. The set of points $\Sigma_\beta(t, X^0_\beta)$, of all solutions of the β-th inclusion at time t, was the level set of a fuzzy set which could be regarded as the solution of the fuzzy DE.

If the interval valued example above is interpreted in this way, all solutions $x(t)$ are bounded, $\underline{x}(t) \le x(t) \le \overline{x}(t)$, where

$$\underline{x}' = -(1+\varepsilon)\underline{x},\ \underline{x}(0) = a; \quad \overline{x}' = -(1-\varepsilon)\overline{x},\ \overline{x}(0) = b.$$

Consequently, the solution is regarded as the interval

$$\Sigma(t) = \left[ae^{-(1+\varepsilon)t}, be^{-(1-\varepsilon)t}\right],$$

which, although retaining uncertainty, does suffer exponential decay to $0 = \{0\}$ as $t \to \infty$. This is much more satisfactory and intuitive behaviour.

This talk surveys some of the recent results using Hüllermeier's interpretation applied to classes of fuzzy operator equations

$$X(t) = \mathcal{O}(X)(t),$$

where $\mathcal{O}(\cdot)$ might be a differential or integral operator, define some sort of stochastic DE or delay DE, or even a control system. Such equations will be interpreted as families of inclusions $x_\beta(t) \in \mathcal{O}(x_\beta)(t)$, $0 \le \beta \le 1$. The next section recalls some fundamental definitions relating to fuzzy sets and inclusions. The third section considers differential operators and the fourth studies Volterra type integral operators. The fifth section briefly surveys some work which is yet to appear and a conclusion sums up the talk.

2 Basic Concepts

2.1 Preliminaries

Let $\mathcal{D}^n$ denote the set of uppersemicontinuous (usc) normal fuzzy sets on $\mathbb{R}^n$ with compact connected support. That is, if $X \in \mathcal{D}^n$, then $X : \mathbb{R}^n \to [0,1]$ is usc, $\operatorname{supp}(X) = \overline{\{\xi \in \mathbb{R}^n : X(\xi) > 0\}} := [X]^0$ is compact and there exists at least one $\xi \in \operatorname{supp}(X)$ for which $X(\xi) = 1$. The $\beta-$level set of X, $0 < \beta \le 1$, is

$$[X]^\beta = \{\xi \in \mathbb{R}^n : X(\xi) \ge \beta\}.$$

For brevity, level sets $[X]^\beta$ will usually be written as X_β. Clearly, for $\alpha \le \beta$, $X_\alpha \supseteq X_\beta$. The level sets are nonempty from normality, and compact by usc and compact support. The metric d_∞ is defined by

$$d_\infty(X,Y) = \sup\left\{d_H\left(X_\beta, Y_\beta\right) :\ 0 \le \beta \le 1\right\},$$

where $X, Y \in \mathcal{D}^n$, and d_H is the Hausdorff metric on compact subsets of $\mathbb{R}^n$. Then $(\mathcal{D}^n, d_\infty)$ is a complete metric space [9]. If 0 is the origin in $\mathbb{R}^n$, $0 \in \mathcal{D}^n$ denotes the fuzzy set such that $0(\xi) = 1$ if $\xi = 0$ and is zero otherwise.

Define $\mathcal{E}^n \subset \mathcal{D}^n$ as those fuzzy sets with convex level sets. In particular, write $\mathcal{E} = \mathcal{E}^1 = \mathcal{D} = \mathcal{D}^1$. If $X \in \mathcal{E}^n$, write $X_\beta = \left[\underline{X}_\beta, \overline{X}_\beta\right]$.

To discuss solutions of inclusions, some function spaces are required.

- $C_n[0,T]$, the space of continuous functions $f:[0,T]\to\mathbb{R}^n$, with the norm of uniform convergence, $\|f-g\|_\infty=\max_{0\le t\le T}|f(t)-g(t)|$.

- $L_n^1[0,T]$ is the space of *integrable functions* from $[0,T]$ to $\mathbb{R}^n$, with metric $\|f-g\|_1=\int_0^T|f(t)-g(t)|\,dt$.

- $L_n^\infty[0,T]$ is the space of measurable functions from $[0,T]$ to $\mathbb{R}^n$, bounded almost everywhere on $[0,T]$, with norm

$$\|f\|_{L^\infty}=\operatorname{ess}\sup_{0\le t\le T}|f(t)|=\inf\{c\ge 0:|f(t)|\le c\text{ a.e. }0\le t\le T\}.$$

- $\mathcal{Z}_T(\mathbb{R}^n)=\{x(\cdot)\in C([0,T];\mathbb{R}^n):x'(\cdot)\in L^\infty([0,T];\mathbb{R}^n)\}$.

2.2 Differential and integral inclusions

Let $\Omega\subset\mathbb{R}\times\mathbb{R}^n$ be an open subset containing $(0,x_0)$ and let $H:\Omega\to\mathcal{K}_C^n$ be a compact, convex setvalued mapping. The differential inclusion

$$x'(t)\in H(t,x(t)),\qquad x(0)=x_0, \tag{2}$$

is said to have a solution $y(t)$ on $[0,T]$ if $y(\cdot)$ is absolutely continuous, $y(0)=x_0$ and $y(\cdot)$ satisfies the inclusion almost everywhere (a.e.) in $[0,T]$.

The integral inclusions are of Volterra type

$$x(t)\in h(t)+\int_0^t k(t,s)G(s,x(s))\,ds\qquad t\in[0,T], \tag{3}$$

where $h:[0,T]\to\mathbb{R}^n$ is continuous, the matrix valued function $k:\{(s,t):0\le s\le t\le T\}\to L^1_{n\times n}[0,T]$, $G:[0,T]\times\mathbb{R}^n\to\mathcal{K}_C^n$ and $G(\cdot,x)$ has measurable selections. The integral is understood to be in the sense of Aumann. A solution to (3) is a continuous function $x(\cdot)\in C[0,T]$ which satisfies the inclusion a.e. in $[0,T]$.

Under mild conditions, (2) has solutions on $[0,\tau]$, $0<\tau\le T$, [1, 3] and so does (3) [2]. In both, denote the set of solutions in $[0,\tau]$ by $\Sigma(h;\tau)$ and the attainability set for each $\tau\in[0,T]$ by $\mathcal{A}(h;\tau)=\{x(\tau):x(\cdot)\in\Sigma(h;\tau)\}$. It is known that $\Sigma(h;\tau)$ and $\mathcal{A}(h;\tau)$ are nonempty compact, connected sets in their respective spaces [1, 2, 3]. For both, $\mathcal{A}(h;\tau)\subset\mathbb{R}^n$. With (2), $\Sigma(h;\tau)$ is a subset of $\mathcal{Z}_\tau(\mathbb{R}^n)$, and for (3) it is a subset of $C_n[0,\tau]$.

3 Fuzzy Differential Equations

A convenient condition which guarantees existence of solutions to (2) is the *boundedness assumption*: there exist $b,\ T,\ M>0$ such that

- the set $Q=[0,T]\times(x_0+(b+MT)B^n)\subset\Omega$, where B^n is the unit ball of $\mathbb{R}^n$;

- H maps Q into the ball of radius M.

Now, suppose that $F : \mathbb{R} \times \mathbb{R}^n \to \mathcal{E}^n$ and consider the fuzzy differential equation

$$x' = F(t,x), \quad x(0) = X^0 \in \mathcal{E}^n,$$

but identify it with the family of differential inclusions

$$x'_\beta(t) \in F_\beta(t, x_\beta(t)) \quad x_\beta(0) = x_0 \in X^0_\beta, \quad 0 \le \beta \le 1. \tag{4}$$

Here, Ω is an open subset of $\mathbb{R}^{n+1}$ containing $(0, X^0_0)$, $\beta \in I := [0,1]$ and $F : \Omega \times I \to \mathcal{K}^n_C$. The boundedness assumption now holds if the set Q is as above and F maps $Q \times I$ into the ball of radius M. Denote the set of all solutions of (4) on $[0,\tau]$ by $\Sigma_\beta(x_0,\tau)$ and the attainable set by $\mathcal{A}_\beta(x_0,\tau) = \{x(\tau) : x(\cdot) \in \Sigma_\beta(x_0,\tau)\}$. Then $\Sigma_\beta(x_1,T)$ exists and is a compact subset of $\mathcal{Z}_T(\mathbb{R}^n)$, and each attainable section $\mathcal{A}_\beta(x_1,\tau)$, $0 < \tau \le T$, is a compact subset of $\mathbb{R}^n$ [1]. It turns out that the $\mathcal{A}_\beta$, Σ_β build level sets of fuzzy sets, which can then be regarded as solutions to the fuzzy DE.

Theorem 3.1 [10] *Let $X^0 \in \mathcal{E}^n$ and let Ω be an open set in $\mathbb{R} \times \mathbb{R}^n$ containing $\{0\} \times \text{supp}\,(X^0)$. Suppose that $F : \Omega \to \mathcal{E}^n$ is usc and write $F_\beta(t,x) \in \mathcal{K}^n_C$ as the β-level set for all $(t,x,\beta) \in \mathbb{R}^{n+1} \times [0,1]$. Let the boundedness assumption, with constants $b,\ M,\ T$, hold for all $x_0 \in \text{supp}\,(X^0)$ and the inclusion*

$$x'(t) \in F_0(t,x), \quad x(0) \in \text{supp}\,(X^0). \tag{5}$$

Then the attainable sets $\mathcal{A}_\beta(X_\beta,T)$, $\beta \in [0,1]$, of the family of inclusions

$$x'_\beta(t) \in F_\beta(t,x_\beta), \quad x_\beta(0) \in X_\beta := [X^0]^\beta,\ \beta \in [0,1], \tag{6}$$

are the level sets of a fuzzy set $\mathcal{A}(X^0,T) \in \mathcal{D}^n$. The solution sets $\Sigma_\beta(X_\beta,T)$ of (6) are the level sets of a fuzzy set $\Sigma(X^0,T)$ defined on $\mathcal{Z}_T(\mathbb{R}^n)$.

Remarks The proof of the theorem relies on the Negoita-Ralescu characterization of fuzzy sets [13]. The major part lies in showing $\bigcap_{i=1}^\infty \Sigma_{\beta_i}(X_{\beta_i},T) = \Sigma_\beta(X_\beta,T)$ for any nondecreasing sequence $\beta_i \to \beta$ in $[0,1]$, in the function space $\mathcal{Z}_T(\mathbb{R}^n)$. This entails a sequential compactness argument. If the condition that F_0 be bounded on $[0,\infty) \times \Gamma$, where $\Gamma \subseteq \mathbb{R}^n$ is open, is added to the conditions of the theorem, the interval of existence and consequences extend to $[0,\infty)$.

This result opens the way for notions of periodicity and stability to be studied in the fuzzy context. See [5] for further details.

4 Fuzzy Volterra Integral Equations

If equalities, based on the Aumann integral, are used at each level set for such equations, similar unsatisfactory behaviour occurs as was the case for fuzzy DEs. This is unsurprising, since each fuzzy DE is equivalent to an Aumann integral equation [9]. However, if the equations are replaced by a family of integral inclusions, a reasonable framework for modelling is obtained.

Suppose $v \in C^n[0,T]$ and $F : [0,T] \times \mathbb{R}^n \to \mathcal{E}^n$ is strongly measurable (that is, each level set mapping is measurable) and majorized by an L^1 function. Consider the fuzzy equation

$$x(t) = V(t) + \int_0^t k(t,s) F(s,x(s))\, ds, \tag{7}$$

interpreted as a family of integral inclusions

$$x_\beta(\cdot) \in V_\beta + \int_0^{\cdot} k(t,s)F_\beta(s,x(s))\,ds, \qquad 0 \le \beta \le 1. \tag{8}$$

Under mild but technical conditions on the kernel $k(t,s)$, the solution sets $\Sigma_\beta(V_\beta\,;t)$ and attainability sets $\mathcal{A}_\beta(V_\beta\,;t)$ are compact and connected [2], and are the level sets of fuzzy sets Σ and $\mathcal{A}$, respectively [6]. Correctly interpreted, the result extends to the case where $V(t)$ is a fuzzy valued set over $C_n[0\,,T]$.

Moreover, using integral inequalities [12], and solving equations at extreme points of the boundary, exact or computational solutions can be found.

Example 2. Let $(a\,;b)_{\mathbf{S}}$ denote a symmetric fuzzy with support $[a,b]$ and consider the fuzzy equation

$$x(t) = (0;t^2/2)_{\mathbf{S}} - \int_0^t (t-\sigma)(x(\sigma)\,;3x(\sigma))_{\mathbf{S}}\,d\sigma,\ quadt \ge 0,$$

or, equivalently for $0 \le \beta \le 1$,

$$x_\beta(t) \in [\beta t^2/4\,,(1/2-\beta/4)t^2] - \int_0^t (t-\sigma)\left[(1+\beta)x_\beta(\sigma)\,,(3-\beta)x_\beta(\sigma)\right]\,d\sigma$$

That is,

$$\begin{aligned} x_\beta(t) &\ge \beta t^2/4 - (3-\beta)\int_0^t (t-\sigma)x_\beta(\sigma)\,d\sigma \\ x_\beta(t) &\le (1/2-\beta/4)t^2 - (1+\beta)\int_0^t (t-\sigma)x_\beta(\sigma)\,d\sigma. \end{aligned}$$

Taking Laplace transforms, noting the convolution integrals, using the result quoted on integral inequalities and simplifying,

$$\frac{\beta/2}{s(s^2+3-\beta)} \le X_\beta(s) \le \frac{1-\beta/2}{s(s^2+1+\beta)}\,.$$

using partial fractions and taking the inverse transforms, the solution set consists of the fuzzy set with β-levels the intervals, for $0 \le \beta \le 1$,

$$\left[\frac{\beta(1-\cos(\sqrt{3-\beta}t))}{2(3-\beta)}\,,\frac{(2-\beta)(1-\cos(\sqrt{1+\beta}t))}{2(1+\beta)}\right].$$

5 Other Topics

5.1 Variation of constants formula

Analogues of this classical result have recently been developed [7]. Basically, what is required is a fuzzy version of the state transition matrix Φ of a system. Recall that if $x' = Ax$, $\Phi(t)$ is the unique matrix satisfying $\Phi'(t) = A\Phi(t),\ \Phi(0) = I$.

Now, given

$$x'(t) = Ax(t) + Bu(t), \qquad x(0) = x_0,$$

which can be rewritten as

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t-s)Bu(s)\,ds.$$

Suppose that the matrices $A,\ B$ have fuzzy number entries and the initial condition $x_0 \in \mathcal{E}^n$ is fuzzy. Then this equation can be considered as a family of integral inclusions, provided that some meaning can be ascribed to $\Phi(t)$. The family of inclusions

$$\Phi'_\beta(t) \in A_\beta \Phi_\beta(t), \qquad \Phi_\beta(0) = I, \quad 0 \leq \beta \leq 1,$$

has solutions, for matrices V in the set of matrices A_β,

$$\Phi_\beta(t) = \left\{ Y(t) : Y' = VY,\ Y(0) = I \right\}.$$

The $Y(t)$ are found in the usual way: find a basis of vector solutions $v_1(t),\ v_2(t), \ldots,\ v_n(t)$ from the eigenvalue-eigenvector problem for V and form the matrix $Z(t) = [v_1\ v_2\ \ldots\ v_n]$. Then $Y(t) = Z(t)Z(0)^{-1}$.

5.2 Controlling fuzzy dynamical systems

Variation of constants can be used for solving open loop control systems. A question arises as to whether some sort of feedback control is meaningful for control systems of the form

$$X'(t) = AX(t) + BU(t), \quad X(0) = X^0,$$

where A, B are matrices and X^0 an n-vector, all with fuzzy set entries over $\mathbb{R}$, in particular with a fuzzy scalar, linear state feedback law of the form $U(t) = K(t)X(t)$.

It turns out that an appropriate framework is given by set-theoretic approaches to robust control [15]. Again using the inclusion approach, the problem can be treated at each level set as a type of calculus of variations problem and can be solved by iterative numerical procedures. Matrix Riccati equations are involved. For the one dimensional case, exact solutions have been found [8]. These compare favourably with fuzzy versions of LQR control for such systems.

5.3 Miscellaneous results

A number of extensions to the above have been made, but are in a very preliminary stage of development. Integral equations of Hammerstein type have been studied, and the Volterra results generalized (Menshing Guo and Xiaoping Xue, personal communication). Very general operator equations involving measures and impulses are also being studied (Guo, *loc. cit.*).

It seems likely that existence theorems for stochastic differential inclusions can be used to study a fusion of the two uncertainties: fuzzy and random. Care has to be taken to specify which stochastic integral is to be used, but some progress has been made using the Ito integral. The results are, as one might expect, more complicated than others discussed here.

Some very recent progress has been made in filtering of fuzzy signals, governed by a fuzzy DE containing a white noise signal with a fuzzy covariance function. The inclusion approach is used, drawing upon results in robust filtering. These provide linear filters upper and lower bounds for the covariance of an otherwise unspecified white noise disturbance. By considering the extremal equations of the system, bounds can be found for the estimator.

6 Conclusion

Recent developments in the theory of fuzzy differential and fuzzy integral equations were surveyed. These treat equations as families of inclusions and interpret the solution sets as level sets of a fuzzy solution to the equation. Some applications were discussed and examples given.

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Nearness-based limits in multidimensonal case

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The concept of nearness-based limits has been introduced at the 7-th Fuzzy Days conference in Dortmund (see [1]). Now, we are going to discuss the properties of such limits for functions having multidimensional domain.

We recall the definition of fuzzy nearness.

Definition $\mathcal{F} : \mathcal{R}^2 \to [0;1]$ is said to be a relation of fuzzy nearness iff

1. for any $x \in \mathcal{R}$ $\quad x\ \mathcal{F}\ x = 1$
2. for any x, y $\quad x\ \mathcal{F}\ y = y\ \mathcal{F}\ x$
3. for any $\alpha \in]0;1[$ and any x there exist unique $x_\alpha > x$ and $x_{-\alpha} < x$ such that $x\ \mathcal{F}\ x_\alpha = \alpha = x\ \mathcal{F}\ x_{-\alpha}$
4. for any x there holds $\lim_{y\to\infty} x\mathcal{F}y = 0$
5. for any $x \leq s \leq t \leq z$ there holds $x\mathcal{F}z \leq s\mathcal{F}t$.

If moreover $x\mathcal{F}y = 1$ iff $x = y$ holds, then we call $\mathcal{F}$ the strict fuzzy nearness.

References

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On set valued and fuzzy stochastic differential equations

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ABSTRACT

We define a set valued stochastic integral with respect to a 1-dimensional Brownian motion and develop multivalued analogs to the theory of singlevalued stochastic integrals. And we study set valued and fuzzy stochastic differential equations.

1. Introduction

The martingale property of the singlevalued stochastic integral is a basic principle for the development of stochastic analysis. Correspondingly, it is important in multivalued stochastic analysis to both define a set valued stochastic integral and demonstrate the martingale property. Following the seminal work of Matheron[12], more recently the study of random sets, that is, set valued random variables, has been developed by Aubin and Frankowska[1], Hiai and Umegaki[5], Papageorgiou[14] and many others. Set valued martingales have been studied by Li and Ogura[12], Papageorgiou[14]. Kim and Kim[9] and Kisielewicz[11] defined set valued stochastic integral, but did not consider the martingale property and L^2-boundedness, which are useful in applications. The theory of ordinary differential equations has been extensively developed in conjunction with fuzzy valued analysis (see [8]). In particular, Bàn[2], Li and Ogura [12], Puri and Ralescu[15] and Stojacovič[17] introduced the concept of fuzzy random variable, fuzzy conditional expectation and fuzzy martingale. This paper develops a more usable definition of the set valued stochastic integral that is more applicable, and derives set valued analogs of some of the most important properties of singlevalued stochastic integrals. And by using the definition and properties of the set valued stochastic integral, we study the theory for set valued stochastic differential equations, stochastic inclusion problems and fuzzy stochastic differential equations. The paper is set out as follow: Section 2 recalls some useful results on set valued random variables, due to Hiai and Umegaki[5] and Theorem 2.6 examines an important example of a set valued martingale. In Section 3, we define a set valued stochastic integral with respect to 1-dimensional Brownian motion, and discuss some of its properties. Section 2 and 3 are results by Jung and Kim[7]. In Section 4 we prove the existence and uniqueness of the solution for a set valued stochastic differential equation. In Section 5, we give fuzzy analogs of Section 3 and 4.

2. Preliminaries

Let R be the set of all real numbers and $(\Omega, \mathcal{A}, P)$ a complete probability space. Denote by $\mathcal{M}(\Omega; 2^R)$ the family of all measurable set valued functions defined on Ω with values in the family of nonempty closed subsets of R.

For $F \in \mathcal{M}(\Omega; 2^R)$, we define

$$S_F^p = \{f \in L^p(\Omega, \mathcal{A}; R) : f(\omega) \in F(\omega) \ \ \text{a.a.}\omega \in \Omega\}, p = 1, 2, 3 \cdots$$

Here $L^p(\Omega, \mathcal{A}; R)$ is the space of all real valued random variables f such that $||f||_p^p = E[|f|^p] < \infty$, where $E[g]$ is the expectation of a random variable g.

THEOREM 2.1. *([5]) Let $F_1, F_2 \in \mathcal{M}(\Omega; 2^R)$ and $1 \leq p \leq \infty$. If $S_{F_1}^p = S_{F_2}^p \neq \phi$, then $F_1 = F_2$ a.s.*

Let M be a set of measurable function $f : \Omega \to R$. We call M *decomposable* with respect to $\mathcal{A}$ if $f_1, f_2 \in M$ and $A \in \mathcal{A}$ imply $1_A f_1 + 1_{A^c} f_2 \in M$. Here 1_A is the indicator function of A.

THEOREM 2.2. *([5]) Let M be a nonempty closed subset of $L^p(\Omega, \mathcal{A}; R)$, $1 \leq p < \infty$. Then there exists an $F \in \mathcal{M}(\Omega; 2^R)$ such that $M = S_F^p$ if and only if M is decomposable with respect to $\mathcal{A}$.*

For any subset $\Gamma \subset L^p(\Omega, \mathcal{A}; R)$, we define the *decomposable closure* of Γ by

$$\begin{aligned}\overline{de}\Gamma = \big\{ g \in L^p(\Omega, \mathcal{A}; R) :\ & \text{for any } \varepsilon > 0, \text{ there exist a finite} \\ & \mathcal{A}\text{-measurable partition } \{A_1, A_2, \cdots, A_n\} \text{ of } \Omega \text{ and} \\ & f_1, f_2, \cdots, f_n \in \Gamma \text{ such that } ||g - \sum_{i=1}^n 1_{A_i} f_i||_p < \epsilon \big\}.\end{aligned}$$

Remark 2.3. Clearly $\overline{de}\Gamma$ is a closed subset of $L^p(\Omega, \mathcal{A}; R)$ and is decomposable with respect to $\mathcal{A}$. Hence by Theorem 2.2, for any $\Gamma \subset L^p(\Omega, \mathcal{A}; R)$, there exists an $F \in \mathcal{M}(\Omega; 2^R)$ such that $\overline{de}\Gamma = S_F^p$. And if Γ is decomposable, then it holds that $\overline{de}\Gamma = cl\Gamma$, where the closure is taken with respect to the norm $|| \cdot ||_p$.

Denote by $\mathcal{K}(R)$ the family of all nonempty closed subsets of R and $\mathcal{K}_c(R) \subset \mathcal{K}(R)$ the totality of all such sets which are also convex and thus intervals.

For any $A, B \in \mathcal{K}(R)$, we define

$$d_H(A, B) = \ \max \left\{ \sup_{x \in A} d(x, B), \ \sup_{y \in B} d(y, A) \right\},$$

where $d(x, B) = \inf_{y \in B} |x - y|$, and define $|||A||| = d_H(A, \{0\}) = \sup_{x \in A} |x|$.

If A and B are bounded, then $d_H(A, B)$ is the Hausdorff metric of A and B. For $A, B, C \in \mathcal{K}(R)$, the equality

$$d_H(A + C, B + C) = d_H(A, B) \tag{2.1}$$

holds (see [16]).

For a random set $F \in \mathcal{M}(\Omega; 2^R)$, F is called *L^p-bounded* if there is $h \in L^p(\Omega, \mathcal{A}; R)$ such that $|x| \leq h(\omega)$ for any x and ω with $x \in F(\omega)$.

Remark 2.4. For any $F \in \mathcal{M}(\Omega; 2^R)$, it is well known that F is L^p-bounded if and only if the function $\omega \mapsto |||F(\omega)|||$ is in $L^p(\Omega, \mathcal{A}; R)$, i.e. $|||F||| \in L^p(\Omega, \mathcal{A}; R)$(see [5],[9] and [11]).

Let $L^p(\Omega, \mathcal{A}; \mathcal{K}(R))$ (resp. $L^p(\Omega, \mathcal{A}; \mathcal{K}_c(R))$) be the space of all $\mathcal{K}(R)$ (resp. $\mathcal{K}_c(R)$) valued L^p-bounded random sets in $\mathcal{M}(\Omega; 2^R)$. For $F \in \mathcal{M}(\Omega; 2^R)$, the *expectation* of F is defined by

$$\mathcal{E}[F] = \left\{ E[f] : f \in S_F^1 \right\}.$$

We call $(F(t))_{t \geq 0}$ a *set valued process* with values in R or, for short, a *set-valued R-process* if $F : [0, \infty) \times \Omega \to 2^R$ is a set valued function such that

(1) for all $t \in [0, \infty), F(t, \omega)$ is closed convex in R a.a.$\omega \in \Omega$,
(2) for any fixed $t \in [0, \infty), F(t, \cdot)$ is a random set, that is, for all Borel sets $A \in \mathcal{B}(R), \{\omega \in \Omega : F(t, \omega) \cap A \neq \phi\} \in \mathcal{A}$.

A set valued R-process $(F(t))_{t \geq 0}$ is called *$\mathcal{A}_t$-adapted* if $F(t)$ is measurable with respect to $\mathcal{A}_t$ for every $t \geq 0$. And a set valued R-process $(F(t))_{t \geq 0}$ is called *measurable* if $\{(t, \omega) \in [0, \infty) \times \Omega : F(t, \omega) \cap A \neq \phi\} \in \mathcal{B}([0, \infty) \times \mathcal{A})$ for $A \in \mathcal{B}(R)$.

Hiai and Umegaki[5] showed that for any $F \in L^p(\Omega, \mathcal{A}; \mathcal{K}(R))$ and sub-σ-field $\mathcal{B} \subset \mathcal{A}$, there exists a unique $G \in L^p(\Omega, \mathcal{B}, P; \mathcal{K}(R))$ such that

$$S_G^p = cl \left\{ E[f|\mathcal{B}] : f \in S_F^p \right\} \tag{2.2}$$

where the closure is taken with respect to the norm in $L^p(\Omega, \mathcal{B}; R)$. This random set G is called the (set valued) *conditional expectation* of F given $\mathcal{B}$ and we denote it by $\mathcal{E}[F|\mathcal{B}]$.

DEFINITION 2.5. A set valued R-process $(F(t))_{t \geq 0}$ is called an *$\mathcal{A}_t$-martingale* if

(1) for any $t \geq 0$, $F(t)$ is L^1-bounded,
(2) $(F(t))_{t \geq 0}$ is $\mathcal{A}_t$-adapted,
(3) for $t \geq s \geq 0$, $\mathcal{E}[F(t)|\mathcal{A}_s] = F(s)$ a.s.

For any random set $F \in L^1(\Omega, \mathcal{A}; \mathcal{K}_c(R))$, $F(t) = \mathcal{E}[F|\mathcal{A}_t]$ defines a set valued $\mathcal{A}_t$-martingale. Let $L^p(R)$ be the space of all $\mathcal{A}_t$-adapted real valued measurable processes $(f(t))_{t \geq 0}$ such that for any $t > 0$, $E[\int_0^t |f(s)|^p ds] < \infty$.

The next result is derived from Example 4.1 and 4.3 in Li and Ogura[12].

THEOREM 2.6. *Assume that $(m(t))_{t \geq 0}$ is a nonnegative martingale in $L^2(R)$. Let $M = [-1, 1]$ and define $(M(t))_{t \geq 0}$ by M(t)=m(t)M a.s.. Then $(M(t))_{t \geq 0}$ is a set valued $\mathcal{A}_t$-martingale.*

proof. Clearly $S_{M(s)}^2$ is given by

$$S_{M(s)}^2 = \left\{ f \in L^2(\Omega, \mathcal{A}_s; FR) : |f| \leq m(s) \quad \text{a.s.} \right\}.$$

Let $t \geq s > 0$ and $g \in S^2_{\mathcal{E}[M(t)|\mathcal{A}_s]}$. Then $g = E[f|\mathcal{A}_s]$ for some $f \in S^2_{M(t)}$. By the properties of the conditional expectation, we have

$$\begin{aligned} |g| = |E[f|\mathcal{A}_s]| &\leq E[|f||\mathcal{A}_s] \\ &\leq E[m(t)|\mathcal{A}_s] = m(s) \quad \text{a.s.} \end{aligned}$$

This shows that $g \in S^2_{M(s)}$, so that

$$S^2_{\mathcal{E}[M(t)|\mathcal{A}_s]} \subset S^2_{M(s)}. \tag{2.3}$$

Now we prove the converse. Clearly $(M(t))_{t\geq 0}$ is given by $M(t,\omega) = [-m(t,\omega), m(t,\omega)]$. Let $g \in S^2_{M(s)}$ and

$$\lambda_s(\omega) = \begin{cases} \frac{m(s,\omega)-g(\omega)}{2m(s,\omega)} & \text{if } m(s,\omega) \neq 0 \\ 0 & \text{if } m(s,\omega) = 0 \end{cases}$$

and define a random variable h by

$$h(\omega) = (1 - 2\lambda_s(\omega))m(t,\omega).$$

Since $0 \leq \lambda_s \leq 1$ a.s., by the convexity argument, it holds $h \in S^2_{M(t)}$ clearly. And we can calculus that

$$\begin{aligned} E[h|\mathcal{A}_s] &= E[(1 - 2\lambda_s)m(t)|\mathcal{A}_s] \\ &= (1 - 2\lambda_s)m(s) \\ &= g \quad \text{a.s.} \end{aligned}$$

This means that

$$S^2_{M(s)} \subset S^2_{\mathcal{E}[M(t)|\mathcal{A}_s]}.$$

Combining this with (2.3) and using Theorem 2.1, $(M(t))_{t\geq 0}$ is a set valued $\mathcal{A}_t$-martingale. The proof is complete.

3. Set valued stochastic integrals

The martingale property of the singlevalued stochastic integral is very important for the study stochastic analysis. Similarly the martingale property of the set valued stochastic integral plays an important role in set valued and fuzzy stochastic analysis. In this section we introduce a slightly different but more convenient definition than those previously defined (see [9],[11]) and prove the martingale property. For a set valued R-process $(F(t))_{t\geq 0}$, an *L^p-selection* of $(F(t))_{t\geq 0}$ is a real valued process $(f(t))_{t\geq 0} \in L^p(R)$ satisfying for every $t \geq 0$, $f(t,\omega) \in F(t,\omega)$ a.a.$\omega \in \Omega$. We denote by $\mathcal{S}_p(F(t))$ the set of all L^p-selections of the set valued R-process $(F(t))_{t\geq 0}$. Distinguish $\mathcal{S}_p(F(t))$ from the set $S^p_{F(t)}$ of all selections $g \in L^p(\Omega, \mathcal{A}_t; R)$ of a random set $F(t)$ for any fixed $t \geq 0$.

A set valued R-process $(F(t))_{t\geq 0}$ is called *L^p-bounded* if there is a process $(h(t))_{t\geq 0} \in L^p(R)$ such that $|x| \leq h(t,\omega)$ for any $t \geq 0$, $\omega \in \Omega$ and x with $x \in F(t,\omega)$.

Remark 3.1. By the same argument as Remark 2.4, we can see that a set valued R-process $(F(t))_{t\geq 0}$ is L^p-bounded if and only if $(|||F(t)|||)_{t\geq 0} \in L^p(R)$.

Let $\mathcal{L}^p(\mathcal{K}(R))$ (resp. $\mathcal{L}^p(\mathcal{K}_c(R))$) be the set of all $\mathcal{K}(R)$ (resp. $\mathcal{K}_c(R)$) valued L^p-bounded $\mathcal{A}_t$-adapted measurable set valued R-processes.

For any $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$, we define

$$\Gamma_t = \left\{ \int_0^t f(s)dw_s : (f(t))_{t\geq 0} \in \mathcal{S}_2(F(t)) \right\},$$

for all $t \geq 0$, where $(w(t))_{t\geq 0}$ is a real valued Brownian motion with $w(0) = 0$ a.s.

THEOREM 3.2. *For any $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$ and $t \geq 0$, there exists an $I_t(F) \in \mathcal{M}(\Omega; 2^R)$ such that $S^1_{I_t(F)} = \overline{de}\Gamma_t$.*

proof. We first show that $\overline{de}\Gamma_t \subset L^1(\Omega, \mathcal{A}_t; R)$. Let $\phi \in \overline{de}\Gamma_t$. Then for any $\epsilon > 0$, there exist a finite $\mathcal{A}_t$-measurable partition $\{A_1, A_2, \cdots, A_n\}$ of Ω and $\{(f_i(t))_{t\geq 0} \in \mathcal{S}_2(F(t)) : i = 1, 2, \cdots, n\}$ such that

$$E\left[\left|\phi - \sum_{i=1}^n 1_{A_i} \int_0^t f_i(s)dw_s\right|\right] < \varepsilon.$$

Thus we have

$$\begin{aligned} E[|\phi|] &\leq E\left[\left|\phi - \sum_{i=1}^n 1_{A_i} \int_0^t f_i(s)dw_s\right|\right] + E\left[\left|\sum_{i=1}^n 1_{A_i} \int_0^t f_i(s)dw_s\right|\right] \\ &= \varepsilon + E\left[\left|\sum_{i=1}^n 1_{A_i} \int_0^t f_i(s)dw_s\right|\right]. \end{aligned}$$

Here we have

$$\begin{aligned} E\left[\left|\sum_{i=1}^n 1_{A_i} \int_0^t f_i(s)dw_s\right|\right] &\leq \sum_{i=1}^n E\left[1_{A_i}\left|\int_0^t f_i(s)dw_s\right|\right] \\ &\leq \sum_{i=1}^n E\left[1_{A_i}\right] E\left[\left|\int_0^t f_i(s)dw_s\right|^2\right]^{\frac{1}{2}} \\ &= \sum_{i=1}^n P(A_i) E\left[\int_0^t |f_i(s)|^2 ds\right]^{\frac{1}{2}}. \end{aligned}$$

Since $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$ and $(f_i(t))_{t\geq 0} \in \mathcal{S}_2(F(t))$, there exists a constant $K > 0$ such that $E[\int_0^t |f_i(s)|^2 ds]^{\frac{1}{2}} < K$ and hence

$$\begin{aligned} E\left[\left|\sum_{i=1}^n 1_{A_i} \int_0^t f_i(s)dw_s\right|\right] &\leq K \sum_{i=1}^n P(A_i) \\ &= KP(\Omega) = K. \end{aligned}$$

Thus we have $E[|\phi|] < \varepsilon + K < \infty$ which implies $\phi \in L^1(\Omega, \mathcal{A}_t; R)$. Therefore $\overline{de}\Gamma_t \subset L^1(\Omega, \mathcal{A}_t; R)$. Now by the definition, $\overline{de}\Gamma_t$ is closed and decomposable with respect to $\mathcal{A}_t$. So by Theorem 2.2, there exists an $I_t(F) \in \mathcal{M}(\Omega; 2^R)$ such that $S^1_{I_t(F)} = \overline{de}\Gamma_t$. The proof is complete.

DEFINITION 3.3. The random set $I_t(F)$ defined by Theorem 3.2 is called a *stochastic integral* of $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$ with respect to a real valued Brownian motion $(w_t)_{t\geq 0}$ and we denote it by $I_t(F) = \int_0^t F(s)dw_s$.

Remark 3.4. Kim and Kim[9] and Kisielewicz[11] defined a set valued integal $I_t(F)$ by

$$I_t(F)(\omega) = \Gamma_t(\omega) = \left\{ \int_0^t f(s,\omega)dw_s : (f(t))_{t\geq 0} \in \mathcal{S}_2(F(t)) \right\}.$$

The set valued conditional expectation of a random set is defined by its selection representation (2.2). This means that our definition by a selection representation $S^1_{I_t(F)} = \overline{de}\Gamma_t$ is more convenient to prove some properties of $(I_t(F))_{t\geq 0}$ like the martingale property.

THEOREM 3.5. *Suppose that $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$. Then there exists a sequence $\{(f^i(t))_{t\geq 0} : i = 1, 2, \cdots\}$ of $\mathcal{S}_2(F(t))$ such that for each $t \geq 0$, $F(t,\omega) = cl\{f^i(t,\omega) : i = 1, 2, \cdots\}$ and*

$$I_t(F)(\omega) = cl\left\{ \int_0^t f^i(s,\omega)dw_s(\omega) : i = 1, 2, \cdots \right\} \quad a.a.\omega \in \Omega. \tag{3.1}$$

proof. For each $t \geq 0$, since $I_t(F) \in \mathcal{M}(\Omega; 2^R)$, there exists a sequence $\{\phi_n : n = 1, 2, \cdots\} \subset S^1_{I_t(F)}$ such that $I_t(F)(\omega) = cl\{\phi_n(\omega) : n = 1, 2, \cdots\}$ a.a.$\omega \in \Omega$(see Lemma 1.1 in [5]). Since

$$\begin{aligned} S^1_{I_t(F)} &= \overline{de}\left\{ \int_0^t f(s)dw_s : (f(t))_{t\geq 0} \in \mathcal{S}_2(F(t)) \right\} \\ &= cl\Big\{ \sum_{k=1}^m 1_{A_k^m} \int_0^t f_k^m(s)dw_s : \{A_k^m : k = 1, 2, \cdots, m\} \\ &\qquad \text{is an } \mathcal{A}_t\text{-measurable partition of } \Omega \\ &\qquad \text{and } (f_k^m(t))_{t\geq 0} \in \mathcal{S}_2(F(t)) \Big\}, \end{aligned}$$

for any $n = 1, 2, \cdots$, there exists $\{\phi_n^m(t) : m = 1, 2, \cdots\}$ such that $||\phi_n - \phi_n^m(t)||_2 \to 0$ as $m \to \infty$ and

$$\phi_n^m(t) = \sum_{k=1}^m 1_{A_{nk}^m} \int_0^t f_{nk}^m(s)dw_s,$$

where $\{A_{nk}^m : k = 1, 2, \cdots, m\}$ is an $\mathcal{A}_t$-measurable partition of Ω and $\{(f_{nk}^m(t))_{t\geq 0} : k = 1, 2, \cdots, m\} \subset \mathcal{S}_2(F(t))$. There exists a subsequence $\{m_j\}$ of $\{1, 2, \cdots\}$ such that $|\phi_n -$

$\phi_n^{m_j}(t)| \to 0$ a.s.. Thus for a.a.$\omega \in \Omega$,

$$\begin{aligned} I_t(F)(\omega) &= cl\,\{\phi_n^{m_j}(t,\omega) : n \geq 1, j \geq 1\} \\ &= cl\left\{\int_0^t f_{nk}^{m_j}(s,\omega)dw_s(\omega) : n \geq 1, \exists k \leq m_j, j \geq 1\right\} \\ &\subset cl\left\{\int_0^t f_{nk}^{m_j}(s,\omega)dw_s(\omega) : n \geq 1, k \leq m_j, j \geq 1\right\} \\ &\subset I_t(F)(\omega). \end{aligned}$$

Since $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$, there exists a sequence $\{(f_l(t))_{t\geq 0} : l = 1, 2, \cdots\}$ of $\mathcal{S}_2(F(t))$ such that for each $t \geq 0, F(t,\omega) = cl\{f_l(t,\omega) : l = 1, 2, \cdots\}$. Put $\{(f^i(t))_{t\geq 0} : i = 1, 2, \cdots\} = \{(f_l(t))_{t\geq 0}, (f_{nk}^{m_j}(t))_{t\geq 0} : n, l, j \geq 1, k \leq m_j\}$. Then this sequence is the desired one. The proof is complete.

Remark 3.6. Theorem 3.5 shows that $(I_t(F))_{t\geq 0}$ is Castaing representable, i.e., there exists a sequence $\{(\phi^i(t))_{t\geq 0} : i = 1, 2, \cdots\}$ of $\mathcal{A}_t$- measurable processes such that

$$I_t(F)(\omega) = cl\,\{\phi^i(t,\omega) : i = 1, 2, \cdots\}$$

for all $t \geq 0$ and a.a.$\omega \in \Omega$.

THEOREM 3.7. *Let $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$. Then for every $t \geq 0$, we have $I_t(F) \in L^2(\Omega, \mathcal{A}_t; \mathcal{K}_c(R))$.*

proof. Since all selections of $I_t(F)$ are $\mathcal{A}_t$-measurable, $I_t(F)$ is an $\mathcal{A}_t$-measurable. To prove $I_t(F)(\omega)$ is a convex set in R a.a.$\omega \in \Omega$, it is sufficient to show that $S^1_{I_t(F)}$ is convex by Corollary 1.6 in [5]. Let $\phi, \psi \in S^1_{I_t(F)}$. Then there are two $\mathcal{A}_t$-measurable partitions $\{A_1, A_2, \cdots, A_n\}$, $\{B_1, B_2, \cdots, B_m\}$ of Ω and $\{\phi_i : i = 1, 2, \cdots, n\}, \{\psi_j : j = 1, 2, \cdots, m\} \subset \Gamma_t \equiv \{\int_0^t f(s)dw_s : (f(t))_{t\geq 0} \in \mathcal{S}_2(F(t))\}$ such that

$$\left|\phi - \sum_{i=1}^n 1_{A_i}\phi_i\right| < \epsilon$$

and

$$\left|\psi - \sum_{j=1}^m 1_{B_j}\psi_j\right| < \epsilon.$$

For any $0 \leq \alpha \leq 1$, we have

$$\begin{aligned} &\left|\alpha\phi + (1-\alpha)\psi - \alpha\sum_{i=1}^n 1_{A_i}\phi_i - (1-\alpha)\sum_{j=1}^m 1_{B_j}\psi_j\right| \\ &\leq \alpha\epsilon + (1-\alpha)\epsilon \\ &= \epsilon \end{aligned}$$

and

$$\alpha\sum_{i=1}^{n}1_{A_i}\phi_i-(1-\alpha)\sum_{j=1}^{m}1_{B_j}\psi_j$$
$$=\sum_{(i,j)=(1,1)}^{(n,m)}1_{D_{(i,j)}}\{\alpha\phi_i+(1-\alpha)\psi_j\}$$

where $D_{(i,j)}=A_i\cap B_j$. Since $\{D_{(i,j)}: i=1,2,\cdots,n, j=1,2,\cdots,m\}$ is an $\mathcal{A}_t$-measurable partition of Ω and $\{\alpha\phi_i+(1-\alpha)\psi_j : 1\le i\le n, 1\le j\le m\}\subset\Gamma_t$, $\alpha\phi+(1-\alpha)\psi\in\overline{de}\Gamma_t=S^1_{I_t(F)}$. Hence $S^1_{I_t(F)}$ is convex. Finally we prove that for every $t\ge 0$, the random set $I_t(F)$ is L^2-bounded. From Theorem 3.5 there exists a sequence $\{(f^i(t))_{t\ge 0} : i=1,2,\cdots\}$ of $\mathcal{S}_2(F(t))$ such that $F(t,\omega)=cl\{f^i(t,\omega): i=1,2,\cdots\}$ and

$$I_t(F)(\omega)=cl\left\{\int_0^t f^i(s,\omega)dw_s(\omega): i=1,2,\cdots\right\}\qquad \text{a.a.}\omega\in\Omega.$$

By Theorem 2.2 in [5], we have

$$\begin{aligned}
E[|||I_t(F)|||^2] &= \int_\Omega |||I_t(F)(\omega)|||^2 dP(\omega)\\
&= \int_\Omega \sup_i\left|\int_0^t f^i(s,\omega)dw_s(\omega)\right|^2 dP(\omega)\\
&= \int_\Omega \sup_{(x(t))_{t\ge0}\in(F(t,\omega))_{t\ge 0}}\left|\int_0^t x(s)dw_s(\omega)\right|^2 dP(\omega)\\
&= \sup_{(f(t))_{t\ge0}\in\mathcal{S}_2(F(t))}\int_\Omega\left|\int_0^t f(s,\omega)dw_s(\omega)\right|^2 dP(\omega)\\
&= \sup_{(f(t))_{t\ge0}\in\mathcal{S}_2(F(t))} E\left[\left|\int_0^t f(s)dw_s\right|^2\right]\\
&= \sup_{(f(t))_{t\ge0}\in\mathcal{S}_2(F(t))} E\left[\int_0^t |f(s)|^2 ds\right]\\
&= E\left[\int_0^t \sup_{x\in F(s,\cdot)}|x|^2 ds\right]\\
&= E\left[\int_0^t |||F(s)|||^2 ds\right].
\end{aligned}$$

By the fact that $(F(t))_{t\ge0}\in\mathcal{L}^2(\mathcal{K}_c(R))$ and Remark 3.1, the right hand side is finite. Hence by Remark 2.4, $I_t(F)$ is L^2-bounded. The proof is complete.

Remark 3.8. In Theorem 3.7 we proved $I_t(F)\in L^2(\Omega,\mathcal{A}_t;\mathcal{K}_c(R))$. This implies that $S^2_{I_t(F)}=S^1_{I_t(F)}$. In fact, assume that $S^2_{I_t(F)}\subsetneq S^1_{I_t(F)}$. Then there exists a $\phi\in S^1_{I_t(F)}$

such that $\phi \notin S^2_{I_t(F)}$, i.e. $E[|\phi|] < \infty$ but $E[|\phi|^2] = \infty$. Since $I_t(F) \in L^2(\Omega, \mathcal{A}_t; 2^R)$, by Remark 2.4, $|||I_t(F)||| \in L^2(\Omega, \mathcal{A}_t; R)$. Thus we have

$$\begin{aligned}\infty = E[|\phi|^2] &\leq \sup_{\psi \in S^1_{I_t(F)}} E[|\psi|^2] \\ &= E[\sup_{x \in I_t(F)(\cdot)} |x|^2] \\ &= E[|||I_t(F)|||^2] < \infty.\end{aligned}$$

This is contradiction. Therefore we can write that $S^2_{I_t(F)} = \overline{de}\Gamma_t$ i.e.,

$$S^2_{\int_0^t F(s)dw_s} = \overline{de}\left\{\int_0^t f(s)dw_s : (f(t))_{t\geq 0} \in \mathcal{S}_2(F(t))\right\}.$$

THEOREM 3.9. *For any $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$, $(I_t(F))_{t\geq 0}$ is d_H-continuous with probability one.*

proof. Let $\{(f^i(t))_{t\geq 0} : i = 1, 2, \cdots\}$ be a sequence given as in Theorem 3.5. Using the equality (2.1), we have

$$\begin{aligned}d_H(I_{t+h}(F)(\omega), I_t(F)(\omega)) &= |||\int_t^{t+h} F(s,\omega)dw_s(\omega)||| \\ &= \sup_i \left|\int_t^{t+h} f^i(s,\omega)dw_s(\omega)\right| \qquad (3.2)\end{aligned}$$

for all $\omega \in \Omega$. Since $(\int_0^t f(s)dw_s)_{t\geq 0}$ is continuous with probability one for all $(f(t))_{t\geq 0} \in L^2(R)$, there are probability zero sets $N_i, i = 1, 2, \cdots$ such that $\int_0^t f^i(s,\omega)dw_s(\omega)$ is continuous for all $\omega \in N_i^c$. Put $N = \cup_i N_i$. Then $P(N) = 0$ and the right hand side of (3.2) converges to 0 as $h \to 0$ for all $\omega \in N^c$. The proof is complete.

To prove the martingale property of a set valued stochastic integral, we use the following condition.

CONDITION (C). The process $(F(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$ is defined by $F(t) = f_0(t)M$ a.s., where $M = [-1, 1]$ and $(f_0(t))_{t\geq 0} \in L^2(R)$ satisfies that there exists a constant $c \geq 0$ such that

$$\int_0^t f_0(s)dw_s + c \geq 0 \quad \text{a.s.} \qquad (3.3)$$

for all $t \geq 0$.

THEOREM 3.10. *Assume that $(F(t))_{t\geq 0}$ satisfies Condition (C). Then $(I_t(F))_{t\geq 0}$ is a set valued $\mathcal{A}_t$-martingale.*

proof. Let $(f_0(t))_{t\geq 0} \in L^2(R)$ be given by (6) and define a martingale $(m_0(t))_{t\geq 0}$ by

$$m_0(t) = \int_0^t f_0(s)dw_s \quad \text{a.s.}$$

and let $m(t) = m_0(t) + c$ a.s.. Then, by Condition (C), $(m(t))_{t\geq 0}$ is a nonnegative martingale. By the definition of $I_t(F)$, we can see that

$$\begin{aligned} S^2_{I_t(F)+cM} &= \overline{de}\Big\{ \int_0^t f(s)dw_s + a : f(t,\omega) \in f_0(t,\omega)M \\ &\qquad\qquad \text{a.a. } (t,\omega) \in [0,\infty) \times \Omega,\ a \in cM \Big\} \\ &= cl\{\phi(t) : \phi(t,\omega) \in (m_0(t,\omega) + c)M \text{ a.a. } (t,\omega) \in [0,\infty) \times \Omega\} \\ &= S^2_{(m_0(t)+c)M} \\ &= S^2_{m(t)M}. \end{aligned}$$

By Theorem 2.1, $I_t(F) + cM = m(t)M$ a.s.. Using Theorem 2.6, $(I_t(F) + cM)_{t\geq 0}$ is a set valued $\mathcal{A}_t$-martingale. ¿From this we have

$$\begin{aligned} \mathcal{E}[I_t(F)|\mathcal{A}_s] + cM &= \mathcal{E}[I_t(F) + cM|\mathcal{A}_s] \\ &= I_s(F) + cM \end{aligned}$$

so that $\mathcal{E}[I_t(F)|\mathcal{A}_s] = I_s(F)$ i.e., $(I_t(F))_{t\geq 0}$ is a set valued $\mathcal{A}_t$-martingale. The proof is complete.

Remark 3.11. For any $(f(t))_{t\geq 0} \in L^2(R)$, we can take $(f(t))_{t\geq 0} \in L^2(R)$ satisfying (3.3) in Condition (C) such that

$$|\int_0^t f(s)dw_s| \leq \int_0^t f_0(s)dw_s.$$

This means that $(\int_0^t f(s)dw_s)_{t\geq 0}$ is a selection of $(\int_0^t f_0(s)Mdw_s)_{t\geq 0}$. So we can say that Condition(C) is not so strong.

Let us now illustrate the results by means of a simple example.

EXAMPLE 3.12. Let $(r(t))_{t\geq 0} \in L^2(R)$ satisfy

$$E\left[\exp\left(\frac{1}{2}\int_0^t |r(s)|^2 ds\right)\right] < \infty$$

for each $t \geq 0$ and define $(x(t))_{t\geq 0}$ by

$$x(t) = \exp\left\{\int_0^t r(s)dw_s - \frac{1}{2}\int_0^t |r(s)|^2 ds\right\}.$$

Clearly $(x(t))_{t\geq 0}$ is nonnegative. And by Theorem III.5.3 in Ikeda and Watanabe[6], this is an $\mathcal{A}_t$-martingale. Using Itô's formula we can see that

$$x(t) = 1 + \int_0^t x(s)r(s)dw_s.$$

Define $(f_0(t))_{t\geq 0}$ by $f_0(t) = x(t)r(t)$ a.s.. Put $M = [-1, 1]$ and $F(t) = f_0(t)M$ a.s.. Then $(f_0(t))_{t\geq 0}$ and $(F(t))_{t\geq 0}$ satisfy (3.3) in Condition (C) with $c = 1$. Thus by Theorem 3.10, $(I_t(F))_{t\geq 0}$ is a set valued $\mathcal{A}_t$-martingale.

THEOREM 3.13. *Let $(F(t))_{t\geq 0},(G(t))_{t\geq 0}\in\mathcal{L}^2(\mathcal{K}_c(R))$ satisfy Condition(C). Then we have*

$$E\Big[\sup_{0\leq t\leq T} d_H^2\Big(\int_0^t F(s)dw_s,\int_0^t G(s)dw_s\Big)\Big]$$
$$\leq 4E\Big[d_H^2\Big(\int_0^T F(s)dw_s,\int_0^T G(s)dw_s\Big)\Big]. \tag{3.4}$$

proof. Let $(M_t)_{t\geq 0},(N_t)_{t\geq 0}$ be two set valued martingales. Using Lemma 2.6 in [4] or Theorem 4.3 in [11], it follows for $t\geq s$,

$$\begin{aligned}E[d_H(M_t,N_t)|\mathcal{A}_s]&\geq d_H(\mathcal{E}[M_t|\mathcal{A}_s](\cdot),\mathcal{E}[N_t|\mathcal{A}_s](\cdot))\\&=d_H(M_s,N_s).\end{aligned}$$

This implies that $(d_H(M_t,N_t))_{t\geq 0}$ is a real valued submartingale. Putting $M_t=\int_0^t F(s)dw_s$ and $N_t=\int_0^t G(s)dw_s$ and using the Doob maximal inequality we get the inequality(3.4). This completes the proof.

THEOREM 3.14. *Let $(F(t))_{t\geq 0},\ (G(t))_{t\geq 0}\in\mathcal{L}^2(\mathcal{K}_c(R))$. Then it holds*

$$E\left[d_H^2\left(\int_0^t F(s)dw_s,\int_0^t G(s)dw_s\right)\right]\leq E\left[\int_0^t d_H^2(F(s),G(s))ds\right].$$

proof. By Theorem 3.5, there exist countable sequences $\{(f^i(t))_{t\geq 0}:i=1,2,\cdots\}$ and $\{(g^j(t))_{t\geq 0}:j=1,2,\cdots\}$ in $\mathcal{S}_2(F(t))$ and $\mathcal{S}_2(G(t))$, respectively such that for any $t\geq 0$, $F(t,\omega)=cl\{f^i(t,\omega):i=1,2,\cdots\}, G(t,\omega)=cl\{g^j(t,\omega):j=1,2,\cdots\}$ and

$$I_t(F)(\omega)=\int_0^t F(s,\omega)dw_s(\omega)=cl\left\{\int_0^t f^i(s,\omega)dw_s(\omega):i=1,2,\cdots\right\}$$

and

$$I_t(G)(\omega)=\int_0^t G(s,\omega)dw_s(\omega)=cl\left\{\int_0^t g^j(s,\omega)dw_s(\omega):j=1,2,\cdots\right\}$$

a.a.$\omega\in\Omega$. Let

$$\begin{aligned}A_t=\Big\{\omega\in\Omega:\sup_i\inf_j&\Big|\int_0^t(f^i(s,\omega)-g^j(s,\omega))dw_s(\omega)\Big|^2\\&\geq\sup_j\inf_i\Big|\int_0^t(f^i(s,\omega)-g^j(s,\omega))dw_s(\omega)\Big|^2\Big\}.\end{aligned}$$

Using Theorem 2.2 in [5] and two basic properties $\max(u,v)=\frac{1}{2}(u+v)+\frac{1}{2}|u-v|$ and $|u-v|=u1_A+v1_{A^c}-u1_{A^c}-v1_A$ for random variables u,v and $A=\{\omega\in\Omega|u(\omega)-v(\omega)\geq 0\}$, we have

$$
\begin{aligned}
&E\Big[d_H^2\Big(\int_0^t F(s)dw_s, \int_0^t G(s)dw_s\Big)\Big] \\
&= \int_\Omega d_H^2\Big(\int_0^t F(s)dw_s, \int_0^t G(s)dw_s\Big)dP \\
&= \int_\Omega \max\Big\{\sup_{x\in I_t(F)(\omega)} \inf_{y\in I_t(G)(\omega)} |x-y|^2, \sup_{y\in I_t(G)(\omega)} \inf_{x\in I_t(F)(\omega)} |x-y|^2\Big\}dP \\
&= \int_\Omega 1_{A_t} \sup_{x\in I_t(F)(\omega)} \inf_{y\in I_t(G)(\omega)} |x-y|^2 dP \\
&\qquad\qquad + \int_\Omega 1_{A_t^c} \sup_{y\in I_t(G)(\omega)} \inf_{x\in I_t(F)(\omega)} |x-y|^2 dP \\
&= \int_\Omega 1_{A_t} \sup_i \inf_j \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\qquad\qquad + \int_\Omega 1_{A_t^c} \sup_j \inf_i \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&= \frac{1}{2}\int_\Omega \sup_i \inf_j \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\quad + \frac{1}{2}\int_\Omega \sup_j \inf_i \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\quad + \frac{1}{2}\int_\Omega 1_{A_t} \sup_i \inf_j \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\quad - \frac{1}{2}\int_\Omega 1_{A_t^c} \sup_i \inf_j \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\quad - \frac{1}{2}\int_\Omega 1_{A_t} \sup_j \inf_i \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\quad + \frac{1}{2}\int_\Omega 1_{A_t^c} \sup_j \inf_i \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&= \frac{1}{2}\int_\Omega \sup_i \inf_j \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\quad + \frac{1}{2}\int_\Omega \sup_j \inf_i \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\quad + \frac{1}{2}\Big|\int_\Omega \sup_i \inf_j \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP \\
&\qquad - \int_\Omega \sup_j \inf_i \Big|\int_0^t (f^i(s)-g^j(s))dw_s\Big|^2 dP\Big| \\
&= \frac{1}{2} \sup_{(f(t))_{t\geq 0}\in \mathcal{S}_2(F(t))} \inf_{(g(t))_{t\geq 0}\in \mathcal{S}_2(G(t))} \int_\Omega \Big|\int_0^t (f(s)-g(s))dw_s\Big|^2 dP
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sup_{(g(t))_{t\geq 0}\in\mathcal{S}_2(G(t))}\inf_{(f(t))_{t\geq 0}\in\mathcal{S}_2(F(t))}\int_\Omega\left|\int_0^t (f(s)-g(s))dw_s\right|^2 dP\\
&\quad+\frac{1}{2}\left|\sup_{(f(t))_{t\geq 0}\in\mathcal{S}_2(F(t))}\inf_{(g(t))_{t\geq 0}\in\mathcal{S}_2(G(t))}\int_\Omega\left|\int_0^t (f(s)-g(s))dw_s\right|^2 dP\right.\\
&\qquad\left.-\sup_{(g(t))_{t\geq 0}\in\mathcal{S}_2(G(t))}\inf_{(f(t))_{t\geq 0}\in\mathcal{S}_2(F(t))}\int_\Omega\left|\int_0^t (f(s)-g(s))dw_s\right|^2 dP\right|\\
=&\frac{1}{2}\sup_{(f(t))_{t\geq 0}\in\mathcal{S}_2(F(t))}\inf_{(g(t))_{t\geq 0}\in\mathcal{S}_2(G(t))}\int_\Omega\int_0^t |(f(s)-g(s))|^2 dsdP\\
&+\frac{1}{2}\sup_{(g(t))_{t\geq 0}\in\mathcal{S}_2(G(t))}\inf_{(f(t))_{t\geq 0}\in\mathcal{S}_2(F(t))}\int_\Omega\int_0^t |(f(s)-g(s))|^2 dsdP\\
&\quad+\frac{1}{2}\left|\sup_{(f(t))_{t\geq 0}\in\mathcal{S}_2(F(t))}\inf_{(g(t))_{t\geq 0}\in\mathcal{S}_2(G(t))}\int_\Omega\int_0^t |f(s)-g(s)|^2 dsdP\right.\\
&\qquad\left.-\sup_{(g(t))_{t\geq 0}\in\mathcal{S}_2(G(t))}\inf_{(f(t))_{t\geq 0}\in\mathcal{S}_2(F(t)}\int_\Omega\int_0^t |f(s)-g(s)|^2 dsdP\right|\\
=&\frac{1}{2}\int_\Omega\int_0^t \sup_{x\in F(s,\omega)}\inf_{y\in G(s,\omega)}|x-y|^2 dsdP\\
&+\frac{1}{2}\int_\Omega\int_0^t \sup_{y\in G(s,\omega)}\inf_{x\in F(s,\omega)}|x-y|^2 dsdP\\
&\quad+\frac{1}{2}\left|\int_\Omega\int_0^t\left\{\sup_{x\in F(s,\omega)}\inf_{y\in G(s,\omega)}|x-y|^2\right.\right.\\
&\qquad\left.\left.-\sup_{y\in G(s,\omega)}\inf_{x\in F(s,\omega)}|x-y|^2\right\}dsdP\right|\\
\leq&\frac{1}{2}\int_\Omega\int_0^t \sup_{x\in F(s,\omega)}\inf_{y\in G(s,\omega)}|x-y|^2 dsdP\\
&+\frac{1}{2}\int_\Omega\int_0^t \sup_{y\in G(s,\omega)}\inf_{x\in F(s,\omega)}|x-y|^2 dsdP\\
&\quad+\frac{1}{2}\int_\Omega\int_0^t\left|\sup_{x\in F(s,\omega)}\inf_{y\in G(s,\omega)}|x-y|^2-\sup_{y\in G(s,\omega)}\inf_{x\in F(s,\omega)}|x-y|^2\right|dsdP\\
=&\int_\Omega\int_0^t \max\left\{\sup_{x\in F(s,\omega)}\inf_{y\in G(s,\omega)}|x-y|^2,\ \sup_{y\in G(s,\omega)}\inf_{x\in F(s,\omega)}|x-y|^2\right\}dsdP\\
=&E\left[\int_0^t d_H^2(F(s),G(s))ds\right].
\end{aligned}
$$

The proof is complete.

Remark 3.15. Suppose that $(F(t))_{t\geq 0},(G(t))_{t\geq 0}\in\mathcal{L}^2(\mathcal{K}_c(R))$ satisfy Condition(C). By combining Theorem 3.13 and Theorem 3.14 we have the following inequality

$$E\left[\sup_{0\le t\le T} d_H^2\left(\int_0^t F(s)dw_s, \int_0^t G(s)dw_s\right)\right]$$
$$\le 4E\left[\int_0^T d_H^2(F(s),G(s))ds\right]. \tag{3.5}$$

If we assume that $G(t) = \{0\}$ for all $t \in [0,T]$ and so $\int_0^t G(s)dw_s = \{0\}$ in the equality (3.5), then we obtain

$$\left[\sup_{0\le t\le T} |||\int_0^t F(s)dw_s|||^2\right] \le 4E\left[\int_0^T |||F(s)|||^2 ds\right]. \tag{3.6}$$

Fritsch[3] proved that the inequality (3.6) as a necessary condition for the martingale property of $(\int_0^t F(s)dw_s)_{t\ge 0}$.

4. Set valued stochastic differential equations

Let R be the set of all real numbers and let $(\Omega, \mathcal{A}, P)$ be a complete probability space with a filtration $(\mathcal{A}_t)_{t\ge 0}$. In this paper we consider the following stochastic differential inclusion on R;

$$\begin{cases} dx(t) \in F(t,x(t))dt + G(t,x(t))dw_t \\ x(0) = x_0 \end{cases} \tag{4.1}$$

and set valued stochastic differential equation on 2^R;

$$\begin{cases} dX(t) = F(t,X(t))dt + G(t,X(t))dw_t, \\ X(0) = X_0, \end{cases} \tag{4.2}$$

where $F : [0,\infty] \times R \to \mathcal{L}^1(\mathcal{K}_c(R))$ and $G : [0,\infty] \times R \to \mathcal{L}^2(\mathcal{K}_c(R))$ satisfy

$$H(t,X(t)) = \bigcup_{(x(t))_{t\ge 0}\in \mathcal{S}_2(X(t))} H(t,x(t)).$$

First we prove the existence and uniqueness of the solution of set valued differential equation (4.2). The solution of (4.2) is defined as follows.

DEFINITION 4.1. By a solution of set stochastic differential equation (4.2), we mean a $\mathcal{K}(R)$ valued fuzzy process $(X(t))_{t\ge 0}$ defined on $(\Omega, \mathcal{A}, P)$ with a reference family $(\mathcal{A}_t)_{t\ge 0}$ such that

(i) there exists a 1-dimensional Brownian motion $(w_t)_{t\ge 0}$ with $w(0) = 0$ a.s.,

(ii) $(X(t))_{t\ge 0}$ is $\mathcal{A}_t$-adapted and continuous in t a.s. i.e., with probability one, $d_H(X(t+h), X(t)) \to 0$ as $h \to 0$,

(iii) $(X(t))_{t\ge 0}$ and $(w_t)_{t\ge 0}$ satisfy.

$$X(t) = X_0 + \int_0^t F(s,X(s))ds + \int_0^t G(s,X(s))dw_s \text{ a.s.}$$

THEOREM 4.2. *Assume that $F, G : [0,\infty) \times R \longrightarrow \mathcal{K}_c(R)$ satisfy the following conditions;*

(i) there exists a $K > 0$ such that

$$d_H^2(F(t,X),F(t,Y)) + d_H^2(G(t,X),G(t,Y)) \le K d_H^2(X,Y), \tag{4.3}$$

$$|||F(t,X)|||^2 + |||G(t,X)|||^2 \le K(1 + |||X|||^2) \tag{4.4}$$

for all $X, Y \in \mathcal{K}_c(R)$ and $t \in [0,\infty)$,

(ii) $(Y(t))_{t\ge0} \in \mathcal{L}^2(\mathcal{K}_c(R))$, there exists a process $(h(t))_{t\ge0} = (h(Y(t);t))_{t\ge0} \in L^2(R)$ such that it satisfies (3.3) in Condition (C) and

$$G(t, Y(t)) = h(t)M,$$

where $M = [-1, 1]$. Then the set valued differential equation (4.2) has a unique solution $(X(t))_{t\ge0}$.

Proof. Let $T > 0$ be any given. Define $X_0(t) = X_0$ a.s. for all $t \in [0,T]$. Then by (4.4), we have

$$E\left[\int_0^T |||G(t,x_0)|\,||^2 dt\right] \le TK\left(1 + E\left[|||X_0|||^2\right]\right) < \infty$$

and hence $(G(t,X_0))_{t\ge0} \in \mathcal{L}^2(\mathcal{K}_c(R))$. By the definition of the set valued integral, we can define

$$\int_0^t G(s,X_0)dw_s$$

and thus a continuous process

$$X_1(t) = X_0 + \int_0^t F(s,X_0)ds + \int_0^t G(s,X_0)dw_s$$

with $E[|||X_1(t)|||^2] < \infty$ for all $t \in [0,T]$. Now assume that continuous processes

$$X_i(t) = X_0 + \int_0^t F(s,X_{i-1}(s))ds + \int_0^t G(s,X_{i-1}(s))dw_s,\ i = 2,3\cdots,n$$

are defined and these satisfy $\sup_{0\le t\le T} E[|||[X_i(t)]^0|||^2] < \infty$. Then by (4.4),

$$\begin{aligned} &E\left[\int_0^t |||G(s,X_n(s))|||^2\right] \\ &\le TK\left(1 + \sup_{0\le t\le T} |||X_n(t)|||^2\right) \\ &< \infty \end{aligned}$$

and hence $(G(t, X_n(t)))_{t\geq 0} \in \mathcal{L}^2(\mathcal{K}_c(R))$.

This shows that

$$\int_0^t G(s, X_n(s))dw_s$$

and thus, a continuous process

$$X_{n+1} = X_0 + \int_0^t F(s, X_n(s))ds + \int_0^t G(s, X_n(s))dw_s$$

can be defined. By mathematical induction, we obtain a sequence $\{(X_n(t))_{t\geq 0}\}, n = 1, 2, \cdots$ of stochastic processes in $\mathcal{L}^2(\mathcal{K}_c(R))$. By Theorem 3.13, Theorem 3.14, and the condition (4.3), it holds

$$\begin{aligned}
&E\left[\sup_{0\leq t\leq T} d_H^2\left(X_n(t), X_{n+1}(t)\right)\right] \\
&\leq 2E\left[\sup_{0\leq t\leq T} d_H^2\left(\int_0^t F(s, X_{n-1}(s))ds, \int_0^t F(s, X_n(s))ds\right)\right] \\
&\quad + 2E\left[\sup_{0\leq t\leq T} d_H^2\left(\int_0^t G(s, X_{n-1}(s))dw_s, \int_0^t G(s, X_n(s))dw_s\right)\right] \\
&\leq 2TE\left[\int_0^T d_H^2\left(F(s, X_{n-1}(s)), F(s, X_n(s))\right)ds\right] \\
&\quad + 8E\left[\int_0^T d_H^2\left(G(s, X_{n-1}(s)), G(s, X_n(s))\right)ds\right] \\
&\leq (2T+8)K\int_0^T E\left[\sup_{0\leq s\leq t} d_H^2(X_{n-1}(s), X_n(s))\right]dt \\
&\leq \{(2T+8)K\}^n \int_0^T\int_0^{t_1}\cdots\int_0^{t_{n-1}} E\left[\sup_{0\leq s\leq t_n} d_H^2(X_0, X_1(s))\right]\, dt_n\cdots dt_2dt_1.
\end{aligned}$$

by the same argument as above and (4.4), we have

$$E\left[\sup_{0\leq t\leq t_n} d_H^2(X_0, X_1(s))\right] \leq (2T+8)KT\left(1 + |||x_0|||^2\right)$$

Thus we have

$$E\left[\sup_{0\leq t\leq T} d_H^2\left(X_n(t), X_{n+1}(t)\right)\right] \leq \frac{\{(2T+8)KT\}^{n+1}}{(n+1)!}\left(1 + |||x_0|||^2\right)$$

By Chebyshev's inequality, we obtain

$$\begin{aligned}
&P\left(\sup_{0\leq t\leq T} d_H\left(X_n(t), X_{n+1}(t)\right) \geq \frac{1}{2^{n+1}}\right) \\
&\leq 4^{n+1}E\left[\sup_{0\leq t\leq T} d_H^2\left(X_n(t), X_{n+1}(t)\right)\right] \\
&\leq C(T)\frac{\{4(2T+8)KT\}^{n+1}}{(n+1)!}
\end{aligned}$$

where $C(T) > 0$ is a constant depending only on X_0 and T. By the Borel-Cantelli lemma, we see that $X_n(t)$ converges uniformly on $[0, T]$ with probability one. Since T was arbitrary, $\lim_{n\to\infty} X_n(t) = X(t)$ determines a continuous process which is clearly a solution of (4.2).

Now to prove the uniqueness of the solution, let $(X(t))_{t\geq 0}$ and $(X'(t))_{t\geq 0}$ be any two solutions of (4.1). Then by the similar calculations as above, we have

$$E\left[d_H^2(X(t), X'(t))\right] \leq 2K(1+T)\int_0^t E\left[d_H^2(X(s), X'(s))\right] ds$$

for all $t \in [0, T]$. By Gronwall's lemma, we obtain

$$E[d_H^2(X(t), X'(t))] = 0$$

for all $t \in [0, T]$. Hence, letting $T \to \infty$, we have $X(t) = X'(t)$ a.s. for all $t \geq 0$. Since $(X(t))_{t\geq 0}$ and $(X'(t))_{t\geq 0}$ are d_H-continuous in t a.s., we can conclude that $X(t) = X'(t)$ for all $t \geq 0$ a.s.. The proof is complete.

The following problem is meaningful as a viability problem but we can not solve yet.

Problem 4.3. Suppose that there exists a solution $(X(t))_{t\geq 0}$ of (4.2). Does there exist a solution $(x(t))_{t\geq 0}$ of (4.1) such that $x(t) \in S^2_{X(t)}$?

5. Fuzzy stochastic differential equations

Let $\mathcal{F}(R)$ denote the family of all fuzzy sets $u : R \to [0, 1]$ such that the level set(or α-cut) $[u]^\alpha = \{r \in R : u(r) \geq \alpha\} \in \mathcal{K}(\mathcal{R})$, for $0 < \alpha \leq 1$, and $[u]^0 = \cup_{\alpha\in(0,1]}[u]^\alpha$ is bounded. For all $0 \leq \alpha \leq \beta \leq 1$

$$[u]^\beta \subset [u]^\alpha \subset [u]^0.$$

For two fuzzy sets $u, v \in \mathcal{F}(R)$, we denote $u \leq v$ if and only if $[u]^\alpha \subset [v]^\alpha$ for every $\alpha \in [0, 1]$.

Let $\mathcal{F}_c(R)$ denote the family of all fuzzy sets in $\mathcal{F}(R)$ with their level sets are contained to $\mathcal{K}_c(R)$. Define a metric D on $\mathcal{F}(R)$ by

$$D(u, v) = \sup_{\alpha\in[0,1]} d_H([u]^\alpha, [v]^\alpha).$$

For $u_i, i = 1, 2, 3, 4 \in \mathcal{F}(R)$, by (2.1) and (2.2), we obtain

$$D(u_1 + u_3, u_2 + u_3) = D(u_1, u_2)$$

and

$$D(u_1 + u_3, u_2 + u_4) \leq D(u_1, u_2) + D(u_3 + u_4).$$

A fuzzy random variable x is called L^p-*bounded* if there exists a function $h \in L^p(\Omega, \mathcal{A}; R)$ such that $|||[x(\omega)]^0||| \leq h(\omega)$ for a.a.$\omega \in \Omega$. Let $L^p(\Omega; \mathcal{F}(R))$ be the set of all L^p-bounded fuzzy random variables and $L^p(\Omega; \mathcal{F}_c(R))$ be the set of all L^p-bounded fuzzy random variables whose level sets belong to $\mathcal{K}_c(R)$. For $x_1, x_2 \in L^p(\Omega; \mathcal{F}(R))$, they are considered to be identical if for all $\alpha \in (0,1]$,

$$[x_1]^\alpha = [x_2]^\alpha \text{ a.s.}$$

The expectation of a fuzzy random variable x, denoted by $E[x]$ also , is a fuzzy set such that, for $\alpha \in (0,1]$,

$$[E[x]]^\alpha = \mathcal{E}[[x]^\alpha] = \{E[g] : g \in S^1_{[x]^\alpha}\}.$$

Stojaković[17] showed that for any $x \in L^1(\Omega; \mathcal{F}(R))$ and σ-field $\mathcal{B} \subset \mathcal{A}$, there exists a unique fuzzy random variable $\Phi \in L^1(\Omega, \mathcal{B}; \mathcal{F}(R))$ such that for $\alpha \in (0,1]$,

$$[\Phi]^\alpha = \mathcal{E}[[x]^\alpha|\mathcal{B}] \quad \text{a.s.}$$

The fuzzy random variable Φ is called the *fuzzy conditional expectation* of X given $\mathcal{B}$ and denote it by $E[x|\mathcal{B}]$.

We call $(y_t)_{t\geq 0}$ a *fuzzy stochastic process* if each level set $[y]^\alpha_t$ is a nonempty, closed and convex set valued random variable and each y_t is a fuzzy random variable. A fuzzy stochastic process $(y_t)_{t\geq 0}$ is called $\mathcal{A}_t$*-adapted* if for each $t \geq 0$, y_t is $\mathcal{A}_t$-measurable, and *measurable* if y is $\mathcal{B}_{[0,\infty)} \otimes \mathcal{A}$-measurable. A fuzzy stochastic process $(y_t)_{t\geq 0}$ is called $\mathcal{L}^2$*-bounded* if there exists a process $(h(t))_{t\geq 0} \in L^2(R)$ such that $|||[y_t(\omega)]^0||| \leq h(t,\omega)$ for a.a(t,ω). Let $\mathcal{L}^2(\mathcal{F}_c(R))$ be the set of $\mathcal{A}_t$-adapted measurable $\mathcal{L}^2$-bounded $\mathcal{F}_c(R)$ valued fuzzy processes.

A fuzzy stochastic process $(y_t)_{t\geq 0}$ is called a *fuzzy martingale* (respectively, *submartingale, supermartingale)* with respect to $\mathcal{A}_t$ if y_t is L^1-bounded and $\mathcal{A}_t$-measurable, and for $t > s$ $E[y_t|\mathcal{A}_s] = y_s$ (resp. $\geq, \leq$) a.s.. By the definition of fuzzy conditional expectation we can see that $(y_t)_{t\geq 0}$ is a fuzzy martingale if and only if $([y]^\alpha_t)_{t\geq 0}$ is a set valued martingale for any $\alpha \in [0,1]$.

By the same method as Theorem 4.6 in [9] we have the following result.

Theorem 5.1. *Let $(y_t)_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(R))$ and $(w_t)_{t\geq 0}$ be a 1-dimensional Brownian motion. Then for any $t > 0$, there exists a unique fuzzy random variable $z_t \in L^2(\Omega, \mathcal{A}_t; \mathcal{F}_c(R))$ such that for all $\alpha \in (0,1]$, $[z_t]^\alpha(\omega) = (\int_0^t [y_s]^\alpha dw_s)(\omega)$ a.a. $\omega \in \Omega$. Moreover, $(z_t)_{t\geq 0}$ is an $\mathcal{A}_t$-adapted measurable fuzzy stochastic process.*

We call z_t a *stochastic integral* of y_t with respect to $(w_t)_{t\geq 0}$ and denote it by $\int_0^t y_s dw_s$.

Theorem 5.2. *Let $(y_t)_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(R))$ and assume that $([y_t]^\alpha)_{t\geq 0}$, $\alpha \in [0,1]$, satisfy Condition (C). Then $(\int_0^t y_s dw_s)_{t\geq 0}$ is a fuzzy martingale.*

Proof. By Theorem 3.10 and the definition of the fuzzy conditional expectation, for all

$\alpha \in [0,1]$ and $t \geq s \geq 0$, we have

$$\begin{aligned}
&\left[E\left[\int_0^t y_u dw_u \,\middle|\, \mathcal{A}_s\right]\right]^\alpha \\
&= \mathcal{E}\left[\left[\int_0^t y_u dw_u\right]^\alpha \,\middle|\, \mathcal{A}_s\right] \\
&= \mathcal{E}\left[\int_0^t [y_u]^\alpha \, dw_u \,\middle|\, \mathcal{A}_s\right] \\
&= \int_0^s [y_u]^\alpha \, dw_u \\
&= \left[\int_0^s y_u dw_u\right]^\alpha \quad \text{a.s.}
\end{aligned}$$

This means

$$E\left[\int_0^t y_u dw_u \,\middle|\, \mathcal{A}_s\right] = \int_0^s y_u dw_u \text{ a.s.}$$

and hence $(\int_0^t y_u dw_u)_{t\geq 0}$ is a fuzzy martingale.

Theorem 5.3. *Let* $(x_t)_{t\geq 0}, (y_t)_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(R))$ *and assume that* $([x_t]^\alpha)_{t\geq 0}$, $([y_t]^\alpha)_{t\geq 0}$, $\alpha \in [0,1]$ *satisfy Condition (C). Then it holds*

$$E\left[\sup_{0\leq t\leq T} D^2\left(\int_0^t x_u dw_u, \int_0^t y_u dw_u\right)\right] \leq 4E\left[D^2\left(\int_0^T x_u dw_u, \int_0^T y_u dw_u\right)\right].$$

Proof. Since $(\int_0^t [y_u^i]^\alpha dw_u)_{t\geq 0}$ $i = 1,2,$ $\alpha \in [0,1]$ are set valued martingales, by Theorem 2.6 in [4], $d_H(\int_0^t [y_u^1]^\alpha dw_u,\ \int_0^t [y_u^2]^\alpha dw_u)$ is a real valued submartingale. Hence we have for all $t \geq s \geq 0$

$$\begin{aligned}
&E\left[D\left(\int_0^t x_u dw_u, \int_0^t y_u dw_u\right) \,\middle|\, \mathcal{A}_s\right] \\
&= E\left[\sup_{\alpha\in[0,1]} d_H\left(\int_0^t [x_u]^\alpha dw_u, \int_0^t [y_u]^\alpha dw_u\right) \,\middle|\, \mathcal{A}_s\right] \\
&\geq \sup_{\alpha\in[0,1]} E\left[d_H\left(\int_0^t [x_u]^\alpha dw_u, \int_0^t [y_u]^\alpha dw_u\right) \,\middle|\, \mathcal{A}_s\right] \\
&\geq \sup_{\alpha\in[0,1]} d_H\left(\int_0^s [x_u]^\alpha dw_u, \int_0^s [y_u]^\alpha dw_u\right) \\
&= D\left(\int_0^s x_u dw_u, \int_0^s y_u dw_u\right)
\end{aligned}$$

and so $\left(D\left(\int_0^t x_u dw_u, \int_0^t y_u dw_u\right)\right)_{t\geq 0}$ is a real valued submartingale. Using the Doob maximal inequality (Theorem 1.2.3 in [18]), the proof is complete.

Theorem 5.4. *Let* $(x_t)_{t\geq 0}, (y_t)_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(R))$. *Then, we have*

$$E\left[D^2\left(\int_0^t x_s dw_s, \int_0^t y_s dw_s\right)\right] \leq E\left[\int_0^t D^2(x_s, y_s) ds\right].$$

Proof. From Theorem 3.13, we have for all $t \geq 0$

$$\begin{aligned}
&E\left[D^2\left(\int_0^t x_s dw_s, \int_0^t y_s dw_s\right)\right] \\
&= E\left[\sup_{\alpha\in[0,1]} d_H^2\left(\int_0^t [x_s]^\alpha dw_s, \int_0^t [y_s]^\alpha dw_s\right)\right] \\
&\leq E\left[\sup_{\alpha\in[0,1]} \int_0^t d_H^2\left([x_s]^\alpha, [y_s]^\alpha\right) ds\right] \\
&\leq E\left[\int_0^t \sup_{\alpha\in[0,1]} d_H^2\left([x_s]^\alpha, [y_s]^\alpha\right) ds\right] \\
&= E\left[\int_0^t D^2\left(x_s, y_s\right) ds\right].
\end{aligned}$$

The proof is complete.

Now we consider the following fuzzy stochastic differential equation;

$$\begin{cases} dz(t) = f(t, z(t))dt + g(t, z(t))dw_t \\ z(0) = z_0 \end{cases} \tag{5.1}$$

where $f : [0, \infty) \times \mathcal{F}_c(\mathcal{R}) \to \mathcal{L}^1(\mathcal{F}_c(\mathcal{R}))$ and $g : [0, \infty) \times \mathcal{F}_c(\mathcal{R}) \to \mathcal{L}^2(\mathcal{L}^2(\mathcal{R}))$.

Definition 5.5. By a solution of fuzzy stochastic differential equation (5.1), we mean a $\mathcal{F}_c(R)$ valued fuzzy process $(z(t))_{t\geq 0}$ defined on $(\Omega, \mathcal{A}, P)$ with a reference family $(\mathcal{A}_t)_{t\geq 0}$ such that

(i) there exists a 1-dimensional Brownian motion $(w_t)_{t\geq 0}$ with $w(0) = 0$ a.s.,

(ii) $(z(t))_{t\geq 0}$ is $\mathcal{A}_t$-adapted and continuous in t a.s. i.e., with probability one, $D(z(t + h), z(t)) \to 0$ as $h \to 0$,

(iii) $(z(t))_{t\geq 0}$ and $(w_t)_{t\geq 0}$ satisfy

$$z(t) = z_0 + \int_0^t f(s, z(s))ds + \int_0^t g(s, z(s))dw_s \text{ a.s.} \tag{5.2}$$

By the same calculation as the proof of Theorem 4.2, we can prove the following theorem(see [10] for detail).

Theorem 5.6. *Assume that $f, g : [0,\infty) \times \mathcal{F}_c(R) \longrightarrow \mathcal{F}_c(R)$ satisfy the following conditions;*

(i) there exists a $K > 0$ such that

$$D^2(f(t,x), f(t,y)) + D^2(g(t,x), g(t,y)) \leq KD^2(x,y),$$

$$|||[f(t,x)]^0|||^2 + |||[g(t,x)]^0|||^2 \leq K(1 + |||[x]^0|||^2)$$

for all $x, y \in \mathcal{F}_c(R)$ and $t \in [0,\infty)$,

(ii) for any $\alpha \in [0,1]$ and $(y(t))_{t\geq 0} \in \mathcal{L}^2(\mathcal{F}_c(R))$, there exists a process $(h(t))_{t\geq 0} = (h(\alpha, y(t); t))_{t\geq 0} \in L^2(R)$ such that it satisfies (3.3) in Condition (C) and

$$[g(t, y(t))]^\alpha = h(t)M,$$

where $M = [-1,1]$. Then the fuzzy stochastic differential equation (5.1) has a unique solution $(x(t))_{t\geq 0}$.

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Probability theory in spaces of fuzzy sets

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1 Introduction

Statistical data are products of measuring predetermined items with respect to objects which are sample points from fixed sample spaces. According to economic items each process of measurement is confronted with several types of vagueness: Stochastical vagueness by selecting objects from the sample space, epistemic (perceptive) vagueness since sample spaces may be specified unproperly, or incorrect results like false responses have been reported. Last but not least, theoretical economic concepts have to be transformed into items to enable the process of measurement, which raises the question of adequacy due to the physical vagueness of many theoretical concepts.

As a consequence data inherit errors in measurement corresponding to the respective types of vagueness: Random errors (stochastical vagueness), sampling errors (stochastical and epistemical vagueness) and adequacy errors (physical vagueness). They can be defined within the terminology of Mathematical Statistics reformulating the measurement of items as parameter estimation. However, within the framework of Mathematical Statistics only the random errors are considered systematically.

In order to integrate different types of vagueness it has been proposed to represent outcomes of measurements by fuzzy subsets rather than vectors of real numbers. A desirable foundation of Mathematical Statistics with fuzzy observations needs then a suitable linkage of probability with fuzzy set theory.

Technically we shall investigate mathematical structures on fuzzy sample spaces $\mathfrak{X}$ which are contained in the space of fuzzy subsets of $\mathbb{R}^k$. We shall point out compatible pairs $(d, \mathcal{F})$ of a metric d to describe the random errors and a $\sigma-$algebra $\mathcal{F}$ to extend the methods of Mathematical Statistics. The $\sigma-$algebra $\mathcal{F}$ should be generated by the metric d. So only Borel-$\sigma-$algebras on already known metrizable topological spaces $(\mathfrak{X}, \tau)$ are considered and compared. The discussion leads to some unified concept of fuzzy-valued random variables called fuzzy random variables. A Strong Law of Large Numbers, a Central Limit theorem and a Glivenko-Cantelli theorem for fuzzy random variables will be formulated simultaneously with respect to some emphasized metrics. All of them are compatible with the $\sigma-$algebra which underlies the notion of fuzzy random variables. This gives reasonable legitimation to use these metrics and the induced concept of fuzzy random variables.

2 $\mathcal{L}_p-$spaces of fuzzy sets

In order to keep the line of reasoning comprehensive we shall restrict ourselves to $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$, the space of normal fuzzy subsets of $\mathbb{R}^k$ with convex, compact positive $\alpha-$cuts. Since every fuzzy set from $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ is uniquely determined by its support function, we can build in an obvious way $\mathcal{L}_p-$subspaces of $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ with respect to the product measure $\lambda^1 \otimes \lambda^{S^{k-1}}$ of the Lebesgue-Borel measure λ^1 on $[0,1]$ and the unit Lebesgue-Borel measure $\lambda^{S^{k-1}}$ on the euclidean unit sphere S^{k-1} in $\mathbb{R}^k$. The space $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ $(p \in [1,\infty[)$ consists of all $\widetilde{A} \in \mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ with support functions being $\lambda^1 \otimes \lambda^{S^{k-1}}-$integrable of order p, whereas $\mathcal{F}^{no}_{coc\infty}(\mathbb{R}^k)$ contains the fuzzy sets from $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ having support functions which are essentially bounded with respect to $\lambda^1 \otimes \lambda^{S^{k-1}}$. It can be shown that the fuzzy sets from $\mathcal{F}^{no}_{coc\infty}(\mathbb{R}^k)$ are exactly those fuzzy sets from $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ with bounded support. Furthermore it turns out that $\mathcal{F}^{no}_{coc\infty}(\mathbb{R}^k)$ is a dense subset of every $\mathcal{L}_p-$space $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ with respect to the respective L_p-metric ρ_p.

3 L_p-Metrics

Identifying each $\widetilde{A} \in \mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ with the $\lambda^1 \otimes \lambda^{S^{k-1}}-$equivalence class of its support function, the $\mathcal{L}_p-$space $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ can be embedded into the L_p-space $L_p([0,1] \times S^{k-1})$ generated by $\lambda^1 \otimes \lambda^{S^{k-1}}$. Then the L_p-norm on $L_p([0,1] \times S^{k-1})$ induces the so called L_p-**metric** ρ_p on $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$. Some fundamental properties of the L_p-metrics will be collected. Especially, they are complete, and they are also separable except ρ_∞.

4 Other metrics on the L_p-subspaces of $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$

Extending, dependent on $p \in [1,\infty]$, the well known $L_{p,\infty}-$metrics from $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ to $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ respectively, we shall introduce another class of metrics. They are all complete and induce, dependent on p, the same topology on $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ as the respective L_p-metric ρ_p.

Also the so called **Skhorod metric** on $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ will be mentioned which induces a separable and completely metrizable topology.

5 $\sigma-$algebras on spaces of fuzzy sets

We shall investigate $\sigma-$algebras on the sample spaces $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ and the $\mathcal{L}_p-$subspaces $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ $(p \in [1,\infty])$. As a key concept a $\sigma-$algebra on $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)$ will be suggested which covers the different $\sigma-$algebras on the $\mathcal{L}_p-$spaces $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)$ generated by the introduced metrics. So we can define a $\mathcal{F}^{no}_{coc}(\mathbb{R}^k)-$valued random variable, called **fuzzy random variable**, in a way which unifies already known alternative concepts. Particularly, the L_p-metrics are compatible with the notion of $\mathcal{F}^{no}_{cocp}(\mathbb{R}^k)-$valued fuzzy random variable for $p \in [1,\infty[$.

6 Aumann expected value of fuzzy random variables

Since $F^{no}_{cocp}(\mathbb{R}^k)-$valued fuzzy random variables can be identified as $L_p([0,1]\times S^{k-1})-$valued random variables for $p \in [1,\infty[$, limit theorems may be obtained by limit theorems for Banach-space-valued random variables. For preparation this section provides conditions of integrability which ensures that the Aumann expected value of some $F^{no}_{cocp}(\mathbb{R}^k)-$valued fuzzy random variable can be identified with the Pettis integral or even the Bochner integral.

7 Limit theorems for fuzzy random variables

Since there exists a general Glivenko/Cantelli-Theorem for separable metric random elements, only Strong Laws of Large Numbers (SLLN) and Central Limit Theorems (CLT) for fuzzy random variables are of interest. Several versions of a SLLN with respect to some of the introduced metrics are already known. Also different versions of CLT have been formulated. What is missing is a SLLN and a CLT with respect to the same separable metric, which ensures a Glivenko/Cantelli-Theorem too.

Applying limit theorems for random elements in separable Banach spaces of type 2, we shall formulate a SLLN and a CLT simultaneously with respect to each L_p-metric ρ_p for $p \in [2,\infty[$.

8 Concluding Remarks

Summarizing the results, for $p \in [2,\infty[$ $F^{no}_{cocp}(\mathbb{R}^k)-$valued fuzzy random variables together with the L_p-metric ρ_p fulfill all the required properties such that probability theory in spaces of fuzzy subsets of $\mathbb{R}^k$ may be regarded as well-founded. Since the limit theorems need the weakest assumptions in the case of $p = 2$ we would like to recommend the usage of $F^{no}_{coc2}(\mathbb{R}^k)-$valued fuzzy random variables and the L_2-metric ρ_2.

For practicable statistical work on basis of fuzzy observations a further investigation of distributions of fuzzy random variable is desirable. It might be useful to find standard classes and to develop general methods of Monte-Carlo simulation.

Convergence theorems for fuzzy valued random variables in the extended Hausdorff metric H_∞

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Embedding method is a typical method to discuss limit theorems of sequences of set valued and fuzzy valued random variables. For the set valued case, for examples, Artstein and Vitale above mentioned used an embedding method to prove SLLN Hiai and Umegaki used the embedding method to obtain martingale convergence theorems in 1975. For the fuzzy valued case, Klement, Puri and Ralescu obtained embedding theorems to prove SLLN in the sense of Extended Hausdorff Metric H_1 and CLT with respect to Extended Hausdorff Metric H_∞ with Lipschitz condition in 1986. Puri and Ralescu also obtained a martingales convergence theorem in H_∞ also with Lipschitz condition in 1991. Diamond and Kloden pointed out that all space of fuzzy sets with compact supports can be embedded into another Banach space in 1989 but they did not discuss isometric property with respect to H_∞, which is not easy to use for the proof of convergence theorems.

In this paper, we mainly give a new embedding method for fuzzy valued random variables with same expectation and use it to prove a SLLN for fuzzy valued martingales in the sense of H_∞. Then we prove the CLT by using this new embedding theorem and the result on empirical process of Van der Vaart and Wellner. We would like to point that using the result on empirical process to prove CLT is Proske's idea but we had some troubles on some inequalities of his proof. Here we rewritten the proof about CLT for fuzzy valued random variables. Throughout this paper we assume that fuzzy sets are not necessary to satisfy the Lipschitz condition.

On some analytical properties of aggregation operators

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Binary non–decreasing operators on $[a,b] \subset \mathbb{R}$ with idempotent boundary points a, b will be simply called *(binary) aggregation operators* [2]. Without loss of generality we will suppose $[a,b] = [0,1]$. During the study of sensitivity of aggregation operators [3] and some construction methods for aggregation operators [5], compare also [8], the following two analytical properties played a crucial role:

(i) *Lipschitz property*

$$|\mathbf{A}(x_1,y_1) - \mathbf{A}(x_2,y_2)| \le |x_1 - x_2| + |y_1 - y_2|, \quad \forall x_1,x_2,y_1,y_2 \in [0,1],$$

(ii) *Kernel property*

$$|\mathbf{A}(x_1,y_1) - \mathbf{A}(x_2,y_2)| \le \max\left(|x_1 - x_2|, |y_1 - y_2|\right), \quad \forall x_1,x_2,y_1,y_2 \in [0,1].$$

Evidently, both these properties are a kind of continuity, and

kernel property $\Rightarrow$ 1–Lipschitz property $\Rightarrow$ continuity

but

kernel property $\not\Leftarrow$ 1–Lipschitz property $\not\Leftarrow$ continuity

Denote by $\mathcal{C}$, $\mathcal{L}$ and $\mathcal{K}$ the classes of all continuous, 1–Lipschitz and kernel aggregation operators, respectively. All three classes are convex and closed under duality

$$\mathbf{A}^d(x,y) = 1 - \mathbf{A}(1-x, 1-y).$$

Moreover, the classes $\mathcal{C}$ and $\mathcal{K}$ are closed under composition $\mathbf{D} = \mathbf{A}(\mathbf{B},\mathbf{C})$. The class $\mathcal{L}$ does not possess this property. However, for any $\mathbf{A} \in \mathcal{K}$ and $\mathbf{B}$, $\mathbf{C} \in \mathcal{L}$ also $\mathbf{D} = \mathbf{A}(\mathbf{B},\mathbf{C}) \in \mathcal{L}$.

We list some other interesting properties of mentioned classes of aggregation operators:

(1) (i) $\mathbf{T}_L \leq \mathbf{A} \leq \mathbf{S}_L$ for all $\mathbf{A} \in \mathcal{L}$ ($\mathbf{T}_L, \mathbf{S}_L$ - the **L**ukasiewicz *t*–norm and *t*–conorm, respectively)

(ii) $\mathbf{T}_M \leq \mathbf{A} \leq \mathbf{S}_M$ for all $\mathbf{A} \in \mathcal{K}$ ($\mathbf{T}_M = \min$, $\mathbf{S}_M = \max$)

(2) (i) 0 (1) is the zero element of $\mathbb{A} \in \mathcal{L}$ if and only if 1 (0) is the neutral element of $\mathbf{A}$

(ii) 0 (1) is the neutral element of $\mathbf{A} \in \mathcal{K}$ if and only if $\mathbf{A} = \mathbf{T}_M$ ($\mathbf{A} = \mathbf{S}_M$)

(3) Any aggregation operator $\mathbf{A} \in \mathcal{K}$ is idempotent, that is, $\mathbf{A}(x,x) = x$ for all $x \in [0,1]$.

The class $\mathcal{K}$ contains all Choquet and Sugeno integral based aggregation operators, including all weighted means and OWA operators. The class $\mathcal{L} \supset \mathcal{K}$ contains also all 2–copulas (dual copulas) [9].

A simple characterization of 1–Lipschitz aggregation operators is the following one:

An aggregation operator $\mathbf{A} \in \mathcal{L}$ *if and only if the function* $\tilde{\mathbf{A}} : [0,1]^2 \to [0,1]$ *satisfying for all* $(x,y) \in [0,1]^2$ *the equation*

$$\mathbf{A}(x,y) + \tilde{\mathbf{A}}(x,y) = x + y$$

is an aggregation operator.

Note that the previous equation is an analogue of the Frank functional equation [4] in the framework of aggregation operators. By this characterization several other problems can be solved, for instance, due to the non–continuity of uninorms, it is immediate that no uninorm can be solution to this equation, compare [1].

Evidently, if $\mathbf{A} \in \mathcal{L}$ then also $\tilde{\mathbf{A}} \in \mathcal{L}$. For all $\mathbf{A} \in \mathcal{K}$ we have $\tilde{\mathbf{A}} \in \mathcal{L}$, $\tilde{\mathbf{A}}$ is idempotent. However, $\tilde{\mathbf{A}} \in \mathcal{K}$ if and only if $\mathbf{A}$ (and hence also $\tilde{\mathbf{A}}$) is a shift invariant aggregation operator.

A crucial role in the structure of kernel aggregation operators plays the boundary of the unit square, $\mathbb{B} = [0,1]^2 \setminus]0,1[^2$. Some of them can be uniquely determined >from their values on the boundary $\mathbb{B}$, compare [7]. In general, if $\mathbf{A} \in \mathcal{K}$ then for all $x \in [0,1]$

$$\mathbf{A}(0,x) \leq x \leq \mathbf{A}(x,1), \qquad \mathbf{A}(0,x) + 1 - x \geq \mathbf{A}(1-x,1),$$

$$\mathbf{A}(x,0) \leq x \leq \mathbf{A}(1,x), \qquad \mathbf{A}(x,0) + 1 - x \geq \mathbf{A}(1,1-x).$$

Vice versa, if the values of a kernel aggregation operator $\mathbf{A}$ are known on the boundary $\mathbb{B}$, then for all $x \in [0,1]$ and $\varepsilon \in [0,1-x]$ we have

$$\mathbf{A}_*(x,x+\varepsilon) \leq \mathbf{A}(x,x+\varepsilon) \leq \mathbf{A}^*(x,x+\varepsilon),$$

where

$$\begin{aligned} \mathbf{A}_*(x,x+\varepsilon) &= \max\left(\mathbf{A}(0,x+\varepsilon), \mathbf{A}(1-\varepsilon,1) - (1-x-\varepsilon)\right), \\ \mathbf{A}^*(x,x+\varepsilon) &= \min\left(\mathbf{A}(x,1), \mathbf{A}(0,\varepsilon) + x\right) \end{aligned}$$

are kernel aggregation operators [6]. Analogous inequalities can be written for $\mathbf{A}_*(x+\varepsilon,x)$ and $\mathbf{A}^*(x+\varepsilon,x)$. The operators $\mathbf{A}_*$ and $\mathbf{A}^*$ are the weakest and the strongest kernel aggregation operators respectively, coinciding with $\mathbf{A}$ on the boundary $\mathbb{B}$.

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Fixed point theory in probabilistic metric spaces

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1 Introduction

We present some recent results on t-norms which are closely related to some generalizations of the fixed point theorems in probabilistic metric spaces, obtained in [3]. We investigate closer the countable extension of t-norms and we introduce a new notion: the geometrically convergent (g-convergent) t-norm, which is closely related to the fixed point property. We prove that t-norms of H-type and some subclasses of Dombi, Acżel-Alsina, Sugeno-Weber families of t-norms are geometrically convergent. We prove also some practical criterion for the geometrically convergent t-norms.

A new approach to the fixed point theory in probabilistic metric spaces is given in Tardiff's paper [13], where some additional growth conditions for the mapping $\mathcal{F} : S \times S \to \mathcal{D}^+$ are assumed, and $T \geq T_{\mathbf{L}}$, where $T_{\mathbf{L}}(x,y) = \max(x+y-1,0)$. V. Radu [10] introduced a stronger growth condition for $\mathcal{F}$ than in Tardiff's paper (under the condition $T \geq T_{\mathbf{L}}$), which enables him to define a metric. The metric approach also allows an estimation of the convergence with respect to the solution.

We present a fixed point theorem for a probabilistic q-contraction $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfies Radu's condition, and T is a g-convergent t-norm (not necessarily $T \geq T_{\mathbf{L}}$). We prove in [3] a fixed point theorem for mappings $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfy a weaker condition than in [10], and T belongs to some subclasses of Dombi, Acżel-Alsina, Sugeno-Weber families of t-norms. We present a fixed point theorem for a generalized probabilistic C-contraction for multivalued mapping with an application in fuzzy metric space.

Notions and notations can be found in [3, 6, 9, 11].

2 Probabilistic B-contraction principles for single-valued mappings

A triangular norm (t-norm for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1]^2 \to [0,1]$ such that it is commutative, associative, monotone and $T(x,1) = x$. t-conorm $\mathbf{S}$ is defined by $\mathbf{S}(x,y) = 1 - T(1-x, 1-y)$.

Let Δ^+ be the set of all distribution functions F such that $F(0) = 0$ $(F$ is a nondecreasing, left continuous mapping from $[0,\infty]$ into $[0,1]$ with $F(\infty) = 1)$.

Definition 1. (a) A probabilistic metric space (in the sense of Šerstnev) is a triple $(S,\mathcal{F},\tau)$ where S is a nonempty set, $\mathcal{F}: S\times S\to\Delta^+$ is given by $(p,q)\mapsto F_{p,q}$, τ is a triangle function, such that the following conditions are satisfied for all p,q,r in S:

(i) $F_{p,p}=H_0$; (ii) $F_{p,q}\neq H_0$ for $p\neq q$; (iii) $F_{p,q}=F_{q,p}$; (iv) $F_{p,r}\geq\tau(F_{p,q},F_{q,r})$.

(b) $(S,\mathcal{F},\tau)$ is proper if $\tau(H_a,H_b)\geq H_{a+b}$ $(a,b\in[0,\infty))$, where for $a\in[0,\infty]$ we have $H_a(x)=0$ for $x\in[0,a]$ and $H_a(x)=1$ for $x\in(a,\infty]$.

A *Menger space* is a triple $(S,\mathcal{F},T)$, where $(S,\mathcal{F})$ is a probabilistic metric space, T is a t-norm and the following inequality holds

$$F_{u,v}(x+y)\geq T(F_{u,w}(x),F_{w,v}(y))$$

for every $u,v,w\in S$ and every $x>0,y>0$.

We assume in the whole paper for the probabilistic metric spaces $(S,\mathcal{F},\tau)$ that Range$(\mathcal{F})\subset\mathcal{D}^+$, where $\mathcal{D}^+=\{F\mid F\in\Delta^+,\lim_{x\to\infty}F(x)=1\}$.

The first fixed point theorem in probabilistic metric spaces was proved by Sehgal and Bharucha-Reid (1972) for mappings $f:S\to S$, where $(S,\mathcal{F},T_{\mathbf{M}})$ is a Menger space, and $T_{\mathbf{M}}=\min$.

Definition 2. Let $(S,\mathcal{F},\tau)$ be a probabilistic metric space. A mapping $f:S\to S$ is a probabilistic q-contraction $(q\in(0,1))$ if

$$F_{fp_1,fp_2}(x)\geq F_{p_1,p_2}\left(\frac{x}{q}\right)$$

for every $p_1,p_2\in S$ and every $x\in\mathbb{R}$.

Theorem 3. *Let $(S,\mathcal{F},T_{\mathbf{M}})$ be a complete Menger space and $f:S\to S$ a probabilistic q-contraction. Then there exists a unique fixed point x of the mapping f and $x=\lim_{n\to\infty}f^np$ for every $p\in S$.*

Further development of the fixed point theory in a more general Menger space $(S,\mathcal{F},T)$ was connected with investigations of the structure of the t-norm T. Very soon the problem was in some sense completely solved. Namely, if we restrict ourselves to complete Menger spaces $(S,\mathcal{F},T)$, where T is a continuous t-norm, then any probabilistic q-contraction $f:S\to S$ has a fixed point if and only if the t-norm is of H-type.

If T is a t-norm, $x\in[0,1]$ then we write $x_T^{(0)}=1$ and for $n\in\mathbb{N},x_T^{(n)}=T\left(x_T^{(n-1)},x\right)$.

Definition 4. A t-norm T is of H-type if the family $(x_T^{(n)})_{n\in\mathbb{N}}$ is equicontinuous at the point $x=1$.

O. Hadžic and M. Budinčevic (1978) proved the following fixed point theorem.

Theorem 5. *Let $(S,\mathcal{F},T)$ be a complete Menger space, T a t-norm of H-type and $f:S\to S$ a probabilistic q-contraction. Then there exists a unique fixed point $x\in S$ of the mapping f and $x=\lim_{n\to\infty}f^np$ for every $p\in S$.*

Since the t-norm $T_{\mathbf{M}}$ is of H-type, the fixed point theorem of Sehgal and Bharucha-Reid follows from Theorem 5.

V. Radu (1994) proved the following fixed point theorem.

Theorem 6. *Any continuous t-norm T with the fixed point property is of H-type, where a t-norm T has the fixed point property if and only if every probabilistic q-contraction $f: S \to S$, where $(S, \mathcal{F}, T)$ is an arbitrary complete Menger space, has a fixed point.*

A very interesting new approach to the fixed point theory in probabilistic metric spaces is given in Tardiff's paper (1992), where some additional growth conditions for the mapping $\mathcal{F}: S \times S \to \mathcal{D}^+$ are assumed, and $T \geq T_{\mathbf{L}}$ (implies $\sup_{x<1} T(x,x) = 1$ and then also continuity at $(1,1)$).

Theorem 7. *Let $(S, \mathcal{F}, T)$ be a complete Menger space and T a t-norm such that $T \geq T_{\mathbf{L}}$. If for every $u, v \in S$*

$$\int_1^\infty \ln(x)\, dF_{u,v}(x) < \infty$$

holds, then any probabilistic q-contraction $f: S \to S$ has a unique fixed point x and $x = \lim_{n\to\infty} f^n p$ for every $p \in S$.

E. Parau and V. Radu (2001) proved the following theorem.

Theorem 8. *If $(S, \mathcal{F}, T)$ is a complete Menger space such that $T \geq T_{\mathbf{L}}$, then a probabilistic q-contraction $f: S \to S$ has a fixed point if and only if there exist $k > 0$ and $p \in S$ such that*

$$\int_0^\infty x^k\, dF_{p,fp}(x) < \infty.$$

3 Countable extension of t-norms

We investigate closer the countable extension of t-norms ([6], for other types of extensions see [8]) and we introduce a new notion: the geometrically convergent t-norm, which is closely related to the fixed point property.

We prove in [3] that t-norms of H-type and some subclasses of Dombi, Acżel-Alsina, Sugeno-Weber families of t-norms are geometrically convergent. We obtained in [3] also some practical criterion for the geometrically convergent t-norms.

In the fixed point theory it is of interest to investigate the classes of t-norms T and sequences $(x_n)_{n\in\mathbb{N}}$ from the interval $[0,1]$ such that $\lim_{n\to\infty} x_n = 1$, and

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} x_i = \lim_{n\to\infty} \mathop{\mathsf{T}}_{i=1}^{\infty} x_{n+i} = 1. \tag{1}$$

It is of special interest for the fixed point theory in probabilistic metric spaces to investigate condition (1) for a special sequence $(1-q^n)_{n\in\mathbb{N}}$ for $q \in (0,1)$.

Proposition 9. *If for a t-norm T there exists $q_0 \in (0,1)$ such that*

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q_0^i) = 1, \textit{then } \lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q^i) = 1, \textit{for every } q \in (0,1).$$

Definition 10. We say that a t-norm T is geometrically convergent (briefly g-convergent, called also q-convergent for some $q \in (0,1)$) if

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} (1-q^i) = 1.$$

for every $q \in (0,1)$.

Since $\lim_{n\to\infty}(1-q^n) = 1$ and $\sum_{n=1}^{\infty}(1-(1-q^n))^s < \infty$ for every $s > 0$ it follows that all t-norms from the family

$$\mathcal{T}_0 = \bigcup_{\lambda\in(0,\infty)} \{T_\lambda^{\mathbf{D}}\} \bigcup \bigcup_{\lambda\in(0,\infty)} \{T_\lambda^{\mathbf{AA}}\} \bigcup \mathcal{T}^H \bigcup_{\lambda\in(-1,\infty]} \{T_\lambda^{\mathbf{SW}}\}$$

are g-convergent, where $\mathcal{T}^H$ is the class of all t-norms of H-type, and for other families see [6].

The following example shows that not every strict t-norm is g-convergent. Let T be the strict t-norm with an additive generator $\mathbf{t}(x) = -\frac{1}{\log(1-x)}$. In this case the series $\sum_{i=1}^{\infty} \mathbf{t}(1-q^i)$ for any $q \in (0,1)$ is not convergent.

4 Fixed point theorems related to g-convergent t-norms

We proved a fixed point theorem for a probabilistic q-contraction $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfies Radu's condition, and T is a g-convergent t-norm (not necessarily the condition $T \geq T_{\mathbf{L}}$). The metric approach allows an estimation of the convergence with respect to the solution.

Theorem 11. *Let $(S, \mathcal{F}, T)$ be a complete Menger space and $f : S \to S$ a probabilistic q-contraction such that for some $p \in S$ and $k > 0$*

$$\sup_{x>0} x^k (1 - F_{p,fp}(x)) < \infty.$$

If t-norm T is g-convergent, then there exists a unique fixed point z of the mapping f and $z = \lim_{n\to\infty} f^n p$.

There are many corrollaries of the preceding theorem related the family $\mathcal{T}_0$ of t-norms, see [3]. Special non-additive measures, so called **S**-decomposable measures (see [9]), generate a probabilistic metric space in which Theorem 11 implies a specific fixed point theorem, see [3].

5 Multivalued contraction and an application in a fuzzy metric space

T. Hicks (1983) considered another notion of probabilistic contraction mapping than probabilistic q-contraction, which is incomparable with probabilistic q-contraction.

Definition 12. Let $(S, \mathcal{F}, \tau)$ be a probabilistic metric space and $f : S \to S$. The mapping f is a probabilistic C-contraction if there exists $k \in (0,1)$ such that for every $p, q \in S$ and for every $t > 0$

$$F_{p,q}(t) > 1 - t \quad \Rightarrow \quad F_{fp,fq}(kt) > 1 - kt.$$

If $f: S \to S$ is a probabilistic C-contraction and $(S, \mathcal{F}, T_{\mathbf{M}})$ is a complete Menger space Hicks proved that f has a unique fixed point.

As a multivalued generalization of the notion of a probabilistic C-contraction we introduced the notion of a (Ψ, C)-contraction, where $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, for $\mathbb{R}_+ = [0, \infty)$.

Let $\mathcal{P}(S)$ be the family of all nonempty subsets of S.

Definition 13. Let $(X, \mathcal{F}, \tau)$ be a probabilistic metric space and $f : S \to \mathcal{P}(S)$. The mapping f is called a (Ψ, C)-contraction, where $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$, if for every $p, q \in S$ and every $x > 0$ we have the following implication:

$$F_{p,q}(x) > 1 - x \Rightarrow \text{ for every } u \in fp \text{ there exists } v \in fq$$

$$\text{such that } F_{u,v}(\Psi(x)) > 1 - \Psi(x).$$

If $\Psi(x) = kx$, $x > 0$, $k \in (0,1)$ then a (Ψ, C)-contraction $f : S \to S$ is a C-contraction. Let $\mathcal{P}(S)_{cl}$ be the family of all nonempty closed subsets of S.

Theorem 14. *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $\sup_{x<1} T(x,x) = 1$, $M \in \mathcal{P}(S)_{cl}$, $f : M \to \mathcal{P}(M)_{cl}$ a (Ψ, C)-contraction, where the series $\sum_{n=1}^{\infty} \Psi^n(s)$ is convergent for some $s > 1$. If f is weakly demicompact or*

$$\lim_{n\to\infty} \mathop{\mathrm{T}}_{i=1}^{\infty} (1 - \Psi^{n+i-1}(s)) = 1,$$

then there exists at least one element $x \in M$ such that $x \in fx$.

For some families of t-norms it can be obtained special fixed point theorems, see [3].

As a consequence of Theorem 14 we shall obtain a result on the existence of a fixed point for a class of multivalued mappings in fuzzy metric spaces.

In [5] Kaleva and Seikkala introduced the notion of a fuzzy metric space. Let $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, nondecreasing in both arguments such that $L(0,0) = 0$, $R(1,1) = 1$, G the set of all nonnegative upper semicontinuous, normal convex fuzzy numbers, X a nonempty set, $d : X \times X \to G$ and $\overline{0}$ be the fuzzy number defined by $\overline{0}(u) = 1$ for $u = 0$ and $\overline{0}(u) = 0$ for $u \neq 0$. Denote $[d(x,y)]_\alpha = [\lambda_\alpha(x,y), \rho_\alpha(x,y)]$ $(x, y \in X,\ \alpha \in [0,1])$. The quadruple (X, d, L, R) is called a *fuzzy metric space* if and only if (i)-(iii) hold, where

(i) $d(x,y) = \overline{0}$ if and only if $x = y$.

(ii) $d(x,y) = d(y,x)$, for all $x, y \in X$.

(iii) $d(x,y)(s+u) \geq L(d(x,z)(s), d(z,y)(u))$, for all $x, y, z \in X$,
whenever

$$s \leq \lambda_1(x,z),\ u \leq \lambda_1(z,y),\ s+u \leq \lambda_1(x,y)$$

and
$d(x,y)(s+u) \leq R(d(x,z)(s), d(z,y)(u))$, for all $x, y, z \in X$,
whenever

$$s \geq \lambda_1(x,z),\ u \geq \lambda_1(z,y),\ s+u \geq \lambda_1(x,y).$$

We shall suppose that R is associative and satisfies the condition $R(a,0) = a$, for every $a \in [0,1]$. This implies that the mapping $T(a,b) = 1 - R(1-a, 1-b)$ $(a,b \in [0,1])$ is a t-norm.

Every Menger space $(X, \mathcal{F}, T)$ is a fuzzy metric space (X,d,L,R) if

$$d(x,y)(u) = \begin{cases} 0, & u < \sup\{s;\ F_{x,y}(s) = 0\} = u_{x,y} \\ 1 - F_{x,y}(u), & u \geq u_{x,y}, \end{cases}$$

and the functions L and R are defined by $L \equiv 0$ and $R(a,b) = 1 - T(1-a, 1-b) \quad (a,b \in [0,1])$, i.e., R is the dual t-conorm for T.

The converse statement does not hold in general. But under an additional condition on d a fuzzy metric space (X,d,L,R) is a Menger space. This condition is the following: for all $x,y \in X$

$$\lim_{u \to \infty} d(x,y)(u) = 0.$$

Then $(X, \mathcal{F}, T)$ is a Menger space, where $T(a,b) = 1 - R(1-a, 1-b)$, for every $a,b \in [0,1]$ and the mapping $\mathcal{F}$ is defined by $(x,y \in X,\ s \in \mathbb{R})$

$$F_{x,y}(s) = 0, \text{ for } s \leq \lambda_1(x,y),\ \ F_{x,y}(s) = 1 - d(x,y)(s), \text{ for } s \geq \lambda_1(x,y).$$

If a fuzzy metric space (X,d,L,R) is given, then we call the above defined Menger space $(X, \mathcal{F}, T)$ the associated Menger space.

Theorem 15. *Let (X,d,L,R) be a complete fuzzy metric space such that*

$$\lim_{u \to \infty} d(x,y)(u) = 0 \text{ for all } x,y \in X,$$

$\lim_{a \to 0^+} R(a,a) = 0$ *and* $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that the series* $\sum_{n=1}^{\infty} \Psi^n(u)$ *is convergent for some* $u > 1$. *Let* $f : X \to \mathcal{P}(X)_{cl}$ *be such a mapping that the following implication holds:*

For every $x,y \in X$ *and every* $u \in fx$ *there exists* $v \in fy$ *such that for every* $s > 0$ $d(x,y)(s) < s \Rightarrow 1 - F_{u,v}(\Psi(s)) < \Psi(s)$. *If* f *is weakly demicompact or* $\lim_{n \to \infty} \mathsf{R}_{i=n}^{\infty} \Psi^i(s) = 0$, *then there exists* $x \in X$ *such that* $x \in fx$.

For the proof see [3].

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T-transitivity and domination

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1 Introduction

Aggregation is a fundamental process in decision making and in any other discipline where the fusion of different pieces of information is of vital interest, e.g. in fuzzy querying.

Flexible (fuzzy) querying systems are usually designed not just to give results that match a query exactly, but to give a list of possible answers ranked by their closeness to the query—which is particularly beneficial if no record in the database matches the query in an exact way [14]. The closeness of a single value of a record to the respective value in the query is usually measured using a fuzzy equivalence relation, that is, a reflexive, symmetric and T-transitive fuzzy relation. Recently, a generalization has been proposed [5] which also allows flexible interpretation of ordinal queries (such as "at least" and "at most") by using fuzzy orderings [3]. In any case, if a query consists of at least two expressions that are to be interpreted vaguely, it is necessary to combine the degrees of matching with respect to the different fields in order to obtain an overall degree of matching. Assume that we have a query $(q_1,\ldots,q_n)$, where each $q_i \in X_i$ is a value referring to the i-th field of the query. Given a data record $(x_1,\ldots,x_n)$ such that $x_i \in X_i$ for all $i = 1,\ldots,n$, the overall degree of matching is computed as

$$\tilde{R}\big((q_1,\ldots,q_n),(x_1,\ldots,x_n)\big) = \mathbf{A}\big(R_1(q_1,x_1),\ldots,R_n(q_n,x_n)\big),$$

where each R_i is a T-transitive binary fuzzy relation on X_i which measures the degree to which the value x_i matches the query value q_i.

It appears natural to require that the function $\mathbf{A}$ is an aggregation operator [7, 9, 13] and moreover, it would be desirable that $\tilde{R}$ is still T-transitive in order to have a clear interpretation of the aggregated fuzzy relation $\tilde{R}$. Therefore, it is necessary to study which aggregation operators are able to guarantee that $\tilde{R}$ maintains T-transitivity.

It turns out that the preservation of T-transitivity in aggregating fuzzy relations is closely related to the dominance of an aggregation operator A with respect to the corresponding t-norm T.

Let us recall some basic definitions.

Definition 1. [8] A (binary) operation $\mathbf{A} : [0,1]^2 \to [0,1]$ is called an *aggregation operator* if $\mathbf{A}$ is non-decreasing and the equalities $\mathbf{A}(0,0) = 0$ and $\mathbf{A}(1,1) = 1$ hold. Moreover, if $\mathbf{A}$ is also associative, symmetric and has 1 as neutral element, then it is called a t-norm.

Definition 2. Consider a binary fuzzy relation R on some universe X and an arbitrary t-norm T. R is called T-transitive if and only if, for all $x,y,z \in X$,

$$T\big(R(x,y), R(y,z)\big) \leq R(x,z). \tag{1}$$

For more details on fuzzy relations, especially fuzzy equivalence relations and fuzzy orderings and their properties, we recommend either original sources as [17, 1, 12], but also [10, 11, 4, 2].

Standard aggregation of fuzzy equivalence relations (fuzzy orderings) preserving T-transitivity is done either be means of T or $T_{\mathbf{M}}(x,y) = \min(x,y)$. Staying in the framework of t-norms, in fact any t-norm T^* dominating T can be applied to preserve T-transitivity, i.e. if R_1, R_2 are two T-transitive, binary relations on a universe X , then also $T^*(R_1,R_2)$ has this property (see [10]). Recall that trivially, for any t-norm T, it holds that T itself and $T_{\mathbf{M}}$ dominate T.

As already mentioned above in several applications, other types of aggregation preserving T-transitivity are required [6]. Especially different weights (degrees of importance) of input fuzzy equivalences (orderings) R_1 and R_2 cannot be properly modeled by aggregation with t-norms, because of the commutativity . Therefore, we have to consider general T-transitivity-preserving aggregation operators.

Note that in the sequel we will deal with the aggregation of two given T-transitive binary fuzzy relations R_1, R_2 acting on the same universe X. Our results can be easily modified for the case of the Cartesian product of T-transitive equivalence relations, as well as to the case of aggregating more than two T-transitive fuzzy relations such that the resulting output fuzzy relation will still be T-transitive.

2 T-Transitivity and Domination

Definition 3. [8] Let $\mathbf{A}, \mathbf{B}$ be two aggregation operators. We say that $\mathbf{A}$ *dominates* $\mathbf{B}$ ($\mathbf{A} \gg \mathbf{B}$), if and only if, for all $x,y,u,v \in [0,1]$,

$$\mathbf{B}\big(\mathbf{A}(x,y), \mathbf{A}(u,v)\big) \leq \mathbf{A}\big(\mathbf{B}(x,u), \mathbf{B}(y,v)\big). \tag{2}$$

Observe that $\mathbf{A} \gg \mathbf{A}$ if and only if $\mathbf{A}$ is bisymmetric. As already mentioned, for any t-norm T, $T \gg T$ and $T_{\mathbf{M}} \gg T$.

Further on we will denote the class of all aggregation operators $\mathbf{A}$ which dominate a given t-norm T with $\mathcal{D}_T = \{\mathbf{A} \mid \mathbf{A} \gg T\}$.

The following theorem generalizes the result from [10].

Theorem 4. *Let $|X| > 2$. An aggregation operator* $\mathbf{A}$ *preserves the T-transitivity of fuzzy relations on X if and only if* $\mathbf{A} \in \mathcal{D}_T$.

In the following, we will focus on the characterization of the system $\mathcal{D}_T$. As already observed, $\{T, T_{\mathbf{M}}\} \subset \mathcal{D}_T$. For any t-norm T, some interesting properties of $\mathcal{D}_T$ can be found.

Proposition 5. *Consider a t-norm T and the corresponding class of dominating aggregation operators $\mathcal{D}_T$. Then the following holds:*

(i) For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{D}_T$, *also* $\mathbf{D} = \mathbf{A}(\mathbf{B}, \mathbf{C}) \in \mathcal{D}_T$.

(ii) If T is a continuous Archimedean t-norm with an additive generator $f : [0,1] \to [0,\infty]$, then for any $p, q \in]0,\infty[$, also the weighted t-norm $T_{p,q} \in \mathcal{D}_T$, with $T_{p,q}(x,y) = T\left(x_T^{(p)}, y_T^{(q)}\right)$ and $x_T^{(p)} = f^{(-1)}(p \cdot f(x))$ (see also [13, 9]).

Recall that $\mathcal{D}_T$ was discussed and characterized also in [16, 15] for the case that T is a continuous Archimedean t-norm T with an additive generator $f : [0,1] \to [0,\infty]$.

Proposition 6. *[16] Under the circumstances given above,* $\mathbf{A} \in \mathcal{D}_T$ *if and only if there is a metric-preserving function $H : [0,\infty]^2 \to [0,\infty]$ such that for all $x, y \in [0,1]$:*

$$f(\mathbf{A}(x,y)) = H(f(x), f(y)).$$

Observe that Proposition 6 is in fact a corollary of Theorem 4. Indeed, H is metric preserving if and only if it is a sub-additive function of two variables, i.e.

$$H(x+y, u+x) \leq H(x,u) + H(y,v)$$

for all $x, y, u, v \in [0,\infty]$, what is, in fact, the domination of the sum operator over H.

3 Special Cases

We will now discuss three special cases of t-norms: $T_{\mathbf{M}} = \min(x,y)$, $T_{\mathbf{P}}(x,y) = x \cdot y$ (by isomorphism any strict t-norm can be covered [13]), $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$ (by isomorphism, covering all nilpotent t-norms).

Proposition 7. *The class of aggregation operators dominating the minimum t-norm $T_{\mathbf{M}}$ is given by*

$$\mathcal{D}_{\min} = \{\min_{f,g} \mid f,g : [0,1] \to [0,1], \textit{ non-decreasing}, \\ f(1) = g(1) = 1, f(0) \cdot g(0) = 0\},$$

where $\min_{f,g} = \min\big(f(x), g(y)\big)$.

Evidently, $\mathbf{A} \in \mathcal{D}_{\min}$ is symmetric if and only if $\mathbf{A}(x,y) = f\big(\min(x,y)\big)$ for some non-decreasing function $f : [0,1] \to [0,1]$ fulfilling $f(0) = 0$ and $f(1) = 1$. Note also, if $\mathbf{A} \in \mathcal{D}_{\min}$, then $\mathbf{A} = \min_{f,g}$, where $f(x) = \mathbf{A}(x,1)$ and $g(y) = \mathbf{A}(1,y)$ (for all $x, y \in [0,1]$).

Concerning $T_{\mathbf{P}}$ and $T_{\mathbf{L}}$, though the classes $\mathcal{D}_{T_{\mathbf{P}}}$ and $\mathcal{D}_{T_{\mathbf{L}}}$ are completely characterized either by Theorem 4 or by Proposition 6, there is no counterpart of Proposition 7 in these cases. However, it is possible to give examples of these members of these classes, and of course, apply Proposition 5 to obtain new members.

Example 8. Observe that $x_{T_{\mathbf{P}}}^{(p)} = x^p$ and thus for all $p,q \in]0,\infty[$, the operator $P_{p,q} : [0,1]^2 \to [0,1], P_{p,q}(x,y) = x^p y^q$ is contained in $\mathcal{D}_{T_{\mathbf{P}}}$. Particularly, if $p+q=1$, then $P_{p,q}$ is a weighted geometric mean (compare also examples from [16, 15]).

However, observing that for all $\lambda \geq 1$, the function $H_\lambda : [0,\infty]^2 \to [0,\infty]$, $H_\lambda(x,y) = (x^\lambda + y^\lambda)^{\frac{1}{\lambda}}$, is metric preserving, also any member of the Aczél-Alsina family of t-norms $(T^{\mathbf{AA}}\lambda)_{\lambda \in [1,\infty]}$ (see [13]), is contained in $\mathcal{D}_{T_{\mathbf{P}}}$ because of Proposition 6.

Example 9. Similarly, for all $p,q \in]0,\infty[, L_{p,q} \in \mathcal{D}_{T_{\mathbf{L}}}$, where $L_{p,q} = T_{\mathbf{L}\,p,q} = \max(0, px+qy+1-p-q)$. In particular, if $p+q=1$, $L_{p,q}(x,y) = px+qy$, i.e. any weighted mean dominates $T_{\mathbf{L}}$ (compare also examples from [16, 15]).

Based on H_λ any Yager t-norm $T^{\mathbf{Y}}\lambda \in \mathcal{D}_{T_{\mathbf{L}}}$ whenever $\lambda \geq 1$.

4 Conclusions

An aggregation operator $\mathbf{A}$ preserves T-transitivity of fuzzy relations if and only if it dominates the corresponding t-norm T ($\mathbf{A} \in \mathcal{D}_T$). Although several methods for constructing aggregation operators within a certain class $\mathcal{D}_T$ have been mentioned, an explicit description of $\mathcal{D}_T$ could only been presented for the minimum t-norm $T_{\mathbf{M}}$.

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Dynamics of fuzzy systems: theory and applications

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Abstract

In this paper I present the dynamics theory which can operate with an imprecise knowledge about system states and with qualitative information about the forces influence. Such situation is typical for biology, economy, social and political science, where we don't have, as rule, complete information about the system state and where our "fair" knowledge about the dynamics laws must be expressed in terms of "preferability", rather than in exact definition of the forces.

This approach was successfully applied to some problems of brain activity and the lymphocyte cells maturation.

1 Introduction

There are a few equations in theoretical physics, which in a slightly different modification cover dynamic problems of most physical systems. The same types of the equations appear, also, in the completely "non-physical" areas like finance (Black-Scholes equation), political science (voting theory), ecology, *etc*. In this paper we show how the fuzzy logic mathematical apparatus applied to common causal principal together with some natural restriction on a system state space resolves this puzzle.

Another goal of this paper is concerned with the necessity to have dynamics theory which can operate with an imprecise knowledge about system states and with qualitative information about the forces influence. This situation is typical for the above mentioned non-physical sciences. There, as a rule, we don't have complete information about the system state and our "fair" knowledge should be expressed in terms of "preferability", rather than in exact definition of the forces. For example, based on the observations and an intuitive experience, most experts will be consistent in formulation of the forces influence in this form: "if the system is in state *A* then after a short time state *B* will be preferable than the others". However the experts will be hindered and inconsistent if we ask them to give a more precise formulation of the dynamics laws, or even if we ask them to assign probabilities for transitions from the given state to the others.

The main assumptions for this study are the following:

i) For most practical problems a system state space can be approximated by a Riemann manifold, but choice of the coordinates on the manifold is not unique, so the dynamics theory should be covariant under appropriate transformations of the coordinates.

ii) A system behavior is described in terms of "possibility" [1] that the system is in the neighborhood U_x of a given state $\mathbf{x}$ at a given time t. This possibility can be represented as a function of the domain and the time: $m(U,t)$. This representation should be consistent with a common logic in a sense that at least for small domains the possibility that the system is in the domain $U = U_1 \bigcup U_2$; $U_1 \bigcap U_2 = \emptyset$ must be equal to the possibility that the system is in the domain U_1 *or* it is in the domain U_2:

$$m(U,t) =$$

$$S(m(U_1,t); m(U_2,t)),$$

where $S(m_1; m_2)$ denotes the logical connective *or*.

iii) The dynamics laws have "causal recursion" form such as: Possibility that a system is in some neighborhood of the state - $\mathbf{x}$ at the time t is determined by the possibility that: the system was in a neighborhood of a state $\mathbf{x}_1$ at time $t-\delta$ *and* the transfer to the $\mathbf{x}$-neighborhood during the time interval δ was possible, *or* it was in a neighborhood of a point $\mathbf{x}_2$ at time $t-\delta$ *and* transfer to the $\mathbf{x}$ - neighborhood during the time δ was possible, *or* ... so on, for all possible states at the time $t-\delta$.

iiii) The dynamics is local, i.e. a system state cannot be changed significantly for a short time period.

In spite of the generality of the forms of the above mentioned assumptions, they dramatically reduce the possible types of the dynamics equations and lead to a deep relation between permissable form of the membership function $m(U_{\mathbf{x}},t)$ and representations of the logical connectives *or* and *and*.

2 Fuzzy Dynamics

Denoting by $P_\delta\left(U_{\mathbf{x}}, U_{\mathbf{x}_i}, t\right)$ the possibility of the transition from the neighborhood $U_{\mathbf{x}_i}$ of the point $\mathbf{x}_i$ to the neighborhood $U_{\mathbf{x}}$ of the point $\mathbf{x}$ during the time δ, we can write the dynamics law *iii)* in the symbolic form:

$$m\left(U_{\mathbf{x}},t\right) = \qquad (1)$$

$$S_{\mathrm{all\,x_i}}\left(...; T\left(P_\delta\left(U_{\mathbf{x}}; U_{\mathbf{x}_i}, t\right); m\left(U_{\mathbf{x}_i}, t-\delta\right)\right); ...\right),$$

where $T(P;m)$ denotes the logical connective *and*. Note, that the possibility of transition $P_\delta\left(U_{\mathbf{x}}, U_{\mathbf{x}_i}, t\right)$ is determined by the forces acting on the system.

In order to find the dynamics equations we have to represent S and T connectives and go to the limit $U_{\mathbf{x}} \to \mathbf{x}$ (that is the membership function and S,T - *connectives* can be determined for small neighborhoods $U_{\mathbf{x}}$ only).

Admissible representations of the S and T connectives depend on the limiting properties of the function $m(U_{\mathbf{x}},t)$ in the limit $U_{\mathbf{x}} \to \mathbf{x}$. Generally, there are two cases:

a)

$$\lim_{U_{\mathbf{x}} \to \mathbf{x}} m\left(U_{\mathbf{x}},t\right) = \mu(\mathbf{x},t) \neq 0.$$

b)

$$\lim_{U_{\mathbf{x}} \to \mathbf{x}} m\left(U_{\mathbf{x}},t\right) \equiv 0.$$

As these cases lead to substantially different descriptions of the system dynamics, they will be considered separately. *Case a) - dynamics of a fuzzy State Grade* The function $\mu(\mathbf{x},t)$ can be interpreted

as *possibility* that a system state is $\mathbf{x}$ at the time t. For definiteness we will call this function "State Grade".

Consider now the relation *ii)* in the case $U_1 \to \mathbf{x}$; $U_2 \to \mathbf{x}$ and, therefor, $U = U_1 \bigcup U_2 \to \mathbf{x}$. In this limit for continuous $\mu(\mathbf{x},t)$ we have:

$$S(\mu,\mu) = \mu.$$

Together with such general properties like monotonicity, commutativity and with usual boundary condition [2] this leads immediately to:

$$S(\mu_1,\mu_2) = \max(\mu_1;\mu_2). \tag{2}$$

For the connective *and*, however, we can take an arbitrary *T-norm*.

In the case *a)* the possibility of transition from $U_{\mathbf{x}_i}$ to $U_{\mathbf{x}}$ becomes a function of the states $\mathbf{x}_i$, $\mathbf{x}$ and the time: $P_\delta(\mathbf{x}_i \to \mathbf{x},t)$. It is convenient to choose this function in the form:

$$P_\delta(\mathbf{x}_i,\mathbf{x},t) = G(\mathbf{v};\mathbf{x},t)),$$

where $\mathbf{v} = \frac{\mathbf{x}-\mathbf{x}_i}{\delta}$ and $G(\mathbf{v};\mathbf{x},t)$ is the possibility that the system has the "velocity" $\mathbf{v}$ at the point $\mathbf{x}$ at the time t. Then, the symbolic expression (1) becomes an equation:

$$\mu(\mathbf{x},t) \approx$$
$$\sup_{\mathbf{v}} T(G(\mathbf{v},\mathbf{x}-\mathbf{v}\delta,t-\delta);\mu(\mathbf{x}-\mathbf{v}\delta,t-\delta)).$$

It can be shown (see [3]-[5] for details) that in the limit $\delta \to 0$ this equation leads to the Liouville-type equation:

$$\frac{\partial\mu}{\partial t} = -\sum_i \frac{\partial H}{\partial p_i}\frac{\partial\mu}{\partial x_i}. \tag{3}$$

The "Hamiltonian" - H is determined as:

$$H = (\mathbf{V}(\mathbf{x},\mathbf{p},t)\cdot\mathbf{p}),$$

with $\mathbf{V}(\mathbf{x},\mathbf{p},t)$ is obtained from the system:

$$\frac{\partial G(\mathbf{V};\mathbf{x},t)}{\partial\mathbf{V}} = \beta\mathbf{p},$$
$$T(G(\mathbf{V};\mathbf{x},t);\mu) = \mu(\mathbf{x},t).$$

Equation (3) is equivalent to the Hamiltonian system for its characteristics:

$$\frac{d\mathbf{x}(t)}{dt} = \frac{\partial H}{\partial\mathbf{p}}, \tag{4}$$

$$\frac{d\mathbf{p}(t)}{dt} = -\frac{\partial H}{\partial\mathbf{x}}. \tag{5}$$

>From equations (3)-(5) it follows that the state grade remains constant on the characteristics $\mathbf{x}(t)$:

$$\mu(\mathbf{x}(t),t) = \mu(\mathbf{x}(0),0),$$

for any t.

Equations (4)-(5) describes a system evolution in an extended *"Fuzzy state space'"*, which consists of the two components:

α**)** the “physical” component **- x**

β**)** the “information” component **- p**

As the equation (5) is similar to the equation for the physical moment, we can call **p** - *the moment of possibility*. The direction of this moment $\mathbf{p}/|\mathbf{p}| = \nabla m(\mathbf{x},t)/|\nabla m(\mathbf{x},t)|$ is the direction >from a given state to the locally most preferable one.

It is important, that the Hamiltonian $H(\mathbf{x},p,t)$ can be obtained directly from the “linguistic” description of the “forces” influence. The Hamiltonian H can be more general than the common Hamiltonians in physics, in particular, it may be a set-valued map, i.e. the system (4)-(5) describes both differential equation and differential inclusion. The particular case $T(G;\mu) = \min(G;\mu)$ has been considered in [3],[4]. *Case b): dynamics of a fuzzy State Density* In accordance with the requirement -

i) the function $m(U,t)$ should depend on invariant measure of U only. The simplest such measure is ρdU, where dU is “volume” of U in a given coordinate system and multiplier ρ makes the product ρdU invariant under the admissible transformations of the coordinates.

In the case *b)* this leads to the asymptotic:

$$\lim_{U_{\mathbf{x}}\to\mathbf{x}} \frac{m(U_{\mathbf{x}},t))}{g(\rho(\mathbf{x},t)dU_{\mathbf{x}})} = 1, \tag{6}$$

where the function $g(...)$ is continuous (but not necessary differentiable) near zero function and $g(0) = 0$. We assume also that ρ is a smooth function of $\mathbf{x}$ and t. As in physics the multiplier like ρ is called - density, we will call $\rho(\mathbf{x},t)$ as *State Density*. Generally speaking, the function g could depend explicitly on point $\mathbf{x}$, but, as we will see later, such dependence must be omitted in order to hold homogeneity of the representations of the logical connectives *or* and *and* for the whole state space.

Consider small domains $U_{\mathbf{x}} = U_{\mathbf{x}+\varepsilon} \bigcup U_{\mathbf{x}-\varepsilon}$ with the volumes:

$$dU_{\mathbf{x}} = dU_{\mathbf{x}-\varepsilon} + dU_{\mathbf{x}+\varepsilon}, \tag{7}$$

Possibility, that the system is in $U_{\mathbf{x}}$, is equal to:

$$g(\rho(\mathbf{x},t)dU_{\mathbf{x}}) = \tag{8}$$

$$S(g(\rho(\mathbf{x}-\varepsilon,t)dU_{\mathbf{x}-\varepsilon}); g(\rho(\mathbf{x}+\varepsilon)dU_{\mathbf{x}+\varepsilon})).$$

Using (7), one has for the smooth $\rho(\mathbf{x},t)$ and small ε:

$$\rho(\mathbf{x})dU_{\mathbf{x}} =$$

$$\rho(\mathbf{x}-\varepsilon)dU_{\mathbf{x}-\varepsilon} + \rho(\mathbf{x}+\varepsilon)dU_{\mathbf{x}+\varepsilon}$$

$$+o(\varepsilon dU).$$

Thus, for the connective *OR* we have for the small and close dU_1, dU_2:

$$S(g(\rho(\mathbf{x}_1)dU_1); g(\rho(\mathbf{x}_2)dU_2)) \approx \tag{9}$$

$$g(\rho(\mathbf{x}_1)dU_1 + \rho(\mathbf{x}_2)dU_2).$$

In order to find out for an admissible form of the possibility of transition, let us assume now that at the time $t-\delta$ the system was in the domain U_0 with the possibility 1 and with the possibility 0 elsewhere. It is obvious, that the possibility that at the time t the system will be in a domain $U_{\mathbf{x}}$ is equal to the possibility that the system transferred to $U_{\mathbf{x}}$ from U_0. That is in such situation the transition possibility - $P_\delta(U_{\mathbf{x}},U_{\mathbf{x}_0},t)$ should be equal to $m(U_{\mathbf{x}},t)$. Because $m(U_{\mathbf{x}},t)$ has of the asymptotic (6) we have:

$$P_\delta(U_{\mathbf{x}},U_{\mathbf{x}_0}) \approx g(\Gamma_\delta dU_{\mathbf{x}}), \tag{10}$$

where Γ_δ depend on $(\mathbf{x}_0;\mathbf{x};t)$, but doesn't depend on dU.

In order to find admissible representation of the connective *and*, consider decomposition of the transition from $U_{\mathbf{x}_0}$ to $U_{\mathbf{x}}$ during the time $\delta=\delta_1+\delta_2$ on the transitions from $U_{\mathbf{x}_0}$ to u_1 during the time δ_1 *and* from u_1 to $U_{\mathbf{x}}$ during δ_2, *or* >from $U_{\mathbf{x}_0}$ to u_2 *and* from u_2 to $U_{\mathbf{x}}$, *or ...* so on, for all possible intermediate domains (see Figure 1).

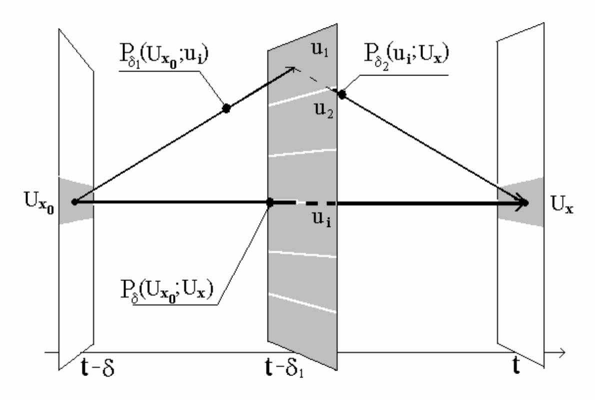

Figure 1: Decomposition of the transition possibility in the case *b)*

In according with (10) we have:

$$P^i_{\delta_1} = P^i_{\delta_1}\left(U_{\mathbf{x}0,u_i}\right) = g\left(\Gamma^i_{\delta_1} du_i\right),$$

$$P^i_{\delta_2} = P^i_{\delta_2}(u_i,U_{\mathbf{x}}) = g\left(\Gamma^i_{\delta_2} dU_{\mathbf{x}}\right).$$

Then, for possibility of the "two-fold" transition from $U_{\mathbf{x}_0}$ to u_i during the time δ_1 *and* >from u_i to $U_{\mathbf{x}}$ during the time δ_2 one obtains:

$$T\left(P^i_{\delta_1},P^i_{\delta_2}\right) = g\left(F\left(\Gamma^i_{\delta_1} du_i;\Gamma^i_{\delta_2} dU_{\mathbf{x}}\right)\right),$$

with some unknown function F. It follows from the figure and (9) that:

$$P_\delta\left(U_{\mathbf{x}0,U_{\mathbf{x}}}\right) = S\left(\ldots,T\left(P^i_{\delta_1},P^i_{\delta_2}\right),\ldots\right) =$$

$$= g\left(\sum_i F\left(\Gamma^i_{\delta_1} du_i,\Gamma^i_{\delta_2} dU_{\mathbf{x}}\right)\right),$$

As $P_\delta = g\left(\Gamma_\delta dU_{\mathbf{x}}\right)$ this implies that:

$$\Gamma_\delta dU_{\mathbf{x}} = \sum_i F\left(\Gamma^i_{\delta_1} du_i,\Gamma^i_{\delta_2} dU_{\mathbf{x}}\right) =$$

$$= \left(\sum_i f\left(\Gamma^i_{\delta_1} du_i\right) \Gamma^i_{\delta_2} \right) dU_{\mathbf{x}}.$$

In order that the last sum will converge for any partition $\{du_i\}$, we have:

$$f\left(\Gamma^i_{\delta_1} du_i\right) = const \cdot \Gamma^i_{\delta_1} du_i.$$

Without loss of generality we can put $const = 1$. So

$$F\left(\Gamma^i_{\delta_1} du_i; \Gamma^i_{\delta_2} dU_{\mathbf{x}}\right) = \Gamma^i_{\delta_1} \Gamma^i_{\delta_2} du_i dU_{\mathbf{x}}.$$

Thus, the logical connective *and* for the case *b)* has to be defined as:

$$T(P_1(\Gamma_1 dU_1); P_2(\Gamma_2 dU_2)) = g(\Gamma_1 dU_1 \Gamma_2 dU_2). \tag{11}$$

If the function g is invertible we can write:

$$S(m_1; m_2) = g(g^{-1}(m_1) + g^{-1}(m_2)), \tag{12}$$

$$T(P_1; P_2) = g(g^{-1}(P_1) \cdot g^{-1}(P_2)), \tag{13}$$

so in the case *b)* the logical connectives *or* and *and*, at least infinitesimally, should be considered as the pseudo-arithmetic sum and product [6],[7].

Therefore, in the given case the symbolic expression (1) leads to an equation:

$$g(\rho(\mathbf{x}, t) d\mathbf{x}) = \tag{14}$$

$$g\left(\left(\int_{V_{x'}} \Gamma\left(\mathbf{x}, \mathbf{x}', t; \delta\right) \rho\left(\mathbf{x}', t - \delta\right) d\mathbf{x}' \right) d\mathbf{x} \right).$$

Decomposing (14) with respect to δ leads to the equation (see [5] for details):

$$\frac{\partial \rho}{\partial t} + \mathbf{V}(\mathbf{x}, t) \cdot \nabla \rho = \left\{ D(\mathbf{x}, t) \nabla^2 + U(\mathbf{x}, t) \right\} \rho, \tag{15}$$

with

$$U(\mathbf{x}, t) = \lim_{\delta \to 0} \frac{1}{\delta} \left(\int \Gamma_\delta (\mathbf{x} - \mathbf{u}, \mathbf{x}, t) d^n \mathbf{u} - 1 \right),$$

$$\mathbf{V}(\mathbf{x}, t) = \lim_{\delta \to 0} \frac{1}{\delta} \int \Gamma_\delta (\mathbf{x} - \mathbf{u}, \mathbf{x}, t) \mathbf{u} d^n \mathbf{u},$$

$$D(\mathbf{x}, t) = \lim_{\delta \to 0} \frac{1}{2\delta} \int \Gamma_\delta (\mathbf{x} - \mathbf{u}, \mathbf{x}, t) |\mathbf{u}|^2 d^n \mathbf{u}.$$

If the manifold consists of M single-connected domains, this equation is transformed to:

$$\frac{\partial \rho_\sigma}{\partial t} + V^{\kappa, i}_\sigma (\mathbf{x}, t) \nabla_i \rho_\kappa = \tag{16}$$

$$\left\{ D^{\kappa, ij}_\sigma (\mathbf{x}, t) \nabla_i \nabla_j + U^\kappa_\sigma (\mathbf{x}, t) \right\} \rho_\kappa,$$

where indexes $\sigma, \kappa = 1, \ldots, M$ describe the different single-connected domains.

The coefficients in these equations have the following meaning: $D(\mathbf{x}, t)$ describe the diffusion, $\mathbf{V}(\mathbf{x}, t)$ - the flows and $U(\mathbf{x}, t)$ - the external fields. For $V_\sigma \equiv 0$ and complex valued ρ equation (16) corresponds to the Schreodinger equation and for $D^\kappa_\sigma \equiv 0$ and quaternion valued ρ it corresponds to the Dirac equation. For real valued ρ we have generalized Fokker-Plank equation.

3 Concluding Remarks

Equations (4)-(5) and (16) cover most dynamics equations successfully used in physics up to the present time. At the resent time the same types of the equations have been successfully applied to the "non-physical problems" in finance, political science, ecology, biology, *etc*.

It is interesting that there are "biological reasons" to use the above-mentioned representations of the logical connectives. It is well known [8] that the elementary operations of a nerve cell are accumulation and amplification of signals that correspond to the pseudo-arithmetic operations. In order to be able to use the *max S-norm* the neuron must be able to compare signals. Genetically a neuron is able to compare several specific signals. In the process of elaboration of conditional reflexes, living creatures develop the ability make a comparison among many other signals. It was shown also that a single neuron could be taught to compare arbitrary signals [9]. Thus, comparison may become a basic neuron operation. For a certain kind of neuron a mechanism that could carry out *max*-operations was proposed in [10].

Application of this approach to some problems of brain activity and the lymphocyte cells maturation were considered in [11] and [4].

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A natural interpretation of fuzzy partitions

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The aim of this talk is to present a natural interpretation of fuzzy partitions, which are naturally one-to-one correspondent to fuzzy equivalence relations.

In [1, 2, 3] we presented a new and natural interpretation of fuzzy sets, fuzzy relations, and fuzzy mappings in a cumulative Heyting valued model for intuitionistic set thoery. By the interpretation we can get most of the standard defining equations of basic notions and operations of fuzzy sets and fuzzy relations, consider notions such as operations of fuzzy subsets of different universes, fuzzy relations and mappings between fuzzy subsets, and make the meaning of Zadeh's extension principle clear.

Let H be a complete Heyting algebra and V^H be the cumulative H-valued model. The *Heyting value* $\|\varphi\|$ and the *check set* $\check{x}$ are defined as usual. For $u,v \in V^H$, u and v are *similar* iff $\|u = v\| = \mathbf{1}$. Every $A \in V^H$ is called an *H-fuzzy set*, and for a crisp set X every subset of $\check{X}$ in V^H is called an *H-fuzzy subset of X*. The *membership function of A on X* is the mapping $\mu_A : X \longrightarrow H;\, x \longmapsto \|\check{x} \in A\|$. There is a natural correspondence between H-fuzzy subsets of X and mappings from X to H, which preserves order and basic set operations. If R is an H-fuzzy subset of $X \times Y$, R is called an *H-fuzzy relation from X to Y*, and it is called an *H-fuzzy relation on X* in case $X = Y$.

Theorem 1. *An H-fuzzy relation R on X is an equivalence relation in the model iff for all* $x,y,z \in X$, $\mu_R\langle xx\rangle = \mathbf{1}$, $\mu_R\langle xy\rangle = \mu_R\langle yx\rangle$, *and* $\mu_R\langle xy\rangle \wedge \mu_R\langle yz\rangle \le \mu_R\langle xz\rangle$ *hold.*

For an H-fuzzy equivalence relation R on X and for every $x \in X$, let $[\check{x}]$ be the equivalence class of $\check{x}$ (in the model) and $P_R = \{[\check{x}]; x \in X\}$. Obviously $[\check{x}]$ is an H-fuzzy subset of X for each $x \in X$ and P_R is the set of all equivalence classes with respect to R.

Definition 2. A family P of H-fuzzy subsets of X is called an *H-fuzzy partition of X* if
(a) for every $A \in P$ there exists an $x \in X$ such that $\mu_A(x) = \mathbf{1}$,
(b) for each $x \in X$ there is a unique $A \in P$ such that $\mu_A(x) = \mathbf{1}$, and
(c) for all $A,B \in P$ and $x,y \in X$, $\mu_A(x) \wedge \mu_A(y) \wedge \mu_B(x) \le \mu_B(y)$.

For an H-fuzzy partition P of X, the corresponding H-fuzzy equivalence relation R_P on X can also be naturally defined, that is, for every $x,y \in X$, $\|\check{x}R_P\check{y}\| = \|\exists A \in P(\check{x} \in A \wedge \check{y} \in A)\|$ holds. For two families P,Q of H-fuzzy sets, P and Q are *equivalent* if for every $A \in P$ there exists a unique $B \in Q$ such that A and B are similar, and vice versa.

Theorem 3. *Let X be a crisp set.*
(1) *If R is an H-fuzzy equivalence relation on X and* $P = P_R$ *is the family of all equivalence classes with respect to R, then P is an H-fuzzy partition of X, and R and* R_P *are similar.*
(2) *If P is an H-fuzzy partition of X and* $R = R_P$ *is the corresponding H-fuzzy equivalence relation on X, then P is equivalent to* P_R*, the family of all equivalence classes with respect to R.*

Hence there is a natural correspondence between H-fuzzy equivalence relations on X and H-fuzzy partition families of X. The definition of H-fuzzy partition seems to be different from any of preceding definitions of fuzzy partition in the literature.

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