LINZ
2003

24th Linz Seminar on
Fuzzy Set Theory

Triangular Norms and Related Operators in Many-Valued Logics

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Abstracts

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Erich Peter Klement, Radko Mesiar
Editors

LINZ 2003

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TRIANGULAR NORMS AND RELATED OPERATORS IN MANY-VALUED LOGICS

ABSTRACTS

Erich Peter Klement, Radko Mesiar
Editors

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Since their inception in 1979, the Linz Seminars on Fuzzy Sets have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work. LINZ 2003 will be already the 24$^{\text{th}}$ seminar carrying on this tradition.

LINZ 2003 will deal with the use of Triangular Norms and Related Operators in Many-Valued Logics and their applications. Though the basic results in the theory of t-norms go back to the Sixties, there is an important growth of interest in the theoretical background of t-norms and related operators (such as copulas, implications, uninorms, etc.) during the last years. Theory and applications of t-norms and related operators influence each other, as can be seen not only in probabilistic metric spaces, but also in many-valued logics, measure and integration theory, preference modeling, etc. For practical purposes, the determination of an appropriate t-norm fitting the observed data becomes an acute problem. The aim of the seminar is an intermediate and interactive exchange of recent results. We expect that the presented talks will provide a comprehensive mathematical framework for the theory and application of triangular norms and related operators.

Erich Peter Klement
Radko Mesiar

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Contents

Ten years later: lessons from a polemics

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1. In 1993, a first version (1) of (2) got a "best paper award" in the Conference of the American Association for Artificial Intelligence and created a remarkable excitement among the community of researchers in fuzzy logic. For example, in (3) one can find some of the correspondence between people working in the field and, specially, the report (4) on the subject.

2. Paper (2) has, in fact, two parts. The first tries to show that the logical formula

$$(p . q')' = q + p' . q' \quad (*)$$

forces fuzzy logic to collapse into classical bivaluate logic. The second tries to criticize some technological achievements of fuzzy logic.

In 1994 the monthly journal IEEE-EXPERT devoted to the controversy a good part of one of its issues (5), with short papers writen by relevant researchers and with both pro and con arguments. In 1996, (6) and (7) appeared in the International Journal of Intelligent Systems, and in 2001 papers (8), (9) and (10) were published in the International Journal of Approximate Reasoning. Paper (6) tries to correct (2), and paper (7) considers the problem of logical equivalence, an important topic that is in the ground of (2). Paper (8) considers formula (*) in a very general fuzzy framework, and papers (9) and (10) are a continuation of the polemics in (4) now motivated by (8).

3. The talk will only consider three problems arising from the first part of (2), namely:

- From where does (*) come as a "classical" logical law?
- Which theories of fuzzy sets admit (*) as a law, and when can it be reached by mixing connectives?
- When is there an implication $\rightarrow$ such that (*) can be rewriten as $p \rightarrow q = q + p' . q'$?

It should be pointed out that the theoretical argument in the first part of (2) is, with numerical truth-values as it is done there, a triviality that says nothing on fuzzy logic, but that with fuzzy sets p and q, the question is not so trivial and formula (*) deserves to be reconsidered. In such a line, the talk will proceed through the following:

CONTENTS

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Universes of fuzzy sets—a survey of the different approaches

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Approaches toward the development of universes of fuzzy sets which are closed under the formation of fuzzy subsets and which know set algebraic operations which are based upon t-norms (or something similar), are intended to provide "closed worlds" for fuzzy set theories and to make precise in this way the notion of fuzzy set of higher level.

The methods to attack this problem of the construction of a fuzzy analogue to the cumulative universe of crisp sets fall essentially into three classes:

- approaches which try to form cumulative universes of fuzzy sets rather similar to the construction of the cumulative universe of sets via an transfinite iteration of the power set operation;
- approaches which intend to give axiomatizations of the theory of fuzzy sets;
- approaches which try to form cumulative universes of fuzzy sets rather similar to Boolean valued models for classical set theory;
- approaches which intend to suitably generalize the categorical characterization of the category SET of all sets and mappings to a similar characterization of some category FSET of all fuzzy sets and of suitable mappings between them.

There is a wealth of such approaches. The most important ones shall be discussed, some recent results and some possibilities for generalizations explained, and some open problems mentioned.

Embedding standard *BL*-algebras into non-commutative pseudo-*BL*-algebras

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BL-algebras are algebras of truth functions of the basic fuzzy logic *BL* [5]. Each continuous *t*-norm defines a standard *BL*-algebra (or *t*-algebra) on the real interval $[0,1]$ (with its standard ordering). As proved in [3], *BL*-tautologies (propositional formulas being tautologies over each *BL*-algebra) are the same as *t*-tautologies (standard *BL*-tautologies). Speaking algebraically, the variety of *BL*-algebra is generated by the class of *t*-algebras.

Di Nola, Georgescu and Iorgulescu [1, 2] introduced and studied pseudo-*BL*-algebras (briefly, *psBL*-algebras), a generalization of *BL*-algebras not assuming commutativity of the semigroup operation (truth function of conjunction). The corresponding propositional logic was established in [6, 7].

As shown in [4], there are no non-commutative standard *psBL*-algebras, i.e. *psBL*-algebras whose lattice reduct is the standard real interval $[0,1]$.In [7] I gave an example of a non-commutative *psBL*-algebra on the "nonstandard" unit interval in which each standard element of $[0,1]$ has continuum of "infinitely near" non-standard elements; $NS[0,1]$ is the set of pairs

$$\begin{array}{ll} \{(0,y)| & y \in Re, y \geq 0\} \cup \\ \{(x,y)| & 0 < x < 1, y \in Re\} \cup \\ \{(1,y)| & y \in Re, y \leq 0\} \end{array}$$

with lexicographic order (standard elements being the pairs $(x,0)$). The example is a pseudo-*MV*-algebra in the terminology of [1] and its standard elements form a standard *BL*-algebra (modulo the representation of $x \in [0,1]$ by the pair $(x,0)$). I asked at the end of [7] if each standard *BL*-algebra is embeddable in this way into a non-commutative *psBL*-algebra on the non-standard unit interval $NS[0,1]$. Our result is the following:

Theorem 1. *For each continuous t-norm $*$ having at least one non-idempotent element there is a non-commutative psBL-algebra* $\mathbf{A}$ *on* $NS[0,1]$ *whose reduct to* $[0,1]*\{0\}$ *is isomorphic to the standard BL-algebra* $[0,1]_*$ *via the identification of* $x \in [0,1]$ *with the pair* $(x,0)$.

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Group-like structures on M-valued sets

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Let $M = (L, \leq, *)$ be a *GL*-algebra. Typical examples are complete Heyting algebras or continuous t-norms on the real unit interval. Further, let *M*-**SET** be the category of *M*-valued sets (cf. [3]). It is not difficult to see that *M*-**SET** is a monoidal category in which the unit object does not coincide with the terminal object. The axioms of group-like structures on *M*-valued sets will make use of this monoidal structure on *M*-**SET**. Among other things we are able to establish the following facts:

1. The axioms of group-like structures are preserved under the so-called tilde-construction which assigns to each *M*-valued set its singleton space (cf. [3]).
2. Fuzzy groups in the sense of J.M. Anthony and H. Sherwood are canonical subgroup-like structures (cf. [1]).
3. In the case of complete Heyting algebras separated presheaves of groups form a natural class of group-like structures (in the case of lattices of open subsets see also [2]) .
4. Probabilistic normed spaces induces group-like structures in a natural way (cf. [5]).

Even though group-like structures are not group structures in the categorical sense of *M*-**SET**, we are convinced that these structres will play a non trivial role in algebraic theories based on nonclassical logics.

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How to construct left-continuous triangular norms—state of the art 2002

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1 Introduction

Triangular norms (t-norms for short) play a crucial role in several fields of mathematics and AI. For an exhaustive overview on t-norms we refer to [23]. Recently an increasing interest of *left-continuous* t-norm based theories can be observed (see e.g. [3, 6, 7, 8, 9, 10, 21]. The condition of left-continuity is a frequently cited property and plays a central role in all the fields that use t-norms. The role of left-continuous t-norms with strong associated negations is even more relevant, since then the negation, which is associated to the t-norm is an involution, and hence one can define a t-conorm via the de Morgan rule. In spite of their significance, the knowledge about left-continuous t-norms was rather poor for a long time; there were no results in the literature where left-continuous t-norms stood as the focus of interest. Moreover, until 1995 there were no known examples for left-continuous t-norms, except for the standard class of continuous t-norms. Continuous t-norms have become well understood from the famous and widely cited paper of Ling, as ordinal sums of continuous Archimedean t-norms [25] and have been used in several applications. The poor knowledge about left-continuous t-norms on one hand and the good understanding of continuous t-norms on the other hand result in the use of continuous t-norms when left-continuity would be sufficient in theory. This very much restricts the freedom of choice when the proper operation has to be found in the mathematical setting in question. In other words, this makes modeling, e.g., in probabilistic metric spaces, in game theory, in the theory of non-additive measures and integrals, in the theory of measure-free conditioning, in fuzzy set theory, in fuzzy logic, in fuzzy control, in preference modeling and decision analysis, and in artificial intelligence much less flexible.

In this paper we discuss in detail the presently existing construction methods which result in left-continuous triangular norms. The methods are (together with their sources):

- annihilation [4, 15, 2] and [23] (Proposition 3.64)
- ordinal sum of t-subnorms [14, 12, 24],
- rotation contruction [17, 11],
- rotation-annihilation construction [18],
- embedding method [20, 9].

An infinite number of left-continuous triangular norms can be generated with these constructions (and with their combinations), which provides a tremendously wide spectrum of choice for e.g. logical and set theoretical connectives in non-classical logic and in fuzzy theory. By using these methods (consecutive combination of them is as well possible) an infinite number of new left-continuous t-norms can be generated. Some of them has the additional advantage that the associated negation of the resulted t-norm is strong, which may be useful in logical applications. The resulted operations can be admitted into the attention of researchers of algebra, probabilistic metric spaces, non-classical measures and integrals, non-classical logics, fuzzy theory and its applications.

Acknowledgements

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Annex

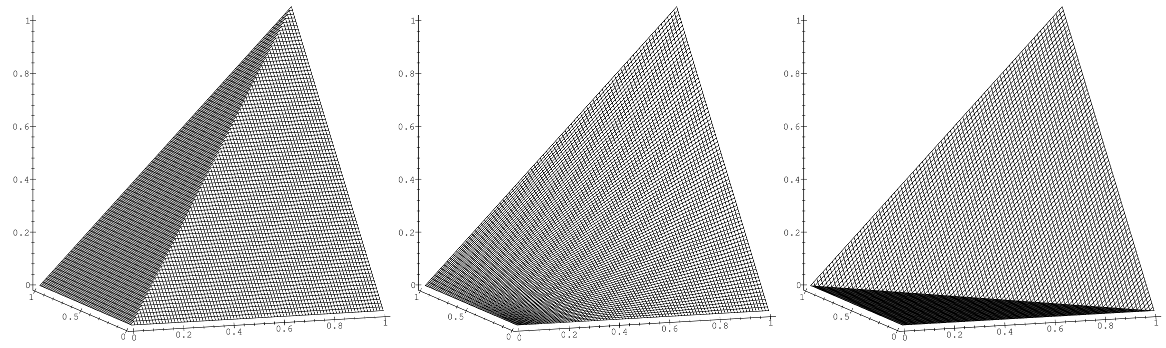

Figure 1: Minimum $T_{\mathbf{M}}$ (left), product $T_{\mathbf{P}}$ (center) and Łukasiewicz t-norms $T_{\mathbf{L}}$ (right)

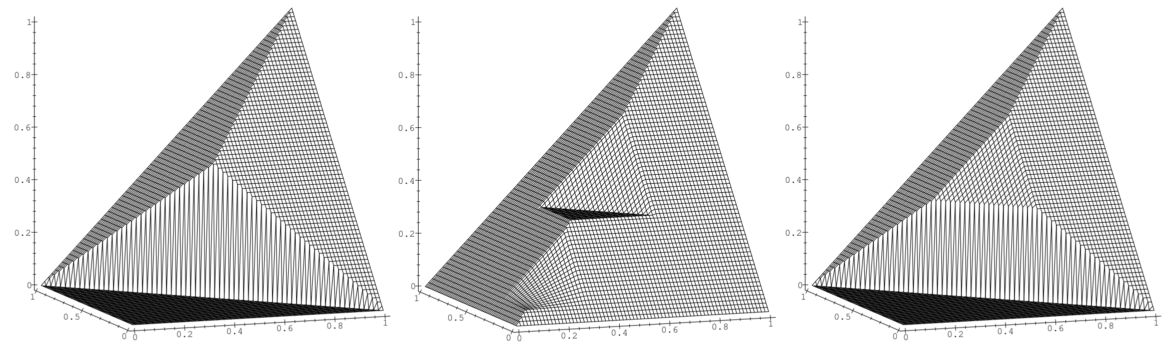

Figure 2: The nilpotent minimum $T_{\mathbf{M}_0}$ (left), a continuous t-norm (center) and its annihilation $T_{\mathbf{J}}$ (right)

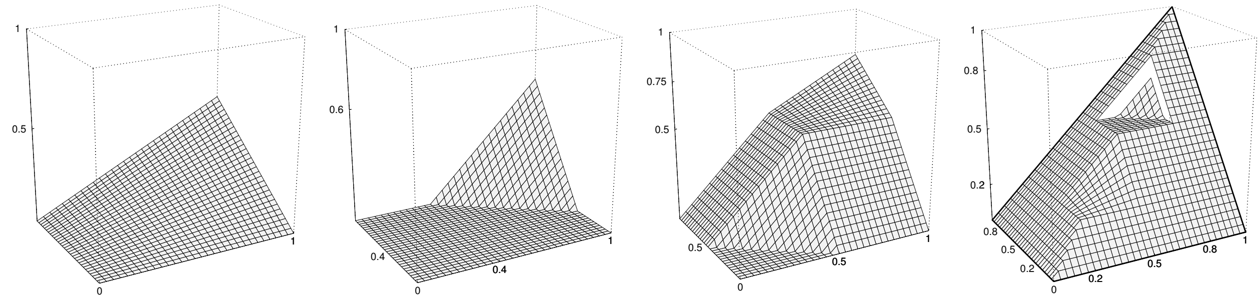

Figure 3: $T_{\mathbf{P}_{0.5}}$ and $T_{\mathbf{L}_{0.4}}$ (left). A t-subnorm and a t-norm, which are ordinal sums of t-subnorms (right).

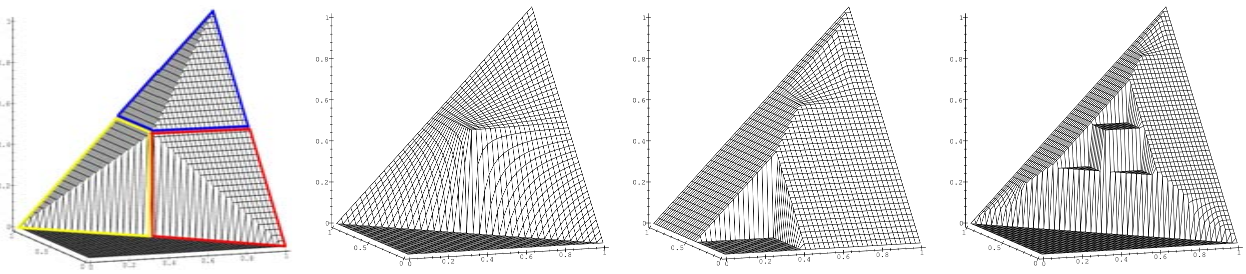

Figure 4: $(T_{\mathbf{M}})_{\mathbf{Rot}}$ and $(T_{\mathbf{P}})_{\mathbf{Rot}}$ (left), a t-norm with zero divisors and its rotation (right)

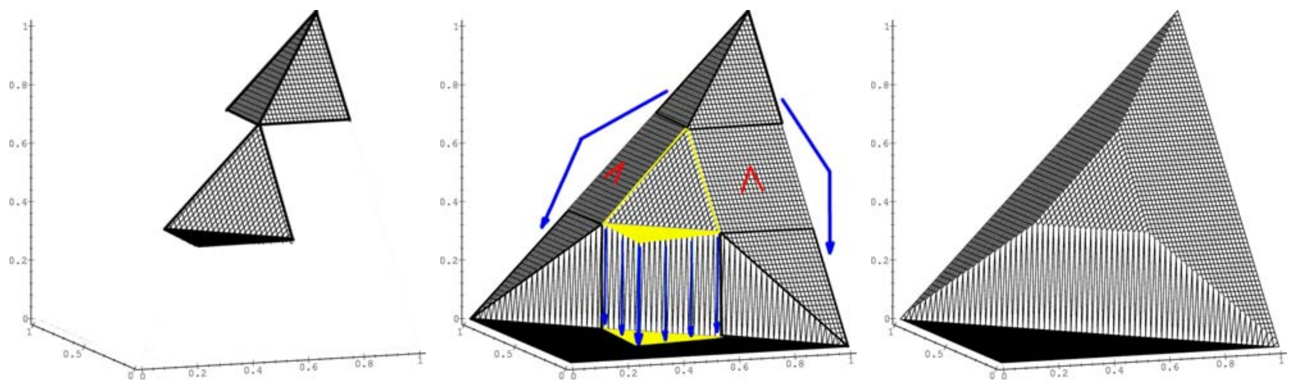

Figure 5: Geometrical explanation of the rotation-annihilation construction

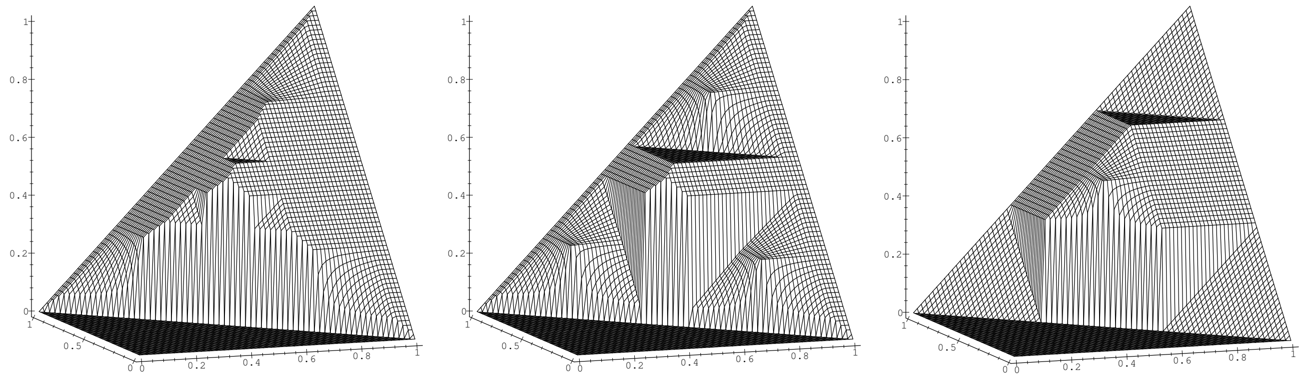

Figure 6: Rotations of ordinal sums

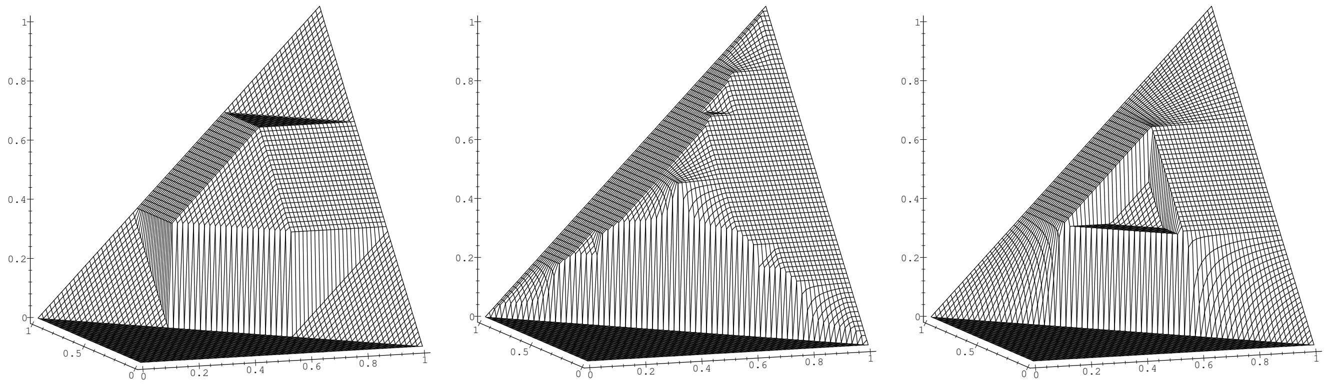

Figure 7: T-norms generated by the rotation-annihilation construction

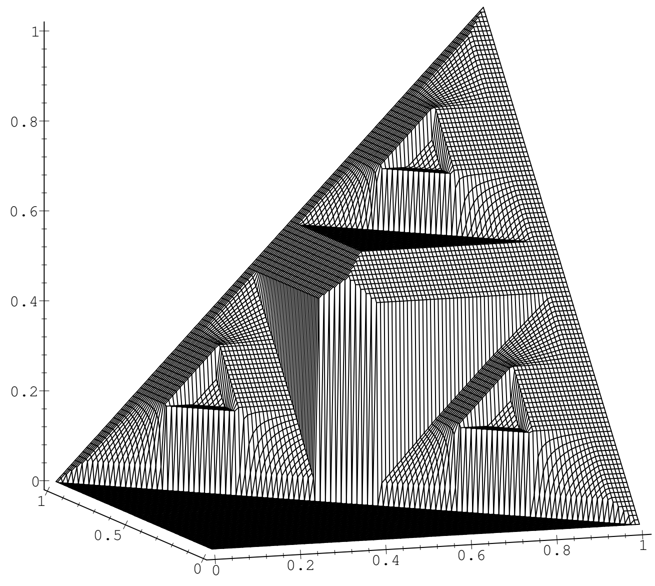

Figure 8: Combination of rotation-annihilation and rotation

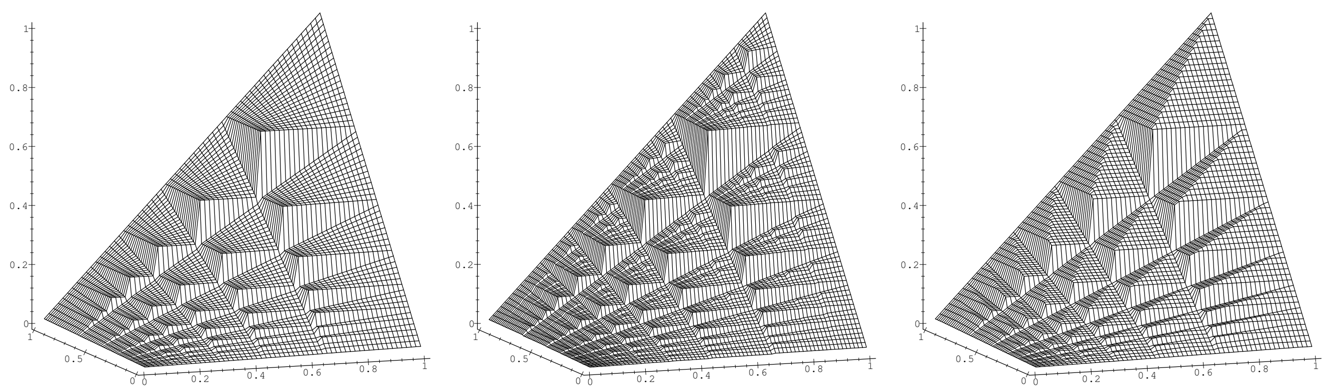

Figure 9: Hájek t-norm $(T_{\mathbf{P}})_{\langle + \rangle}$ (left), $(T_{\mathbf{P}})_{\langle +,+ \rangle}$ (center) and $(T_{\mathbf{M}})_{\langle + \rangle}$ (right

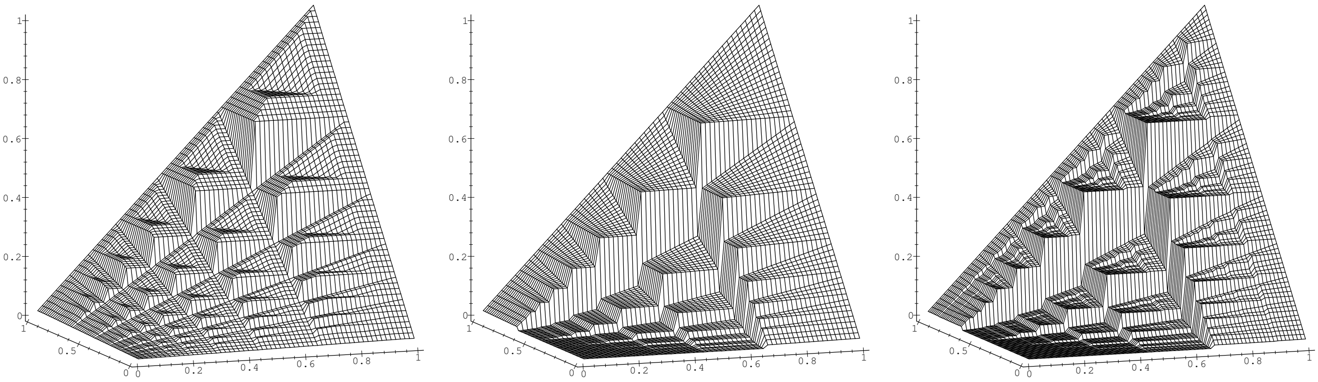

Figure 10: $(T_{\mathbf{os}})_{\langle + \rangle}$ (left), $(T_{\mathbf{P}})_{\langle \oplus_{\mathrm{x}} \rangle}$ (center) and $(T_{\mathbf{P}})_{\langle \oplus_{\mathrm{x}}, \oplus_{\mathrm{x}} \rangle}$ (right)

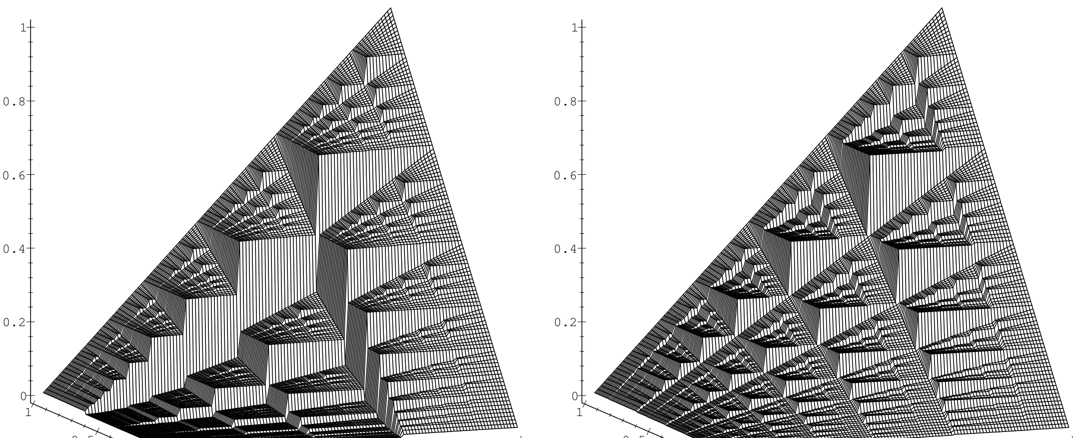

Figure 11: $(T_{\mathbf{P}})_{\langle +, \oplus_{\mathrm{x}} \rangle}$ (left) and $(T_{\mathbf{P}})_{\langle \oplus_{\mathrm{x}}, + \rangle}$ (right)

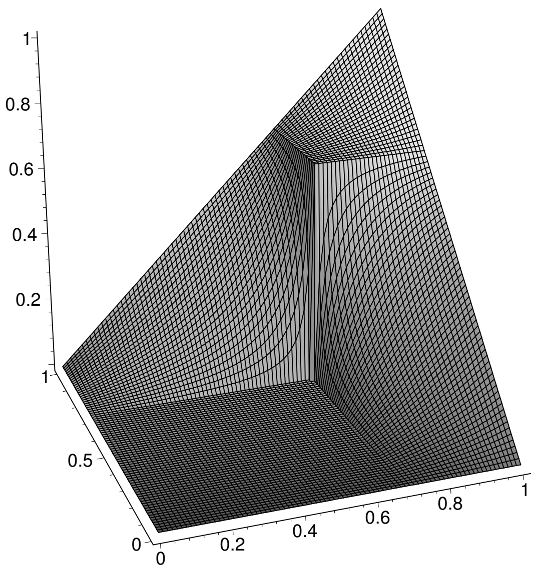

Figure 12: A t-norm which is obtained via rotation of a mean (the 3Pi operator)

Triangular norms, uni- and nullnorms, balanced norms: the cases of the hierarchy of iterative operators

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Abstract

A new hierarchy of the fuzzy operators has been proposed in this paper. This interpretation was based on the observation of non-symmetry of fuzzy operators as, for instance, triangular norms. The starting point of this approach is based on the method of symmetrization relaying on spreading out negative information from the point 0 to the interval $[-1,0)$. Based on this assumption, the normal and weak forms of balanced triangular norms are defined in the paper. Relations between normal form of balanced triangular norms and uninorms and nullnorms are studied. It is shown that balanced triangular norms, uni- and nullnorms are cases of generalized operators, so called iterative triangular norms.

1 Preliminaries

The operators investigated in this paper relay on their axiomatic definitions and differences between these definitions. Thus, in this Chapter definitions and selected properties of triangular norms, uninorms and nullnorms as well as balanced triangular norms are recalled. It is assumed that reader is accustomed with basic knowledge of triangular norms, uni- and nullnorms and balanced triangular norms.

1.1 Triangular norms - definition

Triangular norms, i.e. t-norms and t-conorms, in their classical meaning, are mappings from the unit square $[0,1]\times[0,1]$ onto the unit interval $[0,1]$ satisfying axioms of associativity, commutativity, mononicity and boundary conditions (cf. [5, 7] for details), i.e.:

Definition 1. t-norms and t-conorms are mappings $p:[0,1]\times[0,1]\to[0,1]$, where p stands for both t-norm and t-conorm, satisfying the following axioms:

1.	$p(a,p(b,c)) = p(p(a,b),c)$	associativity
2.	$p(a,b) = p(b,a)$	commutativity
3.	$p(a,b) \le p(b,a)$ if $a \le c$ and $b \le d$	monotonicity
4.	$t(1,a) = a$ for $a \in [0,1]$ and $b \le d$	boundary condition for t-norm
	$s(0,a) = a$ for $a \in [0,1]$ and $b \le d$	boundary condition for t-conorm

t-norms and t-conorms are dual operations in the sense that for any given t-norm t, we have a dual t-conorm s defined by the De Morgan formula $s(a,b) = 1 - t(1-a, 1-b)$ and vice-versa, for any given t-conorm s, we have a dual t-norm t defined by the De Morgan formula $t(a,b) = 1 - s(1-a, 1-b)$. Duality of triangular norms causes duality of their properties. Note that the max/min are pairs of dual t-norms and t-conorms.

1.2 Uninorms and nullnorms

Uni-norms were introduced in [8] as a unification and generalization of the triangular norms. Definition of uninorms is derived from definition of triangular norms with boundary condition varieted. Namely:

Definition 2. Uninorm is a mapping: $u : [0,1] \times [0,1] \to [0,1]$ satisfying the following axioms:

1., 2., 3.	associativity, commutativity and monotonicity	
4.	$(\exists e \in [0,1])$ such that for all $x \in [0,1] u(x,e) = x$	identity element

It is clear that a t-norm is a special uninorm with identity element $e = 1$ and a t-conorm s is a special uninorm with identity element $a = 0$.

The definition of nullnorms differs from the definition of uninorms in boundary condition:

Definition 3. Nullnorm is a mapping: $u : [0,1] \times [0,1] \to [0,1]$ satisfying the following axioms:

1., 2., 3.	associativity, commutativity and monotonicity
4.	$(\exists a \in [0,1])$ such that $(\forall x \in [0,a]) u(x,0) = 0$ and $(\forall x \in [a,1]) u(x,1) = x$ neutral element

Obviously, a t-norm is a special nullnorm with neutral element $a = 0$ and a t-conorm s is a special nullnorm with neutral element $a = 1$. Assuming that u is a uninorm with identity e and if v is defined as $v(x,y) = 1 - u(1-x, 1-y)$, then v is a uninorm with identity $1-e$. v is called the dual uninorm of u. This fact shows that difference between uninorm and its dual analogue is only quantitative. This means that they are similar from the perspective of global properties discussed in the paper. So that duality will not be considered in the paper.

Assuming that u is a uninorm with identity e:

1.	$u(a,0) = 0$	for all $a \le e$	and $u(a,1) = 1$ for all $a \ge e$
2.	$x \le u(x,y) \le y$	for all $x \le e$	and $e \le y$
3.	either $u(0,1) = 0$	or $u(1,0) = 1$	

Uninorms generalize the concept of triangular norms. According to [2], assuming that u is a uninorm with identity $e \in (0,1)$, the mappings t_u and s_u are t-norm and t-conorm respectively:

$$t_u(x,y) = \frac{u(ex,ey)}{e} \qquad and \qquad s_u(x,y) = \frac{u(e+(1-e)x, e+(1-e)y)}{1-e} \tag{1}$$

or equivalently:

$$u_u(x,y) = e\,t\,(\frac{x}{e}, \frac{y}{e}) \quad for \quad x,y \in [0,e] \quad and$$

$$u_u(x,y) = e + (1-e) s_u(\frac{x-e}{1-e}, \frac{y-e}{1-e}) \quad for \quad x,y \in [e,1] \tag{2}$$

The isomorphic mapping $h(x) = 2x - 1$ (and its inverse $h^{-1}(x) = (x+1)/2)$) transforms uninorms and nullnorms into the interval $[-1,1]$ with respective values of unit and neutral elements equal to $\overline{e} = 2e - 1$ and $\overline{a} = 2a - 1$, respectively. It is easily seen that the isomorphic mappings:

$$h(x) = \begin{cases} (x-e)/e \\ (x-e)/(1-e)] \end{cases} \quad and \quad h^{-1}(x) = \begin{cases} e(x+1)/e \\ (1-e)x+e \end{cases} for \begin{cases} x \in [0,e] \\ x \in [e,1] \end{cases} \tag{3}$$

will transform uninorms (and nullnorm) to their symmetrized versions with unity and neutral elements equal to 0.

Comment: nullnorms satisfy similar properties, cf. [5].

1.3 Balanced triangular norms in normal form

The definition of balanced triangular norms in normal form, as introduced in [3], is derived from the definition of triangular norms. The domain of balanced triangular norms is extended to the square $[-1,1] \times [-1,1]$. Balanced triangular norms are identical with classical triangular norms on the unit square $[0,1] \times [0,1]$ and satisfy axioms of associativity, commutativity and monotonicity on the whole domain $[-1,1] \times [-1,1]$, boundary conditions are exactly the same as in case of classical triangular norms. An extra symmetry axiom supplements the definition, also cf. [4]. Additional operator of balanced negation is introduced.

Definition 4. Balanced operators are defined as follow:
Balanced negations is the mapping:

$N : [-1,1] \to [-1,1] \qquad N(x) = -x$

Balanced t-norms and t-conorms are mappings

$P : [-1,1] \times [-1,1] \to [-1,1]$

satisfying the following axioms, where P stands for both balance t-norm T and t-conorm S:

1., 2., 3.	associativity, commutativity and monotonicity	
4.	$T(1,a) = a, \quad S(0,a) = a \quad for \quad a \in [0,1]$	boundary conditions
5.	$P(x,y) = N(P(N(x),N(y)))$	symmetry

Conclusion 5. Axiomatic definition of balanced t-norm and balanced t-conorm restricted to the unit square $[0,1] \times [0,1]$ are equivalent to the classical t-norm and classical t-conorm, respectively.

Conclusion 6. Balanced t-norm and balanced t-conorm restricted to the square $[-1,0] \times [-1,0]$ are isomorphic with the classical t-conorm and classical t-norm, respectively.

Conclusion 7. Balanced t-norm vanishes on the squares $[-1,0] \times [0,1]$ and $[0,1] \times [-1,0]$.

The above conclusions are obvious.

1.4 Balanced triangular norms in weak form

The weak system of the balanced triangular norms satisfies a collection of axioms of the normal system except of the properties 1 and 3, i.e. axioms of associativity and monotonicity of the defintion of balanced t-norm in normal form. The following sets of axioms defining balanced t-norm in its weak form completes the definition.

Definition 8. The weak form of balanced triangular norm, t-norm, satisfies the following set of axioms:

1.	$min(\lvert T(a,T(b,c))\rvert, \lvert T(T(a,b),c)\rvert) \leq$ $\lvert T(a,T(b,c))\rvert, \lvert T(T(a,b),c)\rvert \leq$ $T(T(\lvert a\rvert,\lvert b\rvert),\lvert c\rvert) = T(\lvert a\rvert, T(\lvert b\rvert,\lvert c\rvert))$	semi-associativity
2.	$T(a,b) = T(b,a)$	commutativity
3.	$T(a,b) \leq T(c,d) \quad for \quad 0 \leq a \leq c, 0 \leq b \leq d$	semi-monotonicity
4.	$T(1,a) = a, \quad S(0,a) = a \quad for \quad a \in [0,1]$	boundary conditions
5.	$P(x,y) = N(P(N(x),N(y)))$	symmetry

Comment: Balanced triangular norms in weak form satisfy Conclusion 1.1 and 1.2. However, Conclusion 1.3. is not satisfied.

2 Balanced triangular norms versus uninorms and nullnorms

The balanced t-conorms, as defined in the section 1.3, are special cases of uninorms in the sense of the isomorphism defined in the formula 3. Amazingly, balanced triangular norms as well as uninorms and nullnorms are similar products of two different paths of thinking, paths that begin in two different starting points. Detailed properties of balanced triangular norms and uninorms and nullnorms might differ. Despite of this, the general meaning of balanced triangular norms and of uninorms and nullnorms are the same in the sense of isomorphic mapping between them.

The definition of balanced t-conorm includes the symmetry axiom in addition to other axioms that are common for uninorm and balanced t-conorm: associativity, commutativity, monotonicity and boundary conditions. The extra restriction - i.e. the symmetry axiom - makes that not every uninorm is isomorphic with a balanced t-conorm while every t-conorm is isomorphic with a uninorm. Precisely, every balanced t-conorm is isomorphic with a set of uninorms that satisfy the symmetry axiom and differ in the unit elements. Of course, any two uni-norms of such a set are isomorphic in the sense of an isomorphism analogous to that defined in the formula 3. Two sets of uninorms related to any two balanced t-conorms are disjoint assuming that respective balanced t-conorms are different. Moreover, the set of uninorms that are not isomorphic with any balanced t-conorm and the sets of uninorms related to balanced t-conorms partition the set of all uninorms, i.e. they create equivalence classes of an equivalence relation. The same notes concerns balanced t-norms and nullnorms

The following propositions describe the characteristic of the set of all balanced t-conorms (balanced t-norms in normal form) as a family of equivalence classes of the relation $\approx_S$ ($\approx_T$, respectively) defined on the set of all uninorms (nullnorms, respectively).

Proposition 9. *Let $U = \{u : u \text{ is a uninorm}\}$. Let us consider isomorphic mappings as defined in the formula 3. Then, the pair $(U, \approx_S)$ is an equivalence relation if for every two uninorms u and v, $u \approx_S v$ iff u and v are isomorphic with the same balanced t-conorm S or none of u and v is isomorphic with any balanced t-conorm S.*

Proposition 10. *Let $V = \{v : v \text{ is a nullnorm}\}$. Let us consider isomorphic mappings as defined in the formula 3. Then, the pair $(V, \approx_T)$ is an equivalence relation if for every two uninorms u and v, $u \approx_S v$ iff u and v are isomorphic with the same balanced t-norm T or none of u and v is isomorphic with any balanced t-norm T.*

3 A hierarchy of balanced operators

In this Chapter the method of balanced extension of fuzzy operators is applied to uninorms. Comparing relations between uninorms, balanced t-conorms and balanced uninorms (created with the method of balanced extension), leads to a broader family of balanced operators, so called iterative norms.

3.1 Balanced uninorms

Definition 11. Balance uninorm is a mapping: $U : [-1,1] \times [-1,1] \to [-1,1]$ satisfying the following axioms:

1., 2., 3.	associativity, commutativity and monotonicity	
4.	$(\exists e \in [0,1])$ such that for all $x \in [0,1] u(x,e) = x$	identity element
5.	$U(x,y) = N(U(N(x),N(y)))$	symmetry

As in case of balanced triangular norms, the values of balanced uninorms on the squares $[0,1] \times [0,1]$ and $[-1,0] \times [-1,0]$ are determined by the values of uninorm and symmetry principle. The values of balanced uninorm on the squares $[0,1] \times [-1,0]$ and $[-1,0] \times [0,1]$ are unconstrained and could be defined according to subjective aim of application.

Obviously, similar considerations are valid in case of nullnorms, though the values of balanced nullnorms in the unconstrained area meet different type of border conditions.

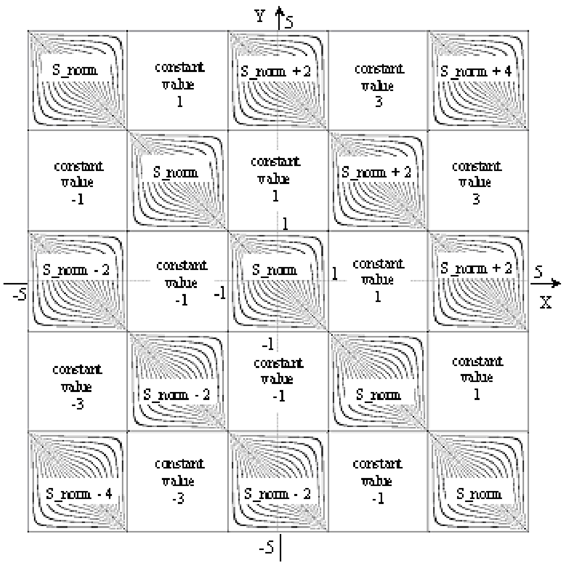

Figure 13: The plot of iterative t-conorm based on the additive generator

3.2 A hierarchy of balanced operators

Balanced triangular norms are isomorphic with uninorms and nullnorms. Thus, the method of balanced uninorms creation (i.e. immersion of classical uninorms in the extended space of fuzzy sets) could be replaced by running the process of double utilization of this method to a classical t-conorm. The first stage of this process creates balanced t-conorm, then - after isomorphic transformation of balanced t-conorm to the unipolar scale, i.e. to the respective uninorm - balanced uninorm could be created.

In light of the idea of balanced extension of fuzzy sets, uninorm (as a fuzzy operator) could be subjected to balanced extension method to produce balanced uninorm. This means that balanced uninorm is a result of two iterations of balanced extension method applied to classical t-conorm. Thus, balanced uninorm is a kind of balanced t-conorm of the higher rank. The process could be continued creating next ranks of balanced t-conorms. It means that balanced triangular norms, uninorms and nullnorms are products of the same process of iterative balanced extension method applied to classical triangular norms. This property explains similarity between balanced triangular norms, on one hand, and uninorms and nullnorms, on the other hand. The process of consecutive applications of balanced extension method creates a hierarchy of balanced triangular norms. A new function, so called iterative t-conorm, will be used as illustration of creation of balanced hierarchy.

Definition 12. The iterative t-conorm is a function $S_{iter} : R \times R \to R$

$$S_{iter}(x,y) = \begin{cases} S(x-2k-2l, y+2k-2l) & (x-2k-2l, y+2k-2l) \in [-1,1] \times [-1,1] \\ & \text{and k.l-integers} \\ 1+2l & (x-2k-2l, y+2k-2l) \in [1,3] \times [-1,1] \\ & \text{and k.l-integers} \end{cases}$$

where S is a balanced t-conorm.

Note: balanced t-conorm S in the above definition could vary for different areas of the domain. Thus, in this case, the formula looks like:

$$S_{iter}(x,y) = \begin{cases} S_{k,l}(x-2k-2l, y+2k-2l) & (x-2k-2l, y+2k-2l) \in [-1,1] \times [-1,1] \\ & \text{and k.l - integers} \\ 1+2l & (x-2k-2l, y+2k-2l) \in [1,3] \times [-1,1] \\ & \text{and k.l - integers} \end{cases}$$

where $S_{k,l}$ is a balanced t-conorm for all values of k and l.

Properties of iterative t-conorm are determined by balanced t-conorm. For instance, continuity of iterative t-conorm S_{iter} is determined by continuity of basic balanced t-conorm. Iterative t-conorm S_{iter} may be non-continuous in all non-continuity points of balanced t-conorm and on the borders of upper-left and bottom-down quarters of the domain squares growing values of balanced t-conorm S. Iterative t-conorm S_{iter} is definitely non-continuous in upper-left and bottom-down vertexes of those squares where balanced t-conorm S is increasing.

Example: since balanced t-conorm S based on the additive generator $f_S(x) = x/(1-|x|)$ is non-continuous in upper-left and bottom-down vertexes of its domain, the respective iterative t-conorm S_{iter} is also non-continuous in all such points. Specifically, S_{iter} is a continuous function in its domain except left-upper and right bottom vertexes of the squares $\{[-1+2k+2l, 1+2k+2l] \times [-1-2k+2l, 1-2k+2l] : k,l - integer\, values\}$

The contour plot of the iterative triangular norm based on the above t-conormis shown in the Figure 1.

The Figure 2illustrate the process of creation of the hierarchy of balanced t-norms and balanced t-conorms based on iterative triangular norms. Because balanced t-norm and balanced t-conorm of any given rank have the square $[-1,1]\times[-1,1]$ as their domain, then a part of the iterative triangular norm defined by respective squares displayed in the Figures 2 and 3 must be transformed in order to satisfy the fuzzy operator domain and co-domain. For instance, a balanced t-conorm of the rank 2 described by the part of iterative triangular norm restricted to the square $[-5,-1]\times[-1,3]$:

$$fun : [-5,-1]\times[-1,3] \qquad fun(x,y) = S_{iter}(x,y) \tag{4}$$

must be transformed using the transformation:

$$\begin{gathered} t_x : [-1,1] \to [-5,-1], t_x(x) = 2x-3 \quad \text{and} \quad t_y : [-1,1] \to [-1,3], t_y(y) = 2y+1 \\ t^{-1} : [-3,1] \to [-1,1], t^{-1}(x) = (x+1)/2 \end{gathered} \tag{5}$$

what means that the balanced t-conorm of rank 2 $S^{(2)}$ respective to the mapping fun is defined as follow:

$$\begin{gathered} S^{(2)} : [-1,1]\times[-1,1] \to [-1,1], \\ S^{(2)}(x,y) = (t^{-1}\circ S_{iter}\circ(t_x,t_y))(x,y) = t^{-1}(fun(t_x(x),t_y(y))) \end{gathered} \tag{6}$$

In other words, the graph of mapping fun included in the cube $[-5,-1]\times[-1,3]\times[-3,1]$ has to be squeezed to the cube $[-1,1]\times[-1,1]\times[-1,1]$ in order to create balanced t-conorm of rank 2.

In the Figure 1 the balanced t-conorm of rank 2 is also marked as uninorm what should be interpreted as rela-tion between balanced t-conorms and uninorms in terms of the Chapter 2. On the other hand, slightly modified iterative t-conorms and t-norms could be used for creation a hierarchy of balanced operators including all uni-norms and nullnorms. This issue, as a subject of potential subject of investigation, is out of the scope of the aim of this paper. So then it will not be developed here.

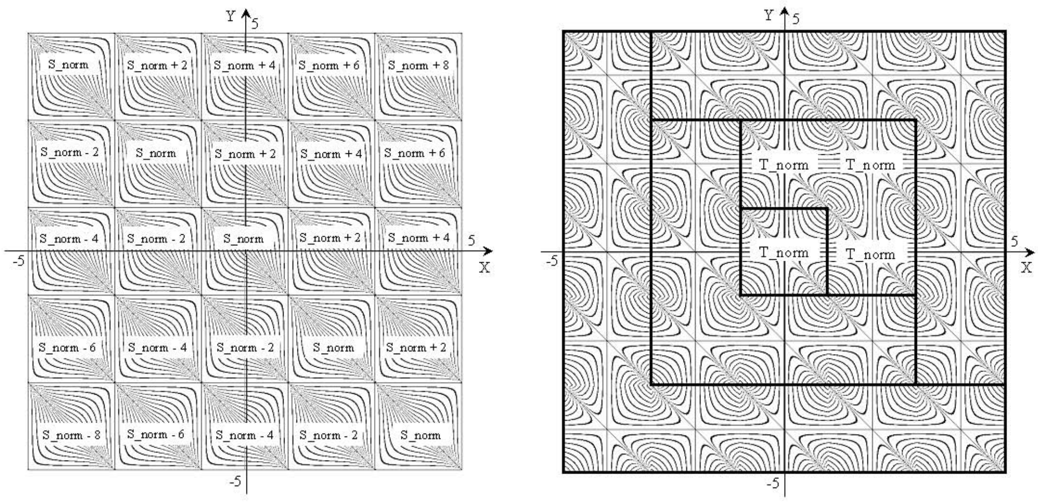

Figure 14: The hierarchy of balanced t-conorms and balanced t-norms

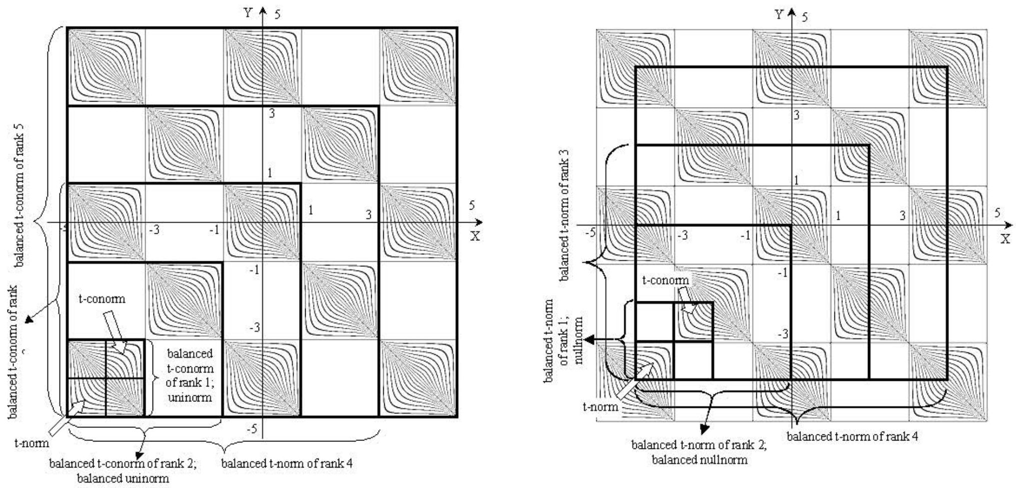

Figure 15: The structure of uniform iterative t-conorms and uniform iterative t-norms in weak form

In the Figure 3 uniform iterative triangular norms S_{iter} and T_{iter} are presented. Uniform norms are based on weak balanced triangular norms. They do not have plain regions, i.e. squares of constant values, as it is in case of ordinary iterative triangular norms based on balanced t-conorms:

$$\begin{aligned} &S_{iter} : R \times R \to R, \\ &S_{iter}(x,y) = S(x-2k, y-2l), \quad (x-2k, y-2l) \in [-1,1] \times [-1,1], k.l-integers \end{aligned} \tag{7}$$

$$\begin{aligned} &T_{iter} : R \times R \to [-1,1], \\ &T_{iter}(x,y) = T(x-2k, y-2l), \quad (x-2k, y-2l) \in [-1,1] \times [-1,1], k.l-integers \end{aligned} \tag{8}$$

4 Conclusions

Relations between different fuzzy operators: triangular norms, uninorms and nullnorms, balanced triangular norms are studied in this paper. Dependencies between uninorms / nullnorms and balanced triangular norms are investigated. The triangular norms, uninorms and nullnorms, balanced triangular norms are subjected to a process of iterations of balanced transformation. The triangular norms, uninorms and nullnorms, balanced triangular norms are placed in the broader hierarchy of iterative operators.

Several topics were signalized in the paper: properties of weak systems of balanced fuzzy sets and balanced triangular norms, properties of iterative triangular norms, relations between balanced operators and iterative triangular norms and other fuzzy operators, applications of balanced systems of fuzzy sets to practical aims. These topics are potential subjects of further studies.

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Triangular subnorms and residual implications

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1 Introduction

Triangular norms were introduced in [19], for an exhaustive overview see the monograph [13]. Applications of triangular norms in fuzzy logic, probabilistic metric spaces, etc., require the left-continuity of the applied t-norm, see e.g. [13]. Recently, several new types of constructions of left-continuous t-norms were introduced, see [11] for an overview. One of these methods is linked to the ordinal sum of t-subnorms introduced in [10]. Note that due to [14], this method is the most general method yielding a t-norm based on Clifford's ordinal sum of semigroups [2].

Observe also that the structure and some constructions of t-subnorms (introduced in [8]) were investigated first in [17], though several important facts about t-subnorms can be straightforwardly derived from results of [13], Chapter 3.

The left-continuity of t-norms is crucial for the existence of the corresponding residual implications. The main aim of this paper is a discussion of these residual implications linked to t-norms which are ordinal sums of semigroups. Recall that the structure of residual implications linked to continuous t-norms, i.e., to ordinal sums of continuous Archimedean t-norms, was studied in [3], where also ordinal sums of residual implications were introduced, compare also [5].

The paper is organized as follows. The next section recalls some results about t-norms, t-subnorms and their ordinal sums. In the third section, the structure of residual implications linked to ordinal sums of left-continuous t-subnorms is studied. Finally, the residual operators related to t-subnorms generated by continuous additive generators are investigated.

2 Triangular norms as ordinal sums of semigroups

Triangular norms as ordinal sums of semigroups in the sense of Clifford [2] have been investigated in [14]. As observed there, these triangular norms can be expressed as ordinal sums of t-subnorms introduced in [8, 10].

Definition 1. A mapping $R : [0,1]^2 \to [0,1]$ is called a t-subnorm whenever it is commutative, associative, non-decreasing and bounded by its arguments, i.e.,

$$R(x,y) \leq x \qquad \text{for all } x,y \in [0,1]. \tag{1}$$

Evidently, each t-norm T is also a t-subnorm. Moreover, for any t-norm T and $c \in]0,1[$, the operation $T_c : [0,1]^2 \to [0,1]$ given by

$$T_c(x,y) = \frac{T(cx,cy)}{c} \tag{2}$$

is a t-subnorm. Note also that because of the commutativity the boundary condition (1) is equivalent to

$$R(x,y) \leq \min(x,y) \qquad \text{for all } x,y \in [0,1]. \tag{3}$$

Note that several notions introduced for t-norms can be directly introduced for t-subnorms, too, and hence we will not define them explicitly. Examples of such notions and properties are: zero divisors, strict monotonicity, Archimedean property, several types of continuities, etc.

Recall that due to [10, 13] each t-norm can be expressed as an ordinal sum of t-subnorms. We present this result for left-continuous t-norms.

Theorem 2. *A mapping $T : [0,1]^2 \to [0,1]$ is a left-continuous t-norm if and only if there is a system $(]\alpha_k, \beta_k[)_{k\in\mathcal{K}}$ of pairwise disjoint non-empty subintervals of [0,1] and a system of left-continuous t-subnorms $(R_k)_{k\in\mathcal{K}}$ such that if either $\beta_k = 1$ for some $k \in \mathcal{K}$ or $\beta_k = \alpha_{k^*}$ for some $k, k^* \in \mathcal{K}$ and R_{k^*} has zero divisors then R_k is a t-norm, so that*

$$T(x,y) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) R_k \left(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k} \right) & \text{if } x,y \in]\alpha_k, \beta_k], \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{4}$$

Observe that the problem of complete characterization of left-continuous t-subnorms is equivalent to the complete characterization of left-continuous t-norms, and thus still unsolved. However, in some special cases such a characterization is already known. Recall the characterization of continuous Archimedean t-norms by means of additive generators [16], which are continuous strictly decreasing from [0,1] to $[0,\infty]$ mappings with value 0 at argument 1 (this fact reflects the property of constant 1 which is neutral element of each t-norm).

Another well-known fact is the representation of continuous t-norms as ordinal sums with Archimedean summands, i.e., the representation in the form (4) where each R_k, $k \in \mathcal{K}$, is a continuous Archimedean t-norm [13, 16].

A similar representation holds for continuous t-subnorms.

Theorem 3 (Mesiarová [18]). *A mapping $R : [0,1]^2 \to [0,1]$ is a continuous t-subnorm but not a t-norm if and only if there is a system $(]\alpha_k, \beta_k[)_{k\in\mathcal{K}}$ of pairwise disjoint non-empty open subintervals of [0,1] and a system $(R_k)_{k\in\mathcal{K}}$ such that there is $k^* \in \mathcal{K}$, for which $\beta_{k^*} = 1$ and R_{k^*} is a continuous Archimedean t-subnorm, which is not a t-norm and for all $k \in \mathcal{K}$, $k \neq k^*$, R_k is a continuous Archimedean t-norm, and*

$$T(x,y) = \begin{cases} \alpha_k + (\beta_k - \alpha_k) R_k \left(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k} \right) & \text{if } x,y \in]\alpha_k, \beta_k], \\ \min(x,y) & \text{otherwise.} \end{cases} \tag{5}$$

However, a representation of continuous Archimedean t-subnorms is not yet known, in general. Applying the results of Aczél [1] on associative functions, we have the following representation.

Theorem 4 (Mesiarová [18]). *A mapping $R : [0,1]^2 \to [0,1]$ is a continuous strictly monotone Archimedean t-subnorm if and only if there is a continuous strictly decreasing mapping $r : [0,1] \to [0,\infty]$, with $r(0) = \infty$, such that*

$$R(x,y) = r^{-1}(r(x) + r(y)). \tag{6}$$

Observe that representation (6) holds also for any strictly monotone (not necessarily continuous) t-subnorm R with no anomalous pair $(a,b) \in]0,1[^2$, i.e., such $a < b$ for which $b > a > R(b,b) > R(a,a) > R(b,b,b) \dots$, see [4], in which case r need not be continuous.

Several other special representation theorems for specific types of continuous Archimedean t-subnorms can be found in [18]. Note that while in the class of t-norms, the subclass of continuous Archimedean t-norms coincides with the subclass of t-norms generated by continuous additive generators, this is no more true in the case of t-subnorms. For the sake of completeness recall that a non-increasing mapping, $t : [0,1] \to [0,\infty]$ $(r : [0,1] \to [0,\infty])$ is called an additive generator of a t-norm T (t-subnorm R) whenever for all $x,y \in [0,1]$,

$$T(x,y) = t^{(-1)}(t(x) + t(y)) \qquad \Big(R(x,y) = r^{(-1)}(r(x) + r(y))\Big), \tag{7}$$

where $t^{(-1)} : [0,\infty] \to [0,1]$ (and similarly $r^{(-1)}$) is the pseudo-inverse of t [12] defined by

$$t^{(-1)}(u) = \sup\{x \in [0,1] \mid t(u) > x\}. \tag{8}$$

Evidently, if $t : [0,1] \to [0,\infty]$ is an additive generator of a t-norm T, then necessarily $t(1) = 0$ and t is strictly decreasing (as a consequence of the fact that $T(x,1) = x$ for all $x \in [0,1]$).

However, an additive generator $r : [0,1] \to [0,\infty]$ of a t-subnorm R need not fulfill $r(1) = 0$ neither it is necessarily strictly decreasing.

Example 5. Vizualizations of the following t-subnorms are given in Figure 16.

(i) The mapping $r : [0,1] \to [0,\infty]$ given by $r(x) = -\ln \frac{x}{2}$ is an additive generator of the t-subnorm $R : [0,1]^2 \to [0,1]$ given by $R(x,y) = \frac{xy}{2}$.

Note that $R = (T_P)_{0.5}$, see expression (2), and that R is a continuous strictly monotone Archimedean t-subnorm. Moreover $r(1) = \ln 2$.

(ii) Let $r : [0,1] \to [0,\infty]$ be given by $r(x) = \max(1-x, a)$, $a \in [0, \frac{1}{2}]$, i.e., $r(1) = a$ and r is not strictly monotone whenever $a \neq 0$. However, r is an additive generator of the continuous Archimedean t-subnorm $R : [0,1]^2 \to [0,1]$ with zero divisors given by

$$R(x,y) = \max(0, \min(x+y-1, x-a, y-a, 1-2a)).$$

Note that $R = T_L$ (the Łukasiewicz t-norm) if $a = 0$, while $R = W$, $W(x,y) = 0$ for all $x,y \in [0,1]$, if $a = \frac{1}{2}$, the weakest t-subnorm.

(iii) Let $r : [0,1] \to [0,\infty]$ be given by $r(x) = \max(0, 1-2x)$. Then r is an additive generator of the t-subnorm $R : [0,1]^2 \to [0,1]$ given by

$$R(x,y) = \max(0, \min(x+y-\frac{1}{2}, x, y, \frac{1}{2})).$$

R is a continuous t-subnorm which is an ordinal sum $\big(\langle 0, \frac{1}{2}, T_L\rangle, \langle \frac{1}{2}, 1, W\rangle\big)$, i.e., R is not Archimedean.

(iv) Let $r : [0,1] \to [0,\infty]$ be given by $r(x) = \min(\frac{1}{2}, 1-x)$. Then r is a continuous additive generator of the non-continuous Archimedean t-subnorm $R : [0,1]^2 \to [0,1]$ given by

$$R(x,y) = \begin{cases} x+y-1 & \text{if } x+y > \frac{3}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that R is a left-continuous Archimedean t-subnorm. Recall that a non-continuous Archimedean t-norm cannot be left-continuous, see [15].

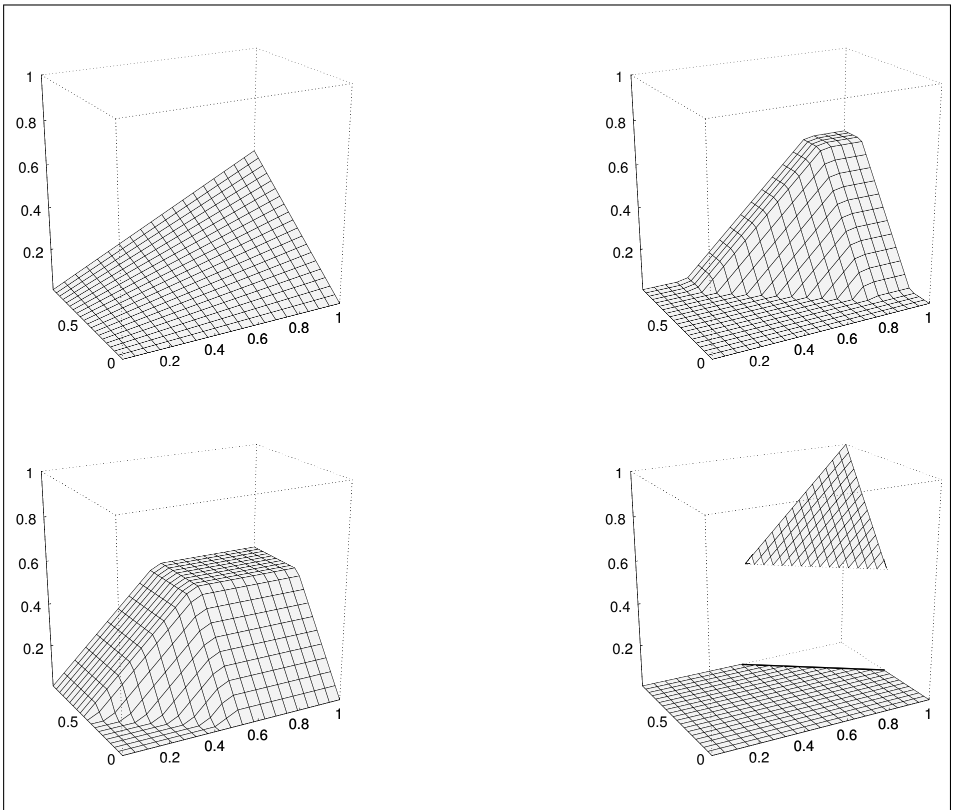

Figure 16: T-subnorms from Example 5.

Though the class of continuous (left-continuous) Archimedean t-subnorms is not yet fully described, and similarly the additive generators of continuous (left-continuous) t-subnorms are not yet completely characterized, we have the following important result shown in [17].

Theorem 6. *Each continuous non-decreasing mapping $r : [0,1] \to [0,\infty]$ is an additive generator of some left-continuous t-subnorm R, i.e., $R(x,y) = r^{(-1)}(r(x)+r(y))$.*

Note that the continuity of the left-continuous t-subnorm R introduced in Theorem 6 is equivalent to the strict monotonicity of r on the interval $[0, r^{(-1)}(2r(1))]$, see [17].

3 Residual implications and ordinal sum t-norms

Recall that, for a given left-continuous t-norm $T : [0,1]^2 \to [0,1]$, the corresponding residual implication $I_T : [0,1]^2 \to [0,1]$ is given by

$$I_T(x,y) = \sup\{z \in [0,1] \mid T(x,z) \le y\}. \tag{9}$$

For more details about residual implications we recommend [5, 6, 13]. Note only that T and I_T are linked by the so called adjunction property

$$T(x,y) \le z \qquad \text{iff} \qquad x \le I_T(y,z), \tag{10}$$

and that

$$T(x,y) = \inf\{z \in [0,1] \mid I_T(x,z) \ge y\}. \tag{11}$$

By means of (9), it is possible to define the residual operator $I_R : [0,1]^2 \to [0,1]$ linked to a left-continuous t-subnorm R, as

$$I_R(x,y) = \sup\{z \in [0,1] \mid R(x,z) \le y\}, \tag{12}$$

so that the adjunction property (10) and equality (11) hold for R and I_R. Obviously, not all properties of residual implications linked to t-norms remain valid for the residual operators linked to t-subnorms. Namely, for any left-continuous t-norm T we have

$$I_T(x,y) = 1 \qquad \text{iff} \qquad x \le y$$

and

$$I_T(1,y) = y \qquad \text{for all} \qquad y \in [0,1].$$

However, for the weakest t-subnorm W (which is continuous) we have

$$I_W(x,y) = 1 \qquad \text{for all} \qquad (x,y) \in [0,1]^2.$$

Now, we turn our attention to left-continuous t-norms which are ordinal sums of semigroups, i.e., t-norms where the summands in their ordinal sum representation are left-continuous t-subnorms.

Theorem 7. *Let $T : [0,1]^2 \to [0,1]$ be a left-continuous t-norm with ordinal sum structure as given in (4) and Theorem 2, i.e., $T = (\langle \alpha_k, \beta_k, R_k \rangle)_{k \in \mathcal{K}}$. Then the corresponding residual implication $I_T : [0,1]^2 \to [0,1]$ is given by*

$$I_T(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ \alpha_k + (\beta_k - \alpha_k)\, I_{R_k}\left(\frac{x-\alpha_k}{\beta_k-\alpha_k}, \frac{y-\alpha_k}{\beta_k-\alpha_k}\right) & \text{if } \alpha_k < y < x \le \beta_k, \\ y & \text{otherwise.} \end{cases} \tag{13}$$

Observe that Theorem 7 applied to continuous t-norms implies the result of [3], see also [5]. Moreover, taking into account the fact that $I_{T_M}(x,y) = y$ whenever $0 \le y < x \le 1$, representation (13) can be understood as an ordinal sum of residual operators. Briefly, residuation of an ordinal sum is just an ordinal sum of residual operators.

Example 8. Let $T : [0,1]^2 \to [0,1]$ be given by

$$T(x,y) = \begin{cases} 0 & \text{if } (x,y) \in [0,\frac{1}{2}]^2, \\ 2xy - x - y + 1 & \text{if } (x,y) \in]\frac{1}{2},1]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

This example was given in [20] : T is a (left-continuous) t-norm fulfilling the diagonal inequality $T(x,x) < x$ for all $x \in]0,1[$ without being Archimedean. As already observed in [10], T is not an ordinal sum of t-norms, but it is an ordinal sum of t-subnorms, $T = \left(\langle 0,\frac{1}{2},W\rangle,\langle \frac{1}{2},1,T_P\rangle\right)$. Because of Theorem 7, the corresponding residual implication $I_T : [0,1]^2 \to [0,1]$ is given by

$$I_T(x,y) = \begin{cases} 1 & \text{if } x \le y, \\ \frac{1}{2} & \text{if } 0 < y < x \le \frac{1}{2}, \\ \frac{y+x-1}{2x-1} & \text{if } \frac{1}{2} < y < x \le 1, \\ y & \text{otherwise.} \end{cases}$$

4 Generated t-subnorms and residual operators

For a generated t-subnorm R, the complete information about R is contained in its additive generator. As we have seen in Theorem 6, the continuity of an additive generator r implies the left-continuity of the corresponding t-subnorm R. Consequently, the residual operator I_R should be expressible by means of r.

Theorem 9. *Let $r : [0,1] \to [0,\infty]$ be a continuous additive generator of the t-subnorm $R : [0,1]^2 \to [0,1]$, i.e., $R(x,y) = r^{(-1)}(r(x)+r(y))$. Then the corresponding residual operator $I_R : [0,1]^2 \to [0,1]$ is given by*

$$I_R(x,y) = r^*(r(y) - r(x)), \tag{14}$$

where $r^* : [-\infty,\infty] \to [0,1]$ is an upper pseudo-inverse of $\widehat{r} : [0,1] \to [-\infty,\infty]$, $\widehat{r}(x) = r(x)$ for all $x \in [0,1]$, given by [12, 21]

$$r^*(u) = \sup\{x \in [0,1] \mid r(x) \ge u\}. \tag{15}$$

Remark 10. Note that for strictly monotone mappings, pseudo-inverses and upper pseudo-inverses coincide. Moreover, if $t : [0,1] \to [0,\infty]$ is a (continuous) additive generator of a continuous Archimedean t-norm T, i.e., if t is continuous, strictly monotone and $t(1) = 0$, then $t^*(u) = t^{(-1)}(\max(0,u)) = t^{-1}(\min(t(0),\max(0,u)))$, and thus $I_T(x,y) = t^{-1}(\max(0,t(y)-t(x)))$, compare e.g. [3, 5].

Example 11. Keeping the notations of Example 5, we get the following residual operators, which are visualized in Figure 17.

(i) $r^*(u) = \min(1, 2e^{-u})$ and $I_R(x,y) = \min(1, \frac{2y}{x})$ with convention $\frac{0}{0} = 1$.

(ii)

$$r^*(u) = \begin{cases} 1 & \text{if } u \le a, \\ \max(0, 1-u) & \text{otherwise,} \end{cases}$$

and

$$I_R(x,y) = \begin{cases} y + \max(a, 1-x) & \text{if } y \leq \min(x-a, 1-2a), \\ 1 & \text{otherwise.} \end{cases}$$

(iii)

$$r^*(u) = \begin{cases} 1 & \text{if } u \leq 0, \\ \max(0, \frac{1-u}{2}) & \text{otherwise,} \end{cases}$$

and

$$I_R(x,y) = \begin{cases} 1 & \text{if } \min(x, \frac{1}{2}) \leq y, \\ \max(y, y + \frac{1}{2} - x) & \text{otherwise.} \end{cases}$$

(iv)

$$r^*(u) = \begin{cases} \min(1, 1-u) & \text{if } u \leq \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$I_R(x,y) = \begin{cases} \min(1, \frac{3}{2} - x) & \text{if } y \leq \frac{1}{2}, \\ \min(1, 1-x+y) & \text{otherwise.} \end{cases}$$

In this case $R(x,y)$ is non-continuous and nilpotent and I_R is continuous.

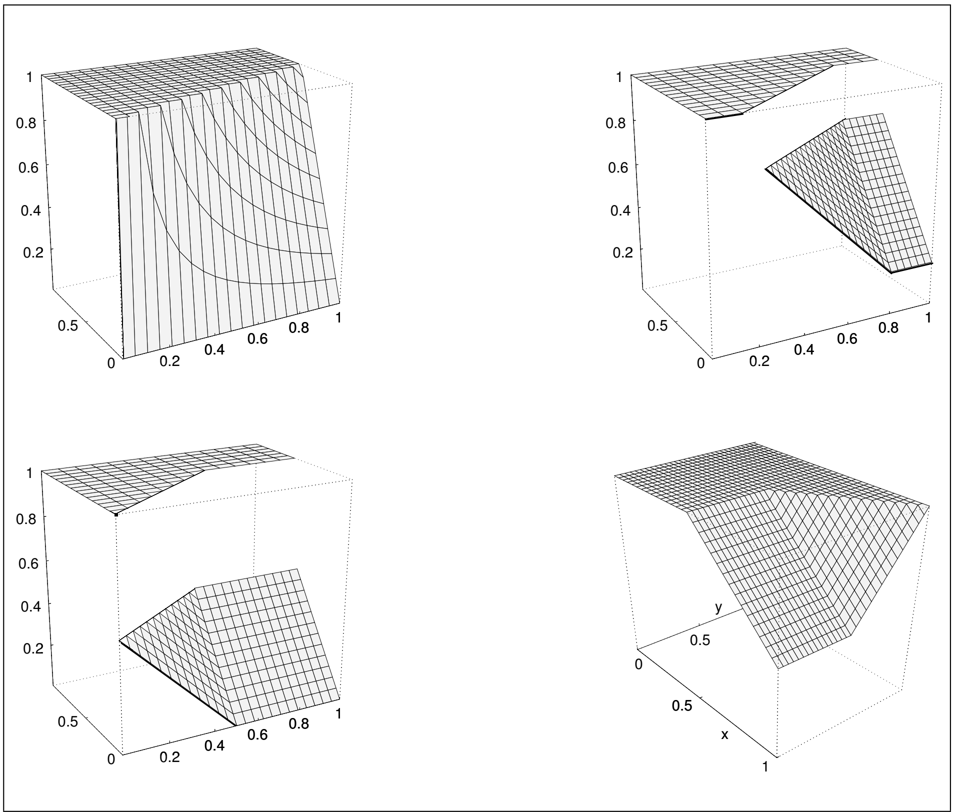

Figure 17: Residual operators from Example 11.

Remark 12. Based on Theorems 2 and 6, we can construct a left-continuous t-norm $T:[0,1]^2 \to [0,1]$ as follows. For an arbitrary system $(]\alpha_k,\beta_k[)_{k\in\mathcal{K}}$ of non-empty pairwise disjoint open subintervals of $[0,1]$, choose an arbitrary system $(r_k)_{k\in\mathcal{K}}$ of non-increasing continuous mappings $r_k:[\alpha_k,\beta_k]\to[0,\infty]$ such that if either $\beta_k = 1$ for some $k\in\mathcal{K}$, or if $\beta_k = \alpha_{k^*}$ for some $k,k^*\in\mathcal{K}$ and $r_{k^*}(\beta_{k^*})$ is finite, then $r_k(\beta_k) = 1$ and r_k is strictly monotone. Then it is suffices to put

$$T(x,y) = \begin{cases} r_k^{(-1)}(r_k(x)+r_k(y)) & \text{if } (x,y)\in]\alpha_k,\beta_k]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Observe that following [12], the pseudo-inverse $r_k^{(-1)}:[0,\infty]\to[\alpha_k,\beta_k]$is given by

$$r_k^{(-1)}(u) = \sup\{x\in[\alpha_k,\beta_k] \mid r_k(x) > u\}.$$

Then the corresponding residual implication $I_T:[0,1]^2\to[0,1]$is given by

$$I_T(x,y) = \begin{cases} 1 & \text{if } x\le y, \\ r_k^*(r_k(y)-r_k(x)) & \text{if } \alpha_k < y < x \le \beta_k, \\ y & \text{otherwise,} \end{cases}$$

where for $u\ge 0$,

$$r_k^*(u) = \sup\{x\in[\alpha_k,\beta_k] \mid r(x)\ge u\}.$$

5 Conclusion

Residual implications linked to the left-continuous ordinal sums of t-subnorms yielding a t-norm were discussed. A new method to construct left-continuous t-norms and the corresponding residual implications based on ordinal sums and additive generators was proposed, and thus some applications in fuzzy logics, as well as in probabilistic metric spaces can be expected.

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Generalizations of some constructions of triangular norms

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In this paper we will generalize some constructions of triangular norms. First, we will put our attention on the constructions of t-norms based on the transformation of a given t-norm by a pair of non-decreasing functions. We will assume:

$$(C)\begin{cases} f,g:[0,1]\to[0,1] \text{ be non-decreasing functions,} \\ T:[0,1]^2\to[0,1] \text{ be a t-norm,} \\ T_{f,g}:[0,1]^2\to[0,1] \text{ be given by the folowing formula:} \\ T_{f,g}(x,y)=\begin{cases} g(T(f(x),f(y))) & \text{if } \max(x,y)<1, \\ \min(x,y) & \text{if } \max(x,y)=1. \end{cases} \end{cases}$$

The conditions under which the function $T_{f,g}$ is a t-norm were discussed in [2], [4] and [7]. We will use notations $f(t_-)$ for $\lim_{x\to t^-} f(x)$, $f(t_+)$ for $\lim_{x\to t^+} f(x)$ and $R(f)$ for the range of a function f. Let us denote:

(1) $g(T(f(x),f(y))\leq\min(x,y)$ for all $x,y\in[0,1)$.

(2) $T(f(x),f(y))\in R(f)\cup[0,f(0_+)]$ for all $x,y\in[0,1)$.

(3) $\forall x,y\in[0,1): T(f(x),f(y))\in R(f)\Rightarrow f(g(T(f(x),f(y))))=T(f(x),f(y))$.

(4) $\forall x,y\in[0,1): T(f(x),f(y))\in[0,f(0_+)]\setminus R(f)\Rightarrow g(T(f(x),f(y)))=0$.

Theorem 1. *Let (C). If (1-4) then $T_{f,g}$ is a t-norm.*

A function g can be for instance a quasi-inverse of a non-decreasing function f or the pseudo-inverse of a non-decreasing function f.

Definition 2. Let $a,b,c,d\in[-\infty,\infty]$, $a<b$, $c<d$ and let $f:[a,b]\to[c,d]$ be a non-decreasing function.

- A function $f^*:[c,d]\to[a,b]$ such that $\forall y\in[c,d]$ the following holds:

  (i) $y\in R(f)\Rightarrow f^*(y)\in f^{-1}(\{y\})=\{x\in[a,b]\mid f(x)=y\}$,
  (ii) $y\notin R(f)\Rightarrow f^*(y)=\sup\{x\in[a,b]\mid f(x)<y\}$, ($\sup\emptyset=a$),

  is called a *quasi-inverse of a non-decreasing function f*.

- A function $f^{(-1)} : [c,d] \to [a,b]$ defined $\forall y \in [c,d]$ by formula:

$$f^{(-1)}(y) = \sup\{x \in [a,b] \mid f(x) < y\},$$

  ($\sup\emptyset = a$), is called the *pseudo-inverse of a non-decreasing function* f.

If $g = f^*$ then we have an immediate consequence of Theorem 1 (see [4]):

Corollary 3. *Let (C) and $g = f^*$. If (1-2) then T_{f,f^*} is a t-norm.*

If $g = f^{(-1)}$ then we have an immediate consequence of Theorem 1 (see [4]):

Corollary 4. *Let (C) and $g = f^{(-1)}$. If (2-3) then $T_{f,f^{(-1)}}$ is a t-norm.*

We can observe that all these results contain the condition (2). We will introduce their generalizations in the following sense: we omit the condition (2) and replace it by a new much more general condition which covers even such cases, when the set

$$M = \{t \in (0,1) \mid \exists x,y \in [0,1) : T(f(x), f(y)) \in [f(t_-), f(t_+)] \setminus R(f)\}$$

is an infinite set.

The second problem we will deal with is the folowing one: Under which conditions a strictly decreasing function $f : [0,1] \to [0,\infty]$, $f(1) = 0$, leads through the formula:

$$T(x,y) = f^{(-1)}(f(x) + f(y)) \qquad \forall x,y \in [0,1],$$

where $f^{(-1)}$ is the pseudo-inverse of a non-increasing function f ($f^{(-1)}(y) = \sup\{x \in [0,1] \mid f(x) > y\}$ for all $y \in [0,\infty]$; ($\sup\emptyset = 0$)), to the associative function $T : [0,1]^2 \to [0,1]$.

The function f is called a *conjunctive additive generator of* T and T is called the *function generated by* f, or briefly a *generated function.* In the case of a t-norm T we will say that f is an *additive generator of a t-norm* T. Some sufficient conditions ensuring associativity of generated functions and some properties of generated functions and their conjunctive additive generators can be found in [3], [5] and [7].

In order to reformulate the above-mentioned problem of associativity of generated functions we introduce the addition operation on $R(f)$ (see [10]):

Let

$$\mathcal{M} = \{A \mid \exists f : [0,1] \to [0,\infty] \text{ strictly monotone, } R(f) = A\}.$$

Definition 5. Let $A \in \mathcal{M}$.

- For all $t \in [0,\infty]$,

$$A \cap [\sup(A \cap [0,t]), \inf(A \cap [t,\infty])]$$

  ($\sup\emptyset = 0$; $\inf\emptyset = \infty$) is always a one-element set.

- A function $F_A : [0,\infty] \to A$,

$$\{F_A(t)\} = A \cap [\sup(A \cap [0,t]), \inf(A \cap [t,\infty])]$$

  is called the *function given by A*.

- A binary operation $\oplus : A \times A \to A$,

$$x \oplus y = F_A(x+y)$$

($+$ is the usual addition on $[0,\infty]$) is called the *addition operation on* A.

The following result holds: Let f be a conjunctive additive generator of T, $R(f) = A$ and let $\oplus$ be the addition operation on A. Then T is a t-norm if and only if $(A, \oplus)$ is a semigroup. This result allow us instead of T and f study the operation $\oplus$ on A.

In this part we will present some constructions of ranges of additive generators of t-norms and we will show the characterization of all additive generators of t-norms which are left-continuous at point 1. Further we will define so called *additive representable semigroups* and we will explain the relations between them and generated t-norms.

Finally, we will introduce the construction of weak additive generators of t-norms. The concept of a weak additive generator of a t-norm was originally introduced by Jenei in [2]. The next definition is its generalization covering the non-continuous case:

Definition 6. Let $f : [0,1] \to [0,\infty]$ be a non-increasing function, $f^{(-1)} : [0,\infty] \to [0,1]$ be the pseudo-inverse of a non-increasing function f and let $T : [0,1]^2 \to [0,1]$ be given by formula:

$$T(x,y) = \begin{cases} f^{(-1)}(f(x) + f(y)) & \text{if } \max(x,y) < 1, \\ \min(x,y) & \text{if } \max(x,y) = 1 \,. \end{cases}$$

Then we will say that f is a *weak conjunctive additive generator of* T.

We will introduce the construction of weak conjunctive additive generators of t-norms starting from conjunctive additive generators of t-norms (see [9]).

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Structure of uninorms with given continuous underlying t-norms and t-conorms

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Uninorms were introduced by Yager and Rybalov [8] as a generalization of t-norms and t-conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in the unit interval.

T-norms do not allow low values to be compensated by high values, while t-conorms do not allow high values to be compensated by low values. Uninorms may allow values separated by their neutral element to be aggregated in a compensating way.

The structure of uninorms was studied by Fodor *et al.* [6]. The unit square (the domain of a uninorm U) is divided into four parts by the neutral element $e \in]0,1[$. In the lower left square $[0,e]^2$ there is an appropriately scaled t-norm, in the upper right square $[e,1]^2$ there is a re-scaled t-conorm. On the rest of the unit square U can be defined in various ways (see [1, 2], and [3, 7] for the important class of representable uninorms).

In this talk we reveal the structure of uninorms with fixed continuous underlying t-norm and t-conorm (for more details see [4, 5]).

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The residuation principle for intuitionistic fuzzy t-norms

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Intuitionistic fuzzy sets defined by Atanassov in 1983 [1] form an extension of fuzzy sets. While fuzzy sets give only a degree of membership, and the degree of non-membership equals one minus the degree of membership, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership that are more or less independent: the only condition is that the sum of the two degrees is smaller than or equal to 1. Formally, an intuitionistic fuzzy set A in a universe U is defined as $A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}$, where μ_A and ν_A are $U - [0,1]$ mappings giving the membership degree and non-membership degree of u in A respectively, and where $\mu_A(u) + \nu_A(u) \leq 1$, for all $u \in U$.

Deschrijver and Kerre [4] have shown that intuitionistic fuzzy sets can also be seen as L-fuzzy sets in the sense of Goguen [6]. Consider the set L^* and the operation $\leq_{L^*}$ defined by :

$$L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice [4]. We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote respectively the first and second component of x, i.e. $x = (x_1, x_2)$. It is easily seen that with every intuitionistic fuzzy set A corresponds an L^*-fuzzy set, i.e. a mapping $A : U \to L^* : u \mapsto (\mu_A(u), \nu_A(u))$. We will also use in the sequel the set $D = \{x \mid x \in L^* \text{ and } x_1 + x_2 = 1\}$.

Using the lattice $(L^*, \leq_{L^*})$, Deschrijver, Cornelis and Kerre have extended the notion of triangular norm to the intuitionistic fuzzy case [2, 3]. An intuitionistic fuzzy triangular norm is a commutative, associative, increasing $(L^*)^2 - L^*$ mapping $\mathcal{T}$ satisfying $\mathcal{T}(1_{L^*}, x) = x$, for all $x \in L^*$. Intuitionistic fuzzy t-norms can be constructed using t-norms and t-conorms on $[0,1]$ in the following way. Let T be a t-norm and S a t-conorm, then the dual t-norm S^* of S is defined by $S^*(a,b) = 1 - S(1-a, 1-b)$, for all $a, b \in [0,1]$. If for all $a, b \in [0,1]$, $T(a,b) \leq S^*(a,b)$, then the mapping $\mathcal{T}$ defined by $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-norm. We call an intuitionistic fuzzy t-norm $\mathcal{T}$ for which such a t-norm T and t-conorm S exist t-representable. Not all intuitionistic fuzzy t-norms are t-representable, e.g. $\mathcal{T}_W(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$ is not t-representable.

An intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle if and only if, for all $x, y, z \in L^*$, $\mathcal{T}(x,y) \leq_{L^*} z \Leftrightarrow y \leq_{L^*} I_{\mathcal{T}}(x,z)$, where $I_{\mathcal{T}}$ denotes the residual implicator generated by $\mathcal{T}$, defined as, for $x, y \in L^*$, $I_{\mathcal{T}}(x,y) = \sup\{\gamma \mid \gamma \in L^* \text{ and } \mathcal{T}(x,\gamma) \leq_{L^*} y\}$.

In the fuzzy case, the residuation principle is equivalent to left-continuity of the t-norm[5]. The intuitionistic fuzzy counterpart of left-continuity is intuitionistic fuzzy left-continuity, defined as follows. Let F be an arbitrary $L^* - L^*$ mapping and $a \in L^*$, then F is called intuitionistic fuzzy left-continuous in a iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in L^*)((d(a,x) < \delta \text{ and } x \leq_{L^*} a) \Rightarrow d(F(x),F(a)) < \varepsilon),$$

where d denotes the Euclidean or Hamming distance of $\mathbb{R}^2$ restricted to L^*.

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. Then $\mathcal{T}$ satisfies the residuation principle if and only if $\sup_{z \in Z} \mathcal{T}(x,z) = \mathcal{T}(x, \sup_{z \in Z} z)$, for all $x \in L^*$ and all $\varnothing \subset Z \subseteq L^*$. Only in the case of t-representable intuitionistic fuzzy t-norms the last property is equivalent to intuitionistic fuzzy left-continuity. So we have that a t-representable intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle if and only if $\mathcal{T}$ is intuitionistic fuzzy left-continuous, but in general we only have that if $\mathcal{T}$ satisfies the residuation principle then $\mathcal{T}$ is intuitionistic fuzzy left-continuous [2].

In general a characterization of intuitionistic fuzzy t-norms satisfying the residuation principle has not yet been established. However, we have the following cases.

For the first representation theorem we will use the following possible properties of an intuitionistic fuzzy t-norm $\mathcal{T}$:

(P.1) $\mathcal{T}(x,x) <_{L^*} x$, for all $x \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$;

(P.2) there exist $x,y \in L^*$ such that x_1 and y_1 are non-zero and such that $\mathcal{T}(x,y) = 0_{L^*}$.

Deschrijver, Cornelis and Kerre have proven that if $\mathcal{T}$ is an $(L^*)^2 - L^*$ mapping, then the following are equivalent [2]:

(*i*) $\mathcal{T}$ is a continuous intuitionistic fuzzy *t*-norm satisfying the residuation principle, the properties (P.1) and (P.2), $I_{\mathcal{T}}(D,D) \subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$;

(*ii*) there exists a continuous increasing permutation φ of $[0,1]$ such that, for all $x,y \in L^*$,

$$\begin{aligned} \mathcal{T}(x,y) \quad = \quad & (\varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)), \\ & 1 - \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(1-y_2) - 1, \varphi(y_1) + \varphi(1-x_2) - 1))); \end{aligned}$$

(*iii*) there exists a continuous increasing permutation Φ of L^* such that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \times \Phi)$.

A more general class of intuitionistic fuzzy t-norms that satisfy the residuation principle is the following. Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm such that, for all $x \in D$, $y_2 \in [0,1]$, $pr_2 \mathcal{T}(x,(0,y_2)) = pr_2 \mathcal{T}(x,(1-y_2,y_2))$. Then $\mathcal{T}$ satisfies the residuation principle if and only if there exist two left-continuous t-norms T_1 and T_2 on $[0,1]$ such that, for all $x,y \in L^*$,

$$\begin{aligned} \mathcal{T}(x,y) \quad = \quad & (T_1(x_1,y_1), \min\{1 - T_2(1 - pr_2 \mathcal{T}((0,0),(0,0)), \\ & T_2(1-x_2, 1-y_2)), 1 - T_2(x_1, 1-y_2), 1 - T_2(y_1, 1-x_2)\}), \end{aligned}$$

and $T_2(x_1,y_1) = T_1(x_1,y_1)$ as soon as $T_2(x_1,y_1) > T_2(1 - pr_2 \mathcal{T}((0,0),(0,0)), T_2(x_1,y_1))$, and $T_1(x_1,y_1) \leq T_2(x_1,y_1)$ else, for all $x_1,y_1 \in [0,1]$.

In the case that $\mathcal{T}(D,D) \subseteq D$, we have the following. Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle such that $\mathcal{T}(D,D) \subseteq D$, T_1 be the $[0,1]^2 - [0,1]$ mapping defined by $T_1(x_1,y_1) = pr_1 \mathcal{T}((x_1, 1-x_1),(y_1, 1-y_1))$, for all $x_1,y_1 \in [0,1]$, and $N_1(x_1) = \sup\{y_1 \mid y_1 \in [0,1] \text{ and } T_1(x_1,y_1) = 0\}$. Assume that range$(N_1) = [0,1]$, and

$$pr_2 \mathcal{T}((0,0),(y_1, 1-y_1)) = 1 \Leftrightarrow y_1 = 0, \quad \forall y_1 \in [0,1].$$

Then, for all $x, y \in L^*$,

$$\begin{aligned}\mathcal{T}(x,y) &= (T_1(x_1,y_1), \min\{1 - T_1(1 - pr_2\mathcal{T}((0,0),(0,0)), T_1(1-x_2, 1-y_2)),\\ &\quad 1 - T_1(1-y_2, x_1), 1 - T_1(1-x_2, y_1)\}).\end{aligned}$$

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On the characterizations of intuitionistic fuzzy implications

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Abstract

The aim of this paper is to present recent results from the theory of intuitionistic fuzzy operators. Besides the known facts we show the characterization theorems for two classes of intuitionistic fuzzy implications: $\mathcal{S}$-implications and $\mathcal{R}$-implications. Based on these characterizations we find the minimal assumptions in the theorem which is dual to the classical Smets-Magrez Theorem: the characterization of the Łukasiewicz implication. Some open problems are presented at the end of the paper.

1 Preliminaries

Intuitionistic fuzzy sets were introduced by Atanassov in 1983 in the following way.

Definition 1 ([1]). An intuitionistic fuzzy set A in a universe X is an object

$$A = \{(x, \mu_A(x), \nu_A(x) : x \in X\}, \tag{1}$$

where functions $\mu_A\colon X \to [0,1]$, $\nu_A\colon X \to [0,1]$ are called, respectively, the membership degree and the non-membership degree. They satisfy the condition $\mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$.

This family can be seen as L-fuzzy set in the sense of Goguen. We use in this paper the following notation presented by Cornelis et al. [6]:

$$L = \{(x_1, x_2) \in [0,1]^2 : x_1 + x_2 \le 1\},$$
$$(x_1, x_2) \le_L (y_1, y_2) \Longleftrightarrow x_1 \le y_1 \wedge x_2 \le y_2, \quad (x_1, x_2), (y_1, y_2) \in L.$$

It can be easily proved that $(L, \le_L)$ is a complete lattice with units $0_l = (0,1)$ and $1_L = (1,0)$. This lattice is not linear.

Like in the fuzzy set theory we can consider the generalizations of classical logical connectives to the lattice L. In last years many papers are dedicated to investigations of these operations. Here we present some results from this theory and we show new facts connecting with intuitionistic fuzzy implications.

Since many characterizations theorem use the increasing bijections, we state now the important result, which shows the dependence between increasing bijections on L and on the unit interval.

Theorem 2 ([6]). *A function $\Phi\colon L \to L$ is an increasing bijection if, and only if, there exists an increasing bijection $\varphi\colon [0,1] \to [0,1]$ such that*

$$\Phi(x) = (\varphi(x_1), 1 - \varphi(1 - x_2)), \qquad x = (x_1, x_2) \in L. \tag{2}$$

Now we present the definitions of fuzzy intuitionistic operators and we recall main results connected with them.

Definition 3. A function $\mathcal{N}\colon L \to L$ is called an intuitionistic fuzzy negation (shortly *IF* negation) if it is decreasing and satisfies $\mathcal{N}(0_L) = 1_L$, $\mathcal{N}(1_L) = 0_L$. If, in addition, $\mathcal{N}$ is an involution, i.e.,

$$\mathcal{N}(\mathcal{N}(x)) = x, \qquad x \in L, \tag{3}$$

then $\mathcal{N}$ is called a strong *IF* negation.

The characterization of strong *IF* negations was first investigated by Bustince et al. [2]. The next result was obtained by Cornelis et al.

Theorem 4 ([4]). *A function $\mathcal{N}\colon L \to L$ is a strong IF negation if, and only if, there exists a strong negation $N\colon [0,1] \to [0,1]$ such that*

$$\mathcal{N}(x) = (N(1 - x_2), 1 - N(x_1)), \qquad x = (x_1, x_2) \in L. \tag{4}$$

The definition of intuitionistic fuzzy t-norms and t-conorms are similar to the classical.

Definition 5. A function $\mathcal{T}\colon L^2 \to L$ is called an intuitionistic fuzzy triangular norm (shortly *IF* t-norm) if it is commutative, associative and increasing operation with the neutral element equal 1_L.
A function $\mathcal{S}\colon L^2 \to L$ is called an intuitionistic fuzzy triangular conorm (shortly *IF* t-conorm)if it is commutative, associative and increasing operation with the neutral element equal 0_L.

The definitions of the algebraic properties (e.g. Archimedean, nilpotent *IF* t-norm) are dual to the classical case (see [8]), so we do not remind them. One of the most important theorems in the classical theory is the representation of continuous, Archimedean t-norms (see [8], Theorem 5.1). Unfortunately, we have not yet the similar result for *IF* t-norms. But for some class of t-norms (and t-conorms) we have the representation.

Theorem 6 (Cornelis et al. [6]). *A function $\mathcal{T}\colon L^2 \to L$ is a continuous, Archimedean, nilpotent IF t-norm which satisfies*

$$\sup_{z \in Z} \mathcal{T}(x, z) = \mathcal{T}(x, \sup_{z \in Z} z), \qquad x \in L, \ Z \subset L \tag{5}$$

if, and only if, there exist an increasing bijection $\Phi\colon L \to L$ such that $\mathcal{T}$ is conjugate with the IF t-norm $\mathcal{T}_W$, i.e.,

$$\begin{aligned} \mathcal{T}(x,y) =& \Phi^{-1}(\mathcal{T}_W(\Phi(x), \Phi(y))) = (\varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)), \\ & 1 - \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(1 - y_2) - 1, \varphi(y_1) + \varphi(1 - x_2) - 1))) \end{aligned} \tag{6}$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in L$ with some increasing bijection $\varphi\colon [0,1] \to [0,1]$.

Theorem 7 (Cornelis et al. [6]). *A function $\mathcal{S}\colon L^2 \to L$ is a continuous, Archimedean, nilpotent IF t-conorm which satisfies*

$$\inf_{z\in Z} \mathcal{S}(x,z) = \mathcal{S}(x, \inf_{z\in Z} z), \qquad x \in L,\ Z \subset L \tag{7}$$

if, and only if, there exist an increasing bijection $\Phi\colon L \to L$ such that $\mathcal{S}$ is conjugate with the IF t-conorm $\mathcal{S}_W$, i.e.,

$$\begin{aligned}\mathcal{S}(x,y) =&\Phi^{-1}(\mathcal{T}_W(\Phi(x),\Phi(y))) = (\varphi^{-1}(\min(1,\varphi(1-x_2)+\varphi(y_1),\varphi(1-y_2)+\varphi(x_1))),\\ &1-\varphi^{-1}(\min(1,\varphi(1-x_2)+\varphi(1-y_2))))\end{aligned} \tag{8}$$

for all $x = (x_1,x_2), y = (y_1,y_2) \in L$ with some increasing bijection $\varphi\colon [0,1] \to [0,1]$.

2 Intuitionistic fuzzy implication

The definition of the implication is based on the notation from fuzzy set theory introduced by Fodor, Roubens [7].

Definition 8. A function $I\colon L^2 \to L$ is called an intuitionistic fuzzy implication (shortly *IF* implication) if it is monotonic with respect to both variables (separately) and fulfills the border conditions

$$I(0_L,0_L) = I(0_L,1_L) = I(1_L,1_L) = 1_L, \qquad I(1_L,0_L) = 0_L. \tag{9}$$

The set of all intuitionistic fuzzy implications is denoted by IFI.

Now we introduce two important classes of *IF* implications which are the generalizations from the fuzzy logic.

Definition 9. Let $\mathcal{S}\colon L^2 \to L$ be an *IF* t-conorm and $\mathcal{N}\colon L \to L$ be an *IF* negation. A function $I_{\int,\mathcal{N}}\colon L^2 \to L$ defined by formula

$$I_{\mathcal{S},\mathcal{N}}(x,y) = \mathcal{S}(\mathcal{N}(x),y), \qquad x,y \in L \tag{10}$$

is called an *IF* $\mathcal{S}$-implication.

The characterization of this family of functions was investigated by Bustinice et al. [3], but their main result was not correct. Our result is the following.

Theorem 10. *A function $I\colon L^2 \to L$ is an IF $\mathcal{S}$-implication based on strong IF negation $\mathcal{N}$ if, and only if $I \in IFI$ satisfies conditions*

$$\begin{aligned} I(1,x) &= x, && x \in L,\\ I(x, I(y,z)) &= I(y, I(x,z)), && z,y,z \in L,\\ I(I(x,0),0) &= x, && x \in L. \end{aligned}$$

Definition 11. Let $\mathcal{T}\colon L^2 \to L$ be an *IF* t-norm which satisfies (5). A function $I_{\mathcal{T}}\colon L^2 \to L$ defined by formula

$$I_{\mathcal{T}}(x,y) = \min\{t \in L : \mathcal{T}(x,t) \le y\}, \qquad x,y \in L \tag{11}$$

is called an *IF* $\mathcal{R}$-implication.

Theorem 12. *A function $I\colon L^2 \to L$ is an IF $\mathcal{R}$-implication if, and only if I satisfies conditions*

$$I(x, I(y,z)) = I(y, I(x,z)), \qquad z,y,z \in L, \tag{12}$$

$$I(x,y) = 1 \Longleftrightarrow x \leq y, \qquad z,y \in L, \tag{13}$$

$$\inf_{z\in Z} I(x,z) = I(x, \inf_{z\in Z} z), \qquad x \in L,\ Z \subset L, \tag{14}$$

$$I(D,D) \subset D, \text{ where } D = \{x \in L : x_1 + x_2 = 1\}. \tag{15}$$

3 Characterization of the intuitionistic Łukasiewicz implication

It is well known that the Łukasiewicz implication $I_{ŁK} = \min(1-x+y,1)$ is the only continuous fuzzy implication (up to a conjugation) which is an *S*-implication and an *R*-implication (cf. [9]). It is a great surprise that for the *IF* implications exists the analogous theorem obtained by Cornelis et al. [5]. Here we want to investigate deeper their result and we want to reduce the needless axioms. As a result we obtain the following theorem.

Theorem 13. *A function $I\colon L^2 \to L$ is continuous and satisfies conditions* (12), (13) *and* (14) *if, and only if there exist an increasing bijection $\Phi\colon L \to L$ such that I is conjugate with the IF Łukasiewicz implication I_{LK}, i.e.,*

$$\begin{aligned} I(x,y) =& \Phi^{-1}(I_{LK}(\Phi(x),\Phi(y))) = \\ & (\varphi^{-1}(\min(1, 1-\varphi(x_1)+\varphi(y_1), 1-\varphi(1-x_2)+\varphi(1-y_2))), \\ & 1-\varphi^{-1}(1-\max(0,\varphi(x_1)-\varphi(1-y_2)))) \end{aligned} \tag{16}$$

for all $x = (x_1,x_2), y = (y_1,y_2) \in L$ with some increasing bijection $\varphi\colon [0,1] \to [0,1]$.

We will in full paper present the examples that these axioms are independent and minimal one.

4 Open problems

Problem 14. An *IF* t-norm $\mathcal{T}$ is called t-representable if there exist a t-norm T and a t-conorm S such that $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$, $x = (x_1,x_2), y = (y_1,y_2) \in L$. What is the characterization of t-representable *IF* t-norm?

The analogous problem can be written for *IF* t-conorms. An *IF* t-conorm $\mathcal{S}$ is called t-representable if there exist a t-norm T and a t-conorm S such that $\mathcal{S}(x,y) = (S(x_1,y_1), T(x_2,y_2))$, $x = (x_1,x_2), y = (y_1,y_2) \in L$.

Problem 15. What is the characterization of t-representable *IF* $\mathcal{S}$-implication, i.e., when *IF* t-conorm $\mathcal{S}$ in Definition 9 is t-representable?

Problem 16. What is the characterization of t-representable *IF* $\mathcal{R}$-implication, i.e., when *IF* t-norm $\mathcal{T}$ in Definition 11 is t-representable?

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t-norms on fuzzy sets of type-2

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1 Introduction

Type-2 fuzzy sets were introduced by Zadeh [12], extending the notion of ordinary fuzzy sets. In [6], [3], [4], [7], [8], [9], and [10] are discussions of both theoretical and practical aspects of type-2 fuzzy sets. We give here a treatment of the mathematical basics of type-2 fuzzy sets that is uncluttered and which uses only standard mathematical notation. One feature is a treatment of t-norms for type-2 sets.

A **fuzzy subset** A of a set S is a mapping $A : S \to [0,1]$. Operations on the set $Map(S,[0,1])$ of all such fuzzy subsets of S come pointwise from operations on $[0,1]$. Common operations on $[0,1]$ of interest in fuzzy theory are $\wedge$, $\vee$, and $'$ given by

$$\begin{aligned} x \wedge y &= \min\{x,y\} \\ x \vee y &= \max\{x,y\} \\ x' &= 1 - x \end{aligned}$$

The constants 0 and 1 are generally considered as part of the algebraic structure, technically being nullary operations. So the algebra basic to fuzzy set theory is $([0,1],\vee,\wedge,',0,1)$. There are operations on $[0,1]$ other than these three that are of special interest in fuzzy matters, such as t-norms and t-conorms.

Interval valued fuzzy sets are mappings of a set S into the algebra $([0,1]^{[2]},\vee,\wedge,',0,1)$, where

$$\begin{aligned} [0,1]^{[2]} &= \{(a,b) : a,b \in [0,1], a \leq b\} \\ (a,b) \vee (c,d) &= (a \vee c, b \vee d) \\ (a,b) \wedge (c,d) &= (a \wedge c, b \wedge d) \\ (a,b)' &= (b',a') \\ 0 &= (0,0) \\ 1 &= (1,1) \end{aligned}$$

The fundamental mathematical properties of this algebra may be found in [1]. Also, t-norms and t-conorms are defined for this algebra, and a theory presented there.

The situation for type-2 fuzzy sets is the same except that fuzzy subsets of type-2 are mappings into a more complicated object than $[0,1]$, namely into $Map([0,1],[0,1])$, the set of all functions from $[0,1]$ to $[0,1]$. Again, operations on type-2 fuzzy sets, that is, on elements of $Map(S,Map([0,1],[0,1]))$, will come point-wise from operations on $Map([0,1],[0,1])$. Operations are put on $Map([0,1],[0,1])$

using operations on both the domain and the range of a function in $Map([0,1],[0,1])$, which are both $[0,1]$. This is where the difficulty of type-2 fuzzy sets lies.

We will put operations on $Map([0,1],[0,1])$ that are of interest in type-2 fuzzy set theory, and develop some of their algebraic properties. Many of these results are known, but our treatment seems simpler and less computational than those heretofore. It follows a systematic pattern, putting this topic in the framework of algebras and their subalgebras. And befitting this meeting, we will emphasize t-norms and t-conorms for this algebra.

2 Type-2 Fuzzy Sets

From now on, denote the unit interval $[0,1]$ simply by I.

Definition 1. Let S be a set. A **type-2 fuzzy subset of** S is a mapping $A : S \to Map(I,I)$.

So for a set S, the set of all type-2 fuzzy subsets of S is $Map(S, Map(I,I))$. We will look at some operations on $Map(I,I)$ commonly defined for type-2 sets. To make the following two definitions, we use the two operations $\wedge$ and $\vee$ on the range and the operation $\vee$ on the domain for the first and the operation $\wedge$ on the domain for the second. Such operations on functions are typically called **convolutions**.

Definition 2. Let f and g be in $Map(I,I)$.

$$(f \sqcup g)(x) = \bigvee_{y \vee z = x} (f(y) \wedge g(z))$$
$$(f \sqcap g)(x) = \bigvee_{y \wedge z = x} (f(y) \wedge g(z))$$

We will denote the convolution of the unary operation $x' = 1 - x$ on the domain of elements of $Map(I,I)$ by *. The formula for it is

$$f^*(x) = \bigvee_{y'=x} f(y) = f(x').$$

For $f \in Map(I,I)$, f' denotes the function given by $f'(x) = (f(x))'$. Denote by $\mathbf{1}$ the element of $Map(I,I)$ defined by $\mathbf{1}(x) = 0$ for all $x \neq 1$, and $\mathbf{1}(1) = 1$. Denote by $\mathbf{0}$ the map defined by $\mathbf{0}(x) = 0$ for all $x \neq 0$, and $\mathbf{0}(0) = 1$. These elements of $Map(I,I)$ can be considered nullary operations, and can be gotten by convolution of the nullary operations 1 and 0 on I.

2.1 The Algebra $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$

At this point, we have the algebra $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$ with the operations $\sqcup$, $\sqcap$, *, $\mathbf{0}$, and $\mathbf{1}$ gotten by convolution using the corresponding operations on the domain, and $\vee$ and $\wedge$ on the image. This is the basic algebra for type-2 fuzzy set theory.

The elements of $Map(I,I)$ have point-wise operations on them coming from operations on the range I. Although we are interested in the algebra $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$, the set $Map(I,I)$ does have the operations $\vee$, $\wedge$, $'$, 0, 1 on it and is a Kleene algebra under these operations. In particular, it is a lattice with order given by $f \leq g$ if $f = f \wedge g$, or equivalently, if $g = f \vee g$. We are at liberty to use these operations in deriving properties of the algebra $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$, and in fact one of our main purposes is to express the operations $\sqcup$ and $\sqcap$ in terms of the simpler pointwise operations. We define two auxiliary unary operations for exactly that purpose.

Definition 3. For $f \in Map(I,I)$, let f^L and f^R be the elements of $Map(I,I)$ defined by

$$f^L(x) = \vee_{y \leq x} f(y)$$
$$f^R(x) = \vee_{y \geq x} f(y)$$

The following theorem expresses the convolution operations $\sqcup$ and $\sqcap$ directly in terms of pointwise operations in two alternate forms.

Theorem 4. *The following hold.*

$$\begin{aligned} f \sqcup g &= \left(f \wedge g^L\right) \vee \left(f^L \wedge g\right) \\ &= (f \vee g) \wedge \left(f^L \wedge g^L\right) \end{aligned}$$

$$\begin{aligned} f \sqcap g &= \left(f \wedge g^R\right) \vee \left(f^R \wedge g\right) \\ &= (f \vee g) \wedge \left(f^R \wedge g^R\right) \end{aligned}$$

Using these unary operations, the basic algebraic properties of the algebra $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$ may be derived rather easily, avoiding more complicated computations with convolutions.

2.2 Two Order Relations

Even though the algebra $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$ is not a lattice under the operations $\sqcup$ and $\sqcap$, these operations have the requisite properties to define partial orders.

Definition 5. $f \sqsubseteq g$ if $f \sqcap g = f$; $f \preceq g$ if $f \sqcup g = g$.

Proposition 6. *The pointwise criteria for $\sqsubseteq$ and $\preceq$ are these:*

1. *$f \sqsubseteq g$ if and only if $f^R \wedge g \leq f \leq g^R$.*
2. *$f \preceq g$ if and only if $f \wedge g^L \leq g \leq f^L$.*

In general, these two partial orders are not the same, but do coincide for some special subalgebras of $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$.

3 Subalgebras of Type-2 Fuzzy Sets

For $a \in [0,1]$, let $\mathbf{a}$ be its characteristic function. That is, $\mathbf{a}(x) = 1$ if $x = a$ and is 0 otherwise.

Theorem 7. *The mapping $a \rightarrow \mathbf{a}$ is an isomorphism from the algebra $([0,1], \vee, \wedge, ', 0, 1)$ to the subalgebra of $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$ of functions of the form $\mathbf{a}$. The mapping $(a,b) \rightarrow \mathbf{a}^L \wedge \mathbf{b}^R$ is an isomorphism from the algebra $([0,1]^{[2]}, \vee, \wedge, ', 0, 1)$ to the subalgebra of $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$ of elements of the form $\mathbf{a}^L \wedge \mathbf{b}^R$.*

This fully legitimizes the claim that type-2 fuzzy sets are generalizations of type-1 and of interval-valued fuzzy sets. But $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$ contains many other subalgebras of interest, and these are investigated. The subalgebra of normal convex functions is one of special interest. A function f in $Map(I,I)$ is normal if $f^{RL} = 1$, and is convex if $f = f^R \wedge f^L$.

Theorem 8. *The subalgebra of* $(Map(I,I), \sqcup, \sqcap, ^*, \mathbf{0}, \mathbf{1})$ *of convex normal functions is a De Morgan algebra.*

4 T-norms for Type-2 Fuzzy Sets

The operations on $Map(I,I)$ resulting from convolutions of t-norms and t-conorms on $[0,1]$, we call type-2 t-norms, and type-2 t-conorms.

Definition 9. Let $\triangle$ be a t-norm, and f and g be elements of $Map(I,I)$.

$$(f \blacktriangle g)(x) = \bigvee_{y \triangle z = x} f(y) \wedge g(z)$$

The convolution $\blacktriangledown$ for a t-conorms $\bigtriangledown$ on [0,1] is defined similarly.

We assume throughout that the t-norms and t-conorms on $[0,1]$ **are continuous**. Of special interest is the interaction of t-norms with the other algebraic operations on $Map(I,I)$. Here are some typical results.

Proposition 10. *The following hold.*

1. $(f \blacktriangle g)^R = f^R \blacktriangle g^R$
2. $(f \blacktriangle g)^L = f^L \blacktriangle g^L$
3. $(f \blacktriangledown g)^R = f^R \blacktriangledown g^R$
4. $(f \blacktriangledown g)^L = f^L \blacktriangledown g^L$

Theorem 11. *The distributive laws*

$$\begin{array}{ll} f \blacktriangle (g \sqcap h) = (f \blacktriangle g) \sqcap (f \blacktriangle h) & f \blacktriangle (g \sqcup h) = (f \blacktriangle g) \sqcup (f \blacktriangle h) \\ f \blacktriangledown (g \sqcap h) = (f \blacktriangledown g) \sqcap (f \blacktriangledown h) & f \blacktriangledown (g \sqcup h) = (f \blacktriangledown g) \sqcup (f \blacktriangledown h) \end{array}$$

hold if and only if f is convex.

Corollary 12. *If f is convex and $g \sqsubseteq h$, then*

$$f \blacktriangle g \sqsubseteq f \blacktriangle h$$
$$f \blacktriangledown g \sqsubseteq f \blacktriangledown h$$

If f is convex and $g \preceq h$, then

$$f \blacktriangle g \preceq f \blacktriangle h$$
$$f \blacktriangledown g \preceq f \blacktriangledown h$$

4.1 T-Norms on the Subalgebra of Characteristic Functions of Points

As we have seen, a copy of the algebra $([0,1],\vee,\wedge,',0,1)$ is contained in the algebra $(Map(I,I),\sqcup,\sqcap,^*,\mathbf{0},\mathbf{1})$, namely the characteristic functions $\mathbf{a}$ for $a\in[0,1]$. The formula

$$(\mathbf{a}\blacktriangle\mathbf{b})(x)=\bigvee_{y\triangle z=x}\mathbf{a}(y)\wedge\mathbf{b}(z)$$

says that $\mathbf{a}\blacktriangle\mathbf{b}$ is the characteristic function of $a\triangle b$, as it should be. This implies the following.

Theorem 13. *For any t-norm $\triangle$, the mapping $a\to\mathbf{a}$ is an isomorphism from the algebra $([0,1],\vee,\wedge,\triangle,',0,1)$ onto the subalgebra of $(Map(I,I),\sqcup,\sqcap,\blacktriangle,^*,\mathbf{0},\mathbf{1})$ of characteristic functions of points.*

4.2 T-Norms on the Subalgebra of Characteristic Functions of Intervals

In [1], t-norms were defined on the set $[0,1]^{[2]}$, and the requirements resulted in exactly that t-norms were calculated coordinatewise on the endpoints of the intervals. That is, t-norms on $[0,1]^{[2]}$ were of the form

$$(a,b)\triangle(c,d)=(a\triangle b,c\triangle d)$$

where $\triangle$ is a t-norm on $[0,1]$. Consider the subalgebra of $(Map(I,I),\sqcup,\sqcap,\blacktriangle,^*,\mathbf{0},\mathbf{1})$ of functions of the form $\mathbf{a}^L\wedge\mathbf{b}^R$ with $a\leq b$, or equivalently of the characteristic functions of closed intervals $[a,b]$. From the formula

$$\left(\mathbf{a}^L\wedge\mathbf{b}^R\right)\blacktriangle\left(\mathbf{c}^L\wedge\mathbf{d}^R\right)(x)=\bigvee_{y\triangle z=x}\left(\mathbf{a}^L\wedge\mathbf{b}^R\right)(y)\wedge\left(\mathbf{c}^L\wedge\mathbf{d}^R\right)(z)$$

it follows that

$$\left(\mathbf{a}^L\wedge\mathbf{b}^R\right)\blacktriangle\left(\mathbf{c}^L\wedge\mathbf{d}^R\right)=\left(\mathbf{a}^L\blacktriangle\mathbf{c}^L\right)\wedge\left(\mathbf{b}^R\blacktriangle\mathbf{d}^R\right)$$

So t-norms on this subalgebra are calculated coordinatewise on the endpoints of the intervals. This results in the following.

Theorem 14. *The mapping $(a,b)\to\mathbf{a}^L\wedge\mathbf{b}^R$ is an isomorphism from the algebra $([0,1]^{[2]},\vee,\wedge,\triangle,',0,1)$ onto the subalgebra of $(Map(I,I),\sqcup,\sqcap,\blacktriangle,^*,\mathbf{0},\mathbf{1})$ of characteristic functions of closed intervals.*

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Defects of properties of the triangular norms

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In the recent book [1] we examine in a systematic way different defects of properties in Sets Theory, Topology, Measure Theory, Real Function Theory, Complex Analysis, Functional Analysis, Algebra, Geometry, Number Theory in a classical or fuzzy context. A discussion on the defects of properties of triangular norms is also initiated in [1], starting from an idea in the paper [4] where the defect of associativity of a binary operation on $[0,1]$ is introduced.

Our purpose is to continue the study of t-norms (and t-conorms) that have not the properties of idempotency, complementarity or distributivity. The deviations from these properties can be evaluated introducing the following global defects of properties:

- defect of idempotency of the t-norm T

$$d_{ID}(T) = \sup\{x - T(x,x); x \in [0,1]\}$$

- defect of complementarity of the t-norm T

$$d_C(T) = \sup\{T(x,1-x); x \in [0,1]\}$$

- defect of distributivity of F with respect to G (F and G t-norms or t-conorms)

$$d_{DIS}(F,G) = \sup\{|F(x,G(y,z)) - G(F(x,y),F(x,z))|; x,y,z \in [0,1]\}.$$

It is obvious that the values of defects are equal to 0 if and only if the respective properties are verified. The defects are calculated for the important families of Frank ($(T_\lambda)_{\lambda\in[0,+\infty]}$), Yager ($(T^\lambda)_{\lambda\in[0,+\infty]}$), Hamacher and Sugeno-Weber t-norms and the basic t-norms $T_M = T_0, T_P = T_1, T_L = T_\infty, T_W = T^0$. For example,

$$d_{ID}(T_\lambda) = \begin{cases} \log_\lambda \frac{\sqrt{\lambda}+1}{2}, & \text{if } \lambda \in (0,1) \cup (1,+\infty) \\ 0, & \text{if } \lambda = 0 \\ \frac{1}{4}, & \text{if } \lambda = 1 \\ \frac{1}{2}, & \text{if } \lambda = +\infty \end{cases}$$

and

$$d_{DIS}(T_M,T_P) = d_{DIS}(T_P,T_P) = d_{DIS}(T_L,T_P) = d_{DIS}(T_W,T_P) = \frac{1}{4}.$$

The above introduced defects are studied in connection with: the dual of a t-norm, the order between t-norms, the reverse of a t-norm, the ordinal sum of a family of t-norms, the properties of Archimedean and strict t-norm, t-norms with threshold, well-founded t-norms, nearly Frank t-norms (see [3]). Partly, the proved properties are generalizations of results already obtained. Thus, the property

$$d_{DIS}\left(S_{T_1},T_2\right)=d_{DIS}\left(T_1,S_{T_2}\right),$$

where S_T denotes the dual t-conorm of t-norm T, can be considered as a generalization of the result proved in [5]: if T is distributive with respect to S_T then S_T is distributive with respect to T. Also, the property

$$d_{DIS}\left(T,S_T\right)\geq d_{ID}\left(T\right)=d_{ID}\left(S_T\right)\geq 0,$$

is a generalization of the result in the same paper [5]: if T is distributive with respect to S_T then T and S_T are idempotent.

Some methods to improve the properties of complementarity and distributivity of t-norms are proposed. Thus, if T_φ is a φ-transform of a t-norm T relative to a standard generator φ (that is $T_\varphi\left(x,y\right)=\varphi^{-1}\left(T\left(\varphi\left(x\right),\varphi\left(y\right)\right)\right),\forall x,y\in[0,1]$, where $\varphi:[0,1]\longrightarrow[0,1]$ is an increasing automorphism with $\varphi\left(x\right)+\varphi\left(1-x\right)=1,\forall x\in[0,1]$ - see e.g. [6]) then

$$d_C\left(T_\varphi\right)=\varphi^{-1}\left(d_C\left(T\right)\right).$$

Choosing the generator φ such that $\varphi^{-1}\left(d_C\left(T\right)\right)<d_C\left(T\right)$ we obtain a t-norm with a better property of complementarity. Also, if T,T' are two t-norms and T_φ,T'_φ are t-norms generated by the pseudo-automorphism φ (that is $T_\varphi\left(x,y\right)=\varphi^{[-1]}\left(T\left(\varphi\left(x\right),\varphi\left(y\right)\right)\right)$, $T'_\varphi\left(x,y\right)=\varphi^{[-1]}\left(T'\left(\varphi\left(x\right),\varphi\left(y\right)\right)\right)$ if $x,y\in[0,1)$ and $T_\varphi\left(x,y\right)=T'_\varphi\left(x,y\right)=\min\left(x,y\right)$ if $\max\left(x,y\right)=1$, where $\varphi:[0,1]\longrightarrow[0,1]$ is a non-decreasing continuous function with $\varphi\left(0\right)=0,\varphi\left(1\right)=1$ and $\varphi^{[-1]}$ is a quasi-inverse of φ - see [2]) and $\varphi^{[-1]}$ is a k-contraction, $k\in(0,1)$, then

$$d_{DIS}\left(T_\varphi,T'_\varphi\right)\leq k d_{DIS}\left(T,T'\right),$$

therefore we obtain t-norms with a better property of distributivity.

Open problems relative to above introduced defects and to other defects of the binary operations constructed by using triangular norms are formulated (the calculus of the defect of associativity of the reverse of a triangular norm, for example). Different defects of properties of t-norms as future themes of research are introduced . As examples, let us consider

- defect of continuous Archimedean t-norm

$$d_A\left(T\right)=\sup\left\{\lim_{n\to\infty}d^n\left(x\right);x\in[0,1)\right\},$$

where d is the diagonal of T and d^n is the composition of n copies of d

- defect of self-reversibility of the t-norm T

$$d_R\left(T\right)=\sup\left\{T\left(x,y\right)-\max\left\{0,x+y-1+T\left(1-x,1-y\right)\right\};x,y\in[0,1]\right\}$$

- defect of Frank t-norm of T

$$d_F\left(T\right)=\sup\left\{\left|T\left(x,y\right)+S_T\left(x,y\right)-x-y\right|;x,y\in[0,1]\right\}.$$

Finally, some possible applications are presented. As example, if we define the defect of vertical $\oplus$-additivity of the integral $\int^{\oplus}$ by

$$\sup\left\{\sup\left\{\left|\int^{\oplus} f_A \odot dm \oplus \int^{\oplus} f_{A^C} \odot dm - \int^{\oplus} f \odot dm\right| ; A \in \mathcal{A}\right\} ; f \in \mathcal{F}\right\},$$

where $f_M(x) = f(x)$ if $x \in M$, $f_M(x) = 0$ if $x \notin M, \oplus$ is a continuous t-conorm, $\odot$ is a left continuous t-norm and m is a $\oplus$-additive fuzzy measure, then an estimation of the defect of vertical $\oplus$-additivity by defect of distributivity of $\odot$ with respect to $\oplus$ can be given.

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Finitely and absolutely non idempotent aggregation operators

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In multicriteria decision making (MDM), the crucial process of combining numerical values into a single one is known as aggregation. In mathematical terms, an aggregation operator can be decribed as a sequence of real functions defined on a n-dimensional domain $D^n \subseteq \mathbb{R}^n$, for some interval $D \subseteq \mathbb{R}$ and for every $n \in \mathbb{N}$. Such operators can be roughly divided into three classes: the conjunctive, the disjunctive and the averaging. The most common families belonging to the first two types are respectively triangular norms and conorms. The aggregators most properly used in MDM are the averaging, also called compensative, because they are always comprised between *minimum* and *maximum*. Generally, they are demanded to satisfy some basic properties as continuity, commutativity, i.e. indifference to the ordering of the arguments, non decreasing monotonicity. What is frequently observed in the literature is that the requested properties characterize every single aggregation function, treating the number n of the arguments not as a variable, but as a static parameter. Rare are the cases of operators which dynamically change their behavior as we add elements to the aggregation (see, for instance, *consistency* in [5]). Furtherly, in many papers, it is clearly decribed the role of a neutral element, that is an element which has no influence on the result of the aggregation. However, this subject is related almost exclusively to the *uninorms*, which belong to the family of logical connectives (see, for a different kind of approach, *self-identity* in [5,7]).

In my talk, I'm interested to introduce a class of operators which possess a neutral element and loose idempotency, usually considered a "natural" requirement, equivalent to compensativity for monotone operators. I will show explicitly that these two simultaneous requests cause an interesting sensitivity to the addition of arguments. Particularly, I will consider two new classes of aggregators in which it occurs respectively a partial and total lackness of idempotency.

In the first case, the operators will be called *finitely non idempotent* and their properties will be investigated in order to prove that they can be seen as an extension of the classical quasi arithmetic means. In particular, I will present a representation theorem of an important subclass of such aggregators, through a definition of a new concept, the *k-decomponibility*, which is a weakening of decomponibility.

In the second case, I will formulate the axioms for an aggregator, called *absolutely non idempotent*, entirely composed of non idempotent functions. Also in this case I will present the main properties and an interesting link to a theoretical aspect such as the non monotonicity of a *generalized mixture operator* (see [2]), which can be considered a generalization of an ordered weighted averaging operator (OWA) [4]. Here it will be important to provide some concrete examples, since explicit examples of such operators are not present in literature and the properties they have to satisfy are really not easy to combine. Moreover, I will give a general procedure to build a general class of such operators.

Finally I will try to explain the characteristics of such operators through a very simple application in which the elements to aggregate are to be intended as degrees of preference in fuzzy sense.

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Fitting aggregation operators to data

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Abstract

Theoretical advances in modelling aggregation of information produced a wide range of aggregation operators, applicable to almost every practical problem. The most important classes of aggregation operators include triangular norms, uninorms, generalised means and OWA operators. With such a variety, an important practical problem has emerged: how to fit the parameters/weights of these families of aggregation operators to observed data? How to estimate quantitatively whether a given class of operators is suitable as a model in a given practical setting?

Aggregation operators are rather special classes of functions, and thus they require specialised regression techniques, which would enforce important theoretical properties, like commutativity or associativity. My presentation will address this issue in detail, and will discuss various regression methods applicable specifically to t-norms, uninorms and generalised means. I will also demonstrate software implementing these regression techniques, which would allow practitioners to paste their data and obtain optimal parameters of the chosen family of operators.

1 Fitting triangular norms

Characterisation theorems (see [4, 7]) provide a way to represent continuous Archimedian t-norms and conorms through their additive generators. Importantly, convergence of a sequence of additive generators is equivalent to convergence of the corresponding sequence of t-norms [4], Ch.8. This result provides a way of fitting t-norms to observation data through the approximation of their additive generators. The additive generator is modelled with a monotone linear spline, and spline coefficients are found by solving a rectangular system of linear equations, subject to non-negativity of the variables. This is a classical problem of non-negative least squares [5], for which fast and robust algorithms are available [3]. There are some technical issues related to non-uniqueness of the additive generators (which are defined up to a positive multiplier), and strict t-norms, which cannot be uniquely identified from the data in on the whole of their domain.

An important class of t-norms that are copulas can also be modelled using additive generators, because of the characterisation theorem [4, 7] that relates copulas to the convexity of additive generators. Thus, additional restrictions are imposed on spline coefficients, which guarantee its convexity.

2 Fitting uninorms

Uninorms behave like t-norms on one part of the domain and like t-conorms on the other. The technique of approximation additive generators can be extended to representable uninorms. For a fixed neutral element e (which is the zero of the additive generator), it is a straightforward adaptation of the above method of monotone splines, now with one additional linear restriction at e. However, the neutral element itself can also be found from the data. To this end, an optimisation problem is solved to find the global optimum of e, in which for every intermediate value of e , spline coefficients are computed using the non-negative least squares method of [3].

3 Fitting generalised means

Quasiarithmetic means also possess additive generators, whose sum is now weighted [1, 2]. Similarly to t-norms and uninorms, one can fit generators to the data, by computing coefficients of a linear monotone spline. The technique is practically the same as the one employed for t-norms, with corresponding weighting of the components of the matrix of the constrained system of linear equations. However, if not only the generator, but the weights of the (generalised) mean need to be found from the data, the problem becomes more complicated. There are two sets of variables in the regression problem: the weights and the spline coefficients. Since for a fixed vector of weights, spline coefficients are found though a non-negative linear least squares problem, one can separate variables: at the outer level the global optimisation problem with respect to weights is solved, and at the inner level (i.e., for every fixed vector of weights) spline coefficients are computed.

A particular instance of this technique, generalised quasilinear means, in which generators are power functions, was discussed in [2]. However the global nature of the optimisation problem was not recognised.

4 Extensions

Similarity of representation of t-norms, uninorms and means through the univariate generator functions prompts one to consider these operators in one framework, as instances of the same class of functions satisfying $\sum_{i=1}^{n} a_i g(x_i) = g(y)$ In case of t-norms and uninorms, all weights $a_i = 1$, for means $a_i = 1/n$, for generalised means $\sum a_i = 1$. Condition $\sum a_i = n$ is used to introduce degrees of importance of arguments into t-norms ([1], Eq.(34)). Intermediate cases result from a weaker restriction $a_i \geq 0$.

Given the generator function g explicitly, the weights a_i can be determined from the data using non-negative least squares procedure. Otherwise, both the generator (i.e., its spline coefficients) and the weights can be found from data in a manner used for generalised means, with one less restriction.

Further, commutativity of the aggregation operator can be ensured by ordering the arguments xi in decreasing order, like it is done in OWA operators (these are so-called pseudo-OWA [6], Eq.(17)). The usual OWA operator becomes a special case of $g(x) = x$. No changes to the regression procedure are necessary, except the reordering of arguments in the observation data.

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Recent results on rotation and rotation-annihilation: symmetrization and compensative operators

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The talk will include two different topics which are somewhat related to each other. These topics are described in Sections 1 and 2.

1 On the relationship between the rotation construction and Abelian groups

We call the construction of extending the operation from the positive cone of an ordered group into the whole group symmetrization. The aim of this section is twofold. First, the rotation construction [8] – a method, which is a much less understood than symmetrization – shall be related to symmetrization, thus providing a better understanding of the rotation-construction. In fact, the rotation-construction is described as a kind of semi-symmetrization. Second, the symmetrization of t-conorms (and t-norms) is defined analogously. We shall symmetrize t-conorms on $[\frac{1}{2}, 1]$ in order to obtain operations on $[0,1]$. The subclass of t-conorms shall be characterized which results in associative operations via symmetrization. In fact, associativity of such an operation, which is constructed from a t-conorm by symmetrization, is equivalent to that it is a uninorm. In addition, a characterization is given for those t-conorms in terms of a set of equations.

The results are illustrated by three-dimensional plots.

1.1 Rotation versus symmetrization

Standing assumption: Unless otherwise specified, throughout the paper we fix an arbitrary strong negation $'$, and denote its (unique) fixed point by t. Further, we denote $I^- = [0,t[$, $I^+ = [t,1]$, $I_- = [0,t]$ and $I_+ =]t,1]$. We shall consider the following properties:

$(A1)$	*Commutativity*	$x \circledast y = y \circledast x$
$(A2)$	*Associativity*	$x \circledast (y \circledast z) = (x \circledast y) \circledast z$
$(A3)$	*Monotonicity*	$x \circledast y \leq x \circledast z$ whenever $y \leq z$
$(A4)$	*Conjunctive nature*	$x \circledast y \leq \min(x,y)$.

The rotation construction and the rotation-annihilation construction for t-norms were introduced in [8] and [9], respectively. Their a far-leading generalization to the setting of partially-ordered semigroups is in [6]. These general results applying to our topic and by using the terminology of the present paper are quoted in Theorems 1 and 8.

Theorem 1. (Rotation) *Let $\circledast$ be a left-continuous operation on $[t,1]$ satisfying (A1), (A2) and (A3). Define $\circledast_{\mathbf{r}}$ (of type $[0,1]\times[0,1]\to[0,1]$) by*

$$x\circledast_{\mathbf{r}}y=\begin{cases} x\circledast y & \text{if } x,y\in I_+ \\ (x\to_{\circledast}y')' & \text{if } x\in I_+ \text{ and } y\in I_- \\ (y\to_{\circledast}x')' & \text{if } x\in I_- \text{ and } y\in I_+ \\ 0 & \text{if } x,y\in I_- \end{cases} \tag{1}$$

$\circledast_{\mathbf{r}}$ is a left-continuous rotation invariant operation satisfying (A1), (A2) and (A3) if and only if either

C1. $x\circledast y=0$ *implies* $\min(x,y)=0$ *or*

C2. there exists $c\in]0,1]$ *such that* $x\circledast y=0$ *iff* $x,y\leq c$.

In addition, $\circledast_{\mathbf{r}}$ satisfies (A4) if and only if $\circledast$ satisfies (A4).

By applying Theorem 1 to t-conorms (which always satisfy condition *C1*) we obtain:

Corollary 2. *Let $\oplus$ be a left-continuous t-conorm on $[t,1]$. The operation $\oplus_{\mathbf{r}}$ (of type $[0,1]\times[0,1]\to[0,1]$) given by*

$$x\,\mathrm{rot}\oplus y=\begin{cases} x\oplus y & \text{if } x,y\in I_+ \\ (x\to_{\oplus}y')' & \text{if } x\in I_+ \text{ and } y\in I_- \\ (y\to_{\oplus}x')' & \text{if } x\in I_- \text{ and } y\in I_+ \\ 0 & \text{if } x,y\in I_- \end{cases} \tag{2}$$

is a left-continuous, rotation invariant operation satisfying (A1), (A2) and (A3).

Since taking the dual operation preserves properties (A1), (A2) and (A3), we proceed as follows: By taking the dual operation $\odot$ of $\oplus$ with respect to $'$ (that is, the de Morgan identity $x\odot y=(x'\oplus y')'$ holds) we deduce the following statement from Corollary 2:

Corollary 3. *Let $\odot$ be a right-continuous t-norm on $[0,t]$. The operation $\odot^{\mathbf{r}}$ (of type $[0,1]\times[0,1]\to[0,1]$) given by*

$$x\odot^{\mathbf{r}}y=\begin{cases} 1 & \text{if } x,y\in I^+ \\ (y\leftarrow_{\odot}x')' & \text{if } x\in I^+ \text{ and } y\in I^- \\ (x\leftarrow_{\odot}y')' & \text{if } x\in I^- \text{ and } y\in I^+ \\ x\odot y & \text{if } x,y\in I^- \end{cases} \tag{3}$$

is a left-continuous, rotation invariant operation satisfying (A1), (A2) and (A3).

Still assuming that $\oplus$ and $\odot$ are duals, (which is equivalent to $x\to_{\oplus}y=(x'\leftarrow_{\odot}y')'$, as it is easy to verify) we obtain that the operation in (3) is equal to

$$x(\oplus_d)^{\mathbf{r}}y=\begin{cases} 1 & \text{if } x,y\in I^+ \\ y'\to_{\oplus}x & \text{if } x\in I^+ \text{ and } y\in I^- \\ x'\to_{\oplus}y & \text{if } x\in I^- \text{ and } y\in I^+ \\ (x'\oplus y')' & \text{if } x,y\in I^- \end{cases} \tag{4}$$

Since the operation $\odot$ is dual to $\oplus$ (in notation, $\odot=\oplus_d$) it is not confusing to denote $\odot^{\mathbf{r}}$ by $(\oplus_d)^{\mathbf{r}}$.

At this point, we are ready to define the *symmetrization* of $\oplus$. One may call the operation defined in (5) the symmetrization of $\odot$ as well (in notation $\odot^{\mathbf{s}}$).

Definition 4. Let $\oplus$ be a left-continuous t-conorm on $[t,1]$. Let $\oplus_{\mathbf{r}}$ and $(\oplus_d)^{\mathbf{r}}$ be defined by (2) and (4), respectively. Define the binary operation $\oplus_{\mathbf{r}}$ on $[0,1]$ by

$$x \oplus_{\mathbf{s}} y = \begin{cases} x \operatorname{rot}\oplus y & \text{if } x,y \in I^+ \text{ or } (x \in I^+, y \in I_-, x \le y') \text{ or } (x \in I_-, y \in I^+, x \le y') \\ x(\oplus_d)^{\mathbf{r}} y & \text{if } x,y \in I_- \text{ or } (x \in I^+, y \in I_-, x > y') \text{ or } (x \in I_-, y \in I^+, x > y') \end{cases} \tag{5}$$

In a more detailed form:

$$x \oplus_{\mathbf{s}} y = \begin{cases} x \oplus y & \text{if } x,y \in I^+ \\ (x \rightarrow_{\oplus} y')' & \text{if } x \in I^+ \text{ and } y \in I_- \text{ and } x \le y' \\ y' \rightarrow_{\oplus} x & \text{if } x \in I^+ \text{ and } y \in I_- \text{ and } x > y' \\ (y \rightarrow_{\oplus} x')' & \text{if } x \in I_- \text{ and } y \in I^+ \text{ and } x \le y' \\ x' \rightarrow_{\oplus} y & \text{if } x \in I_- \text{ and } y \in I^+ \text{ and } x > y' \\ (x' \oplus y')' & \text{if } x,y \in I_- \end{cases} \tag{6}$$

(5) points out that rotation can be considered as a kind of semi-symmetrization. In order to illustrate it with a figure, let $x' = 1 - x$. Denote by $\odot_{\mathbf{P}}$ the product t-norm on $[0, \frac{1}{2}]$, by $\oplus_{\mathbf{P}}$ its dual t-conorm on $[\frac{1}{2}, 1]$. Figure 18 shows the relation between the rotation- and the symmetrization constructions.

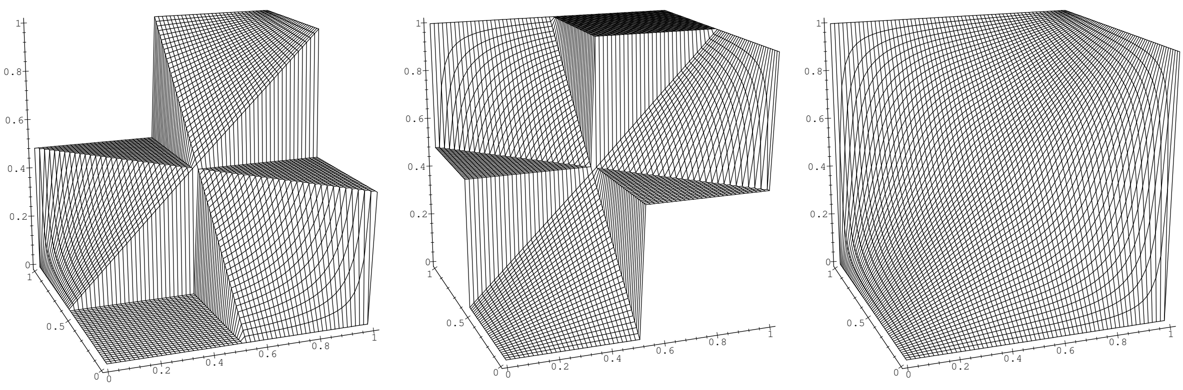

Figure 18: $(\oplus_{\mathbf{P}})_{\mathbf{r}}$ (left), its dual $(\odot_{\mathbf{P}})^{\mathbf{r}}$ (center), and $(\oplus_{\mathbf{P}})_{\mathbf{s}}$ (right).

Lemma 5. $\oplus_{\mathbf{s}}$ *is a uninorm iff it is associative.*

1.2 Symmetrizing t-conorms

Theorem 6. *Let $\oplus$ be a left-continuous t-conorm. $\oplus_{\mathbf{s}}$ is associative if and only if one of the following is true:*

1. *$\oplus$ is isomorphic to the dual of the product t-norm.*

2. *$\oplus$ is isomorphic to the dual of the minimum t-norm.*

3. *$\oplus$ is isomorphic to the dual of an ordinal sum with summands all being product t-norms.*

Example 7. Let $x' = 1 - x$. At the first row of Figure 19 the rotation of the maximum t-conorm (left), its dual (center), and the symmetrization of the maximum t-conorm. In the bottom row and in Figure 18 an example is depicted in the same style corresponding to items 3 and 1 in Theorem 6, respectively.

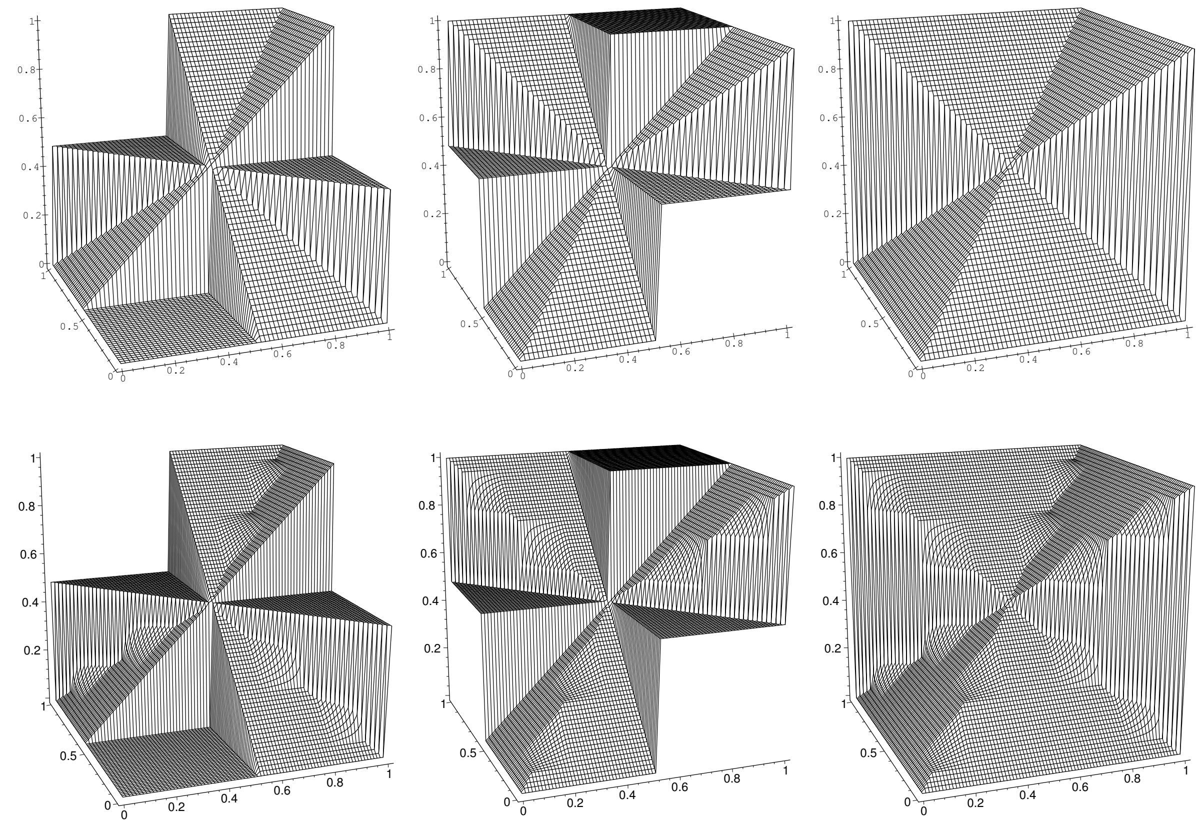

Figure 19: Illustration for items 2 and 3 of Theorem 6, see Example 7

2 Partially compensative associative operators by rotation and rotation-annihilation

2.1 Associativity versus compensation

Many authors have tried to find operators that are associative and compensative at the same time. As pointed out in [3] uninorms admit partial compensation (that is, at least on some subdomain of $[0,1]^2$ they have compensative nature). We shall point out in this talk that associativity and compensative nature can not be satisfied simultaneously. In fact, the proper definition on the diagonal and its neighborhood is problematic. As a way out, the rotation construction and the rotation-annihilation construction, in their most general forms [6], allow us to define wide families of associative aggregation operations, which admit partial compensation. Thus, the here-defined operators are similar to uninorms, a class which is being investigated intensively in the literature. The method is illustrated with several 3D plots.

However, partial compensation is possible. We say that M is *compensative* on a subset X of $[0,1]^2$, if for $(x,y) \in X$, $\min(x,y) \leq M(x,y) \leq \max(x,y)$, and *strictly compensative* if for $(x,y) \in X$, $\min(x,y) < M(x,y) < \max(x,y)$.

Dombi has introduced a class of aggregative operators [1]. A remarkable member of this family is

$$M(x,y) = \frac{xy}{xy + (1-x)(1-y)} \tag{7}$$

called “Three Pi” operator after Yager. This class of aggregative operators is a special class of the so-called uninorms. Uninorms were introduced in [12]. They generalize the notions of t-norms and t-conorms by allowing the neutral element e to lay in the open unit interval $]0,1[$. A first description of the structure of uninorms is in [3]. It has turned out that a subclass of uninorms, called representable uninorms, coincides with the class of aggregative operators of Dombi. Further, any uninorm U has an underlying t-norm T and t-conorm S acting on the subdomains $[0,e] \times [0,e]$, and $[e,1] \times [e,1]$ of $[0,1]^2$, respectively. Therefore, compensation is possible only on the remaining subdomains

$$[0,e] \times [e,1] \ \text{ and } \ [e,1] \times [0,e] \tag{8}$$

and in fact, any uninorm is compensative on that subdomain.

Moreover, any member of the class of *representable* uninorms is strictly compensative on those subdomains. Fodor et. al. [2] have recently characterized all the possible uninorm operations M acting on $[0,e] \times [e,1] \cup [e,1] \times [0,e]$, provided that the underlying t-norm T and t-conorm S are both continuous. The result says, among others, that M has strictly compensative nature only on those subdomains $[a,b] \times [c,d]$ of $[0,e] \times [e,1]$ (and, of course, symmetrically) where $[a,b]$ and $[c,d]$ correspond to *strict* summands in the ordinal sum representation of T and S, respectively (see Fig. 20).

Thus, representable uninorms are the best candidates of uninorms in terms of compensability.

Our aim in this section is to introduce a new class of operators, which – similar to uninorms – admits partial compensation. We shall achieve this goal by using a generalization of the rotation construction [8] and the rotation-annihilation construction [9] for t-norms. The properties of the introduced operators will be discussed and several illustrative examples will be given.

Theorem 8. (Rotation-annihilation) *Let $'$ be a strong negation, t its unique fixed point, $d \in]t,1[$ and define a strong negation by $N_d(x) = \frac{x\cdot(d-d')+d''-d'}{d-d'}$. Let M be a left-continuous operation on $[0,1]$ satisfying (A1), (A2) and (A3).*

C1. If $x,y > 0$ implies $M(x,y) > 0$ then let M_2 be a left-continuous t-subnorm which admits the rotation invariance property w.r.t. N_d. Further, let $I^- = [0,d'[$, $I^0 = [d',d]$ and $I^+ =]d,1]$.

C2. If there are $x,y > 0$ such that $M(x,y) = 0$ then let M_2 be a left-continuous t-norm which admits the rotation invariance property w.r.t. N_d (equivalently, let M_2 be a left-continuous t-norm with associated negation N_d). Further, let $I^- = [0,d']$, $I^0 =]d',d[$ and $I^+ = [d,1]$.

Let M_3 be the linear transformation of M_1 into $[d,1]$, M_4 be the linear transformation of M_2 into $[d',d]$ and M_5 be the annihilation of M_4 given by

$$M_5(x,y) = \begin{cases} 0 & \text{if } x,y \in [d',d] \text{ and } x \leq y' \\ M_4(x,y) & \text{if } x,y \in [d',d] \text{ and } x > y' \end{cases} .$$

Define $M_{\mathbf{ra}} : [0,1] \times [0,1] \to [0,1]$ *by*

$$M_{\mathbf{ra}}(x,y) = \begin{cases} M_3(x,y) & \text{if } x,y \in I^+ \\ I_{M_3}(x,y')' & \text{if } x \in I^+, y \in I^- \\ I_{M_3}(y,x')' & \text{if } x \in I^-, y \in I^+ \\ 0 & \text{if } x,y \in I^- \\ M_5(x,y) & \text{if } x,y \in I^0 \\ y & \text{if } x \in I^+ \text{ and } y \in I^0 \\ x & \text{if } x \in I^0 \text{ and } y \in I^+ \\ 0 & \text{if } x \in I^- \text{ and } y \in I^0 \\ 0 & \text{if } x \in I^0 \text{ and } y \in I^- \end{cases}, \tag{9}$$

Then $M_{\mathbf{ra}}$ *is a left-continuous rotation invariant operation satisfying (A1), (A2) and (A3), and called the* rotation-annihilation *of* M *and* M_2.

In addition, $M_{\mathbf{ra}}$ *satisfies (A4) if and only if* M *satisfies (A4).*

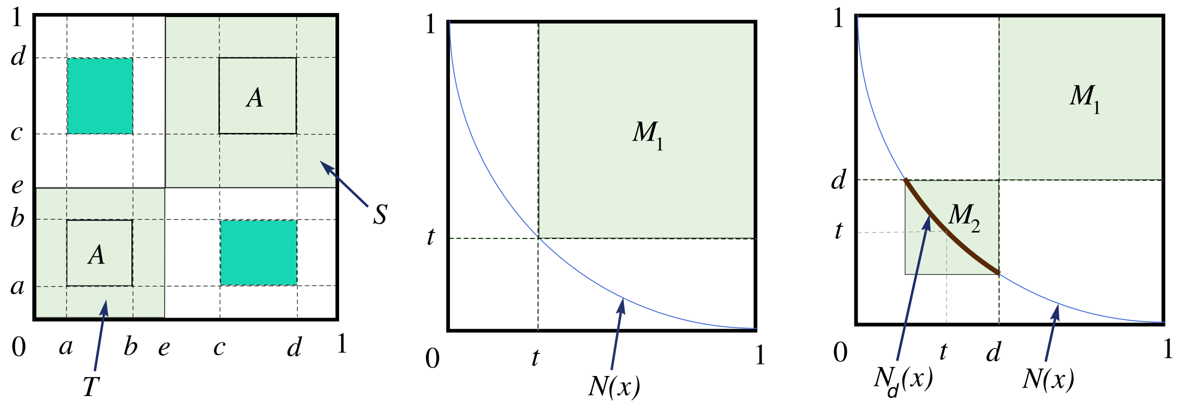

Figure 20: Strictly compensative domains of a uninorm (dark grey) with strict summands A (left). Illustration for the rotation construction (center) and for the rotation-annihilation construction (right)

2.2 Operators by rotation

The last assertion of Theorem 1 points out, that willing to construct compensative operators, t-subnorms (and hence also t-norms) are not suitable to play the role of M.

2.2.1 Rotations of t-conorms and t-superconorms

By observing that condition *C1* is always satisfied by any t-superconorm (hence also by any t-conorm), we obtain that any t-superconorm (hence also any t-conorm) can play the role of M in Theorem 1. We shall investigate the compensative nature of the resulted operator.

Theorem 9. *Let* $'$ *be a strong negation,* t *its unique fixed point and* S *be a left-continuous t-superconorm. Then* $\min \leq S_{\mathbf{r}} \leq \max$ *holds on the domains* $[0,t] \times]t,1]$ *and* $]t,1] \times [0,t]$.

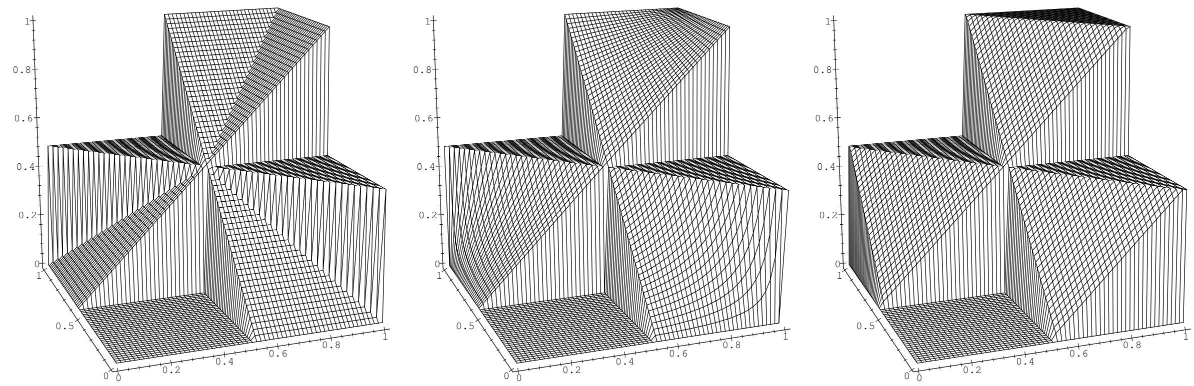

Figure 21: $(S_{\mathbf{M}})_{\mathbf{rot}}$ (left), $(S_{\mathbf{P}})_{\mathbf{rot}}$ (right) and $(S_{\mathbf{L}})_{\mathbf{rot}}$ (bottom)

Example 10. Let $x' = 1 - x$. The rotations of the three basic conorms given by $S_{\mathbf{M}}(x,y) = \max(x,y)$, $S_{\mathbf{P}}(x,y) = 1-(1-x)(1-y)$ and $S_{\mathbf{L}}(x,y) = \min(1, x+y-1)$ are plotted in Figure 21.

Further, let S be the t-conorm defined by the following two summands:

$$S = \{([0.15, 0.45], S_{\mathbf{P}}), ([0.55, 0.85], S_{\mathbf{L}})\}$$

S and its rotation are plotted in Figure 22.

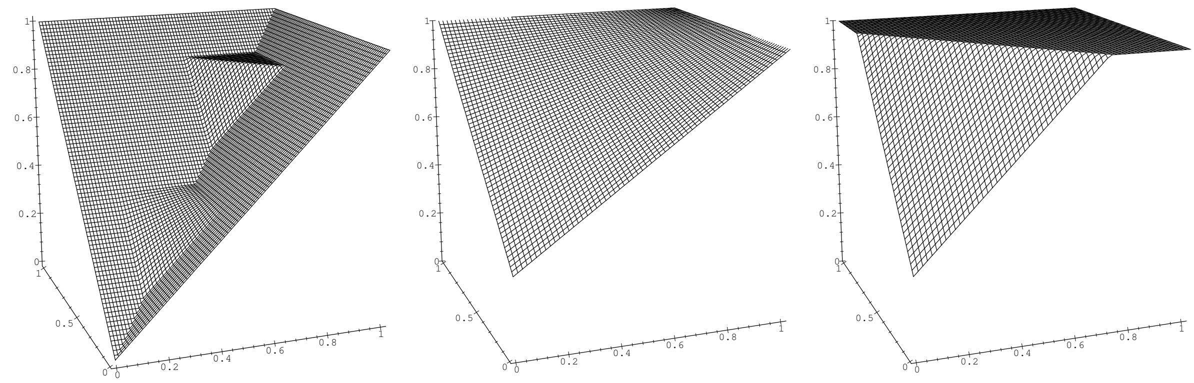

Figure 22: S, see Example 10 (left), $S_{\mathbf{P}_{0.7}}$ (center) and $S_{\mathbf{L}_{0.3}}$ (right)

Example 11. Consider the following two family t-superconorms: Let ε be any real number from $[0,1]$ and define

$$S_{\mathbf{P}_\varepsilon}(x,y) = 1 - \varepsilon(1-x)(1-y), \qquad S_{\mathbf{L}_\varepsilon}(x,y) = \min(1, x+y+\varepsilon).$$

$S_{\mathbf{L}_0}$ and $S_{\mathbf{P}_1}$ are equal to $S_{\mathbf{L}}$, the Łukasiewicz t-conorm and $S_{\mathbf{P}}$, the product t-conorm, respectively. Let $x' = 1 - x$. Two members from these families are plotted together with their rotations in Figure 23.

2.2.2 Rotations of uninorms

Taking into account Theorem 1 we see that not every uninorm is suitable for playing the role of M. The uninorms that can be rotated (i.e. those, which result in associative operation via rotation) are

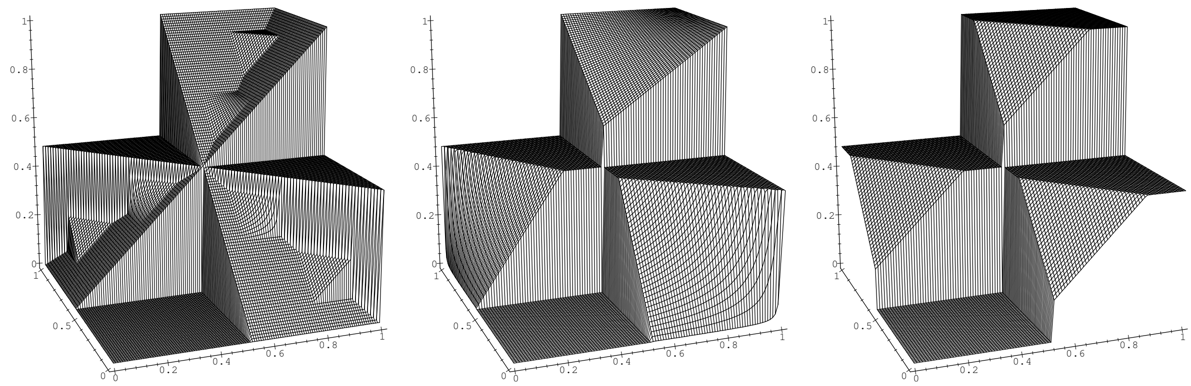

Figure 23: The rotations of the t-norms in Figure 22, respectively

precisely the class of uninorms such that their underlying t-norm admits one of conditions *C1* and *C2*.

Theorem 12. *Let $'$ be a strong negation, t its unique fixed point and U be a left-continuous uninorm with neutral element e. Let U_1 be the linear transformation of U into $[t,1]$ and denote the image of e under this linear transformation by e^*. Then*

1. *the rotation $U_{\mathbf{rot}}$ of a uninorm U is a uninorm with neutral element e^*.*
2. $\min \leq U_{\mathbf{rot}} \leq \max$ *holds on the domains $[0,e^*]\times]e^*,1]$ and $]e^*,1]\times[0,e^*]$.*

Example 13. Let $x' = 1-x$. The "Three Pi" operation defined in (7) and its rotation are presented in Figure 24.

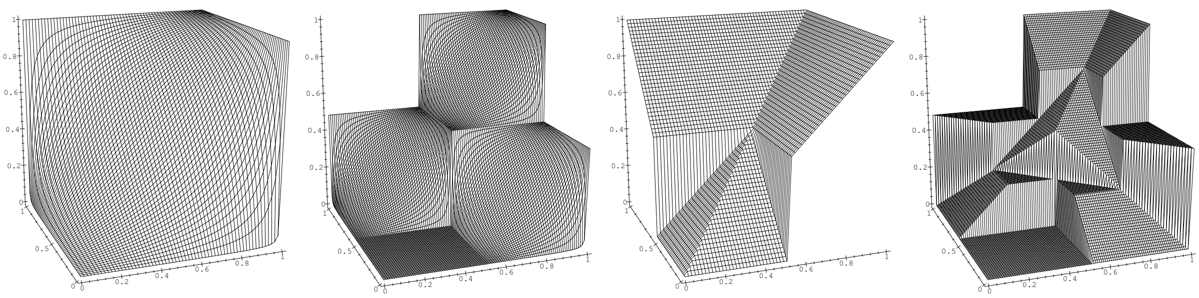

Figure 24: "Three Pi" and its rotation (left), see Example 13. U and its rotation (right), see Example 14

Example 14. Let $x' = 1-x$. Figure 24 shows the uninorm defined below together with its rotation.

$$U(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) \leq \frac{1}{2}, \\ \max(x,y) & \text{otherwise.} \end{cases} \tag{10}$$

Remark 15. Since uninorms have an underlying t-norm, this method – as a by-product – results in a new method for constructing left-continuous (and non-continuous) t-norms.

2.3 Operators by rotation-annihilation

The last assertion of Theorem 8 points out, that willing to construct compensative operators, t-subnorms (and hence also t-norms) are not suitable to play the role of M_1.

Standing assumption: Throughout this section M_2 will be an operation chosen as in Theorem 8 (depending on the zero values of M_1), we change the operation M_1 only.

Theorem 16. *Let $'$ be a strong negation, t its unique fixed point, $d \in]t,1[$. Let $M_{\mathbf{ra}}$ be the rotation-annihilation of a left-continuous t-superconorm and M_2. Then* $\min \le M_{\mathbf{ra}} \le \max$ *holds on the domains* $[0,d] \times]d,1]$ *and* $]d,1] \times [0,d]$.

Theorem 17. *Let $'$ be a strong negation, t its unique fixed point, $d \in]t,1[$. Let U be a a left-continuous uninorm with neutral element e and denote by e^* the image of e under the increasing linear transformation which maps $[0,1]$ onto $[d,1]$. Let $M_{\mathbf{ra}}$ be the rotation-annihilation of U and M_2. Then* $\min \le M_{\mathbf{ra}} \le \max$ *holds on the domains* $[0,e^*] \times]e^*,1]$ *and* $]e^*,1] \times [0,e^*]$.

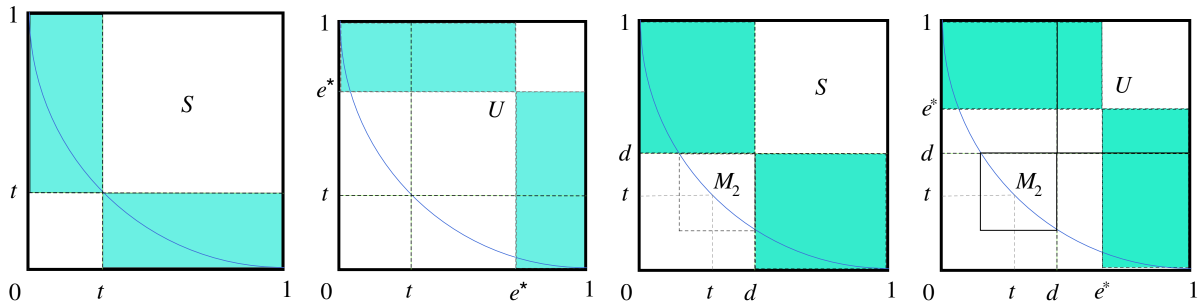

Figure 25: Rotation of t-superconorms and uninorms (left), rotation-annihilations of t-superconorms S and M_2 and uninorms U and M_2 (right). The compensative parts of the domains are highlighted

Example 18. Let $x' = 1 - x$, $d = \frac{2}{3}$. The dual of $S_{\mathbf{L}}$ (called the Łukasiewicz t-norm) is defined by $T_{\mathbf{L}}(x,y) = \min(0, x+y-1)$. Three operators, which are results of rotation-annihilation with M_1 being a t-conorm are in Figure 26.

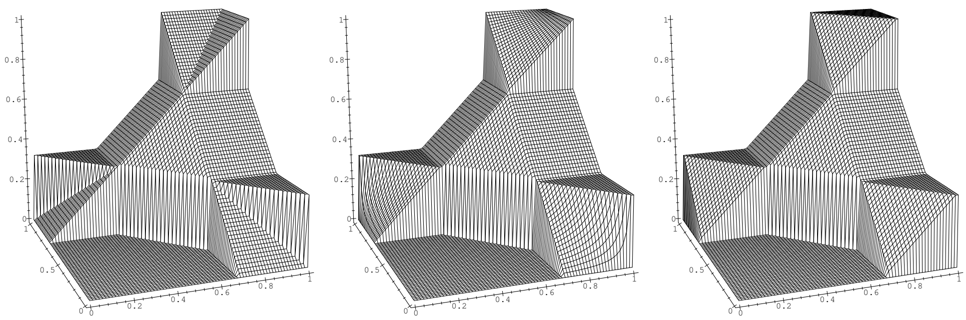

Figure 26: Rotation-annihilations of $S_{\mathbf{M}}$ and $T_{\mathbf{L}}$ (left), $S_{\mathbf{P}}$ and $T_{\mathbf{L}}$ (right) and $S_{\mathbf{L}}$ and $T_{\mathbf{L}}$ (bottom), see Example 18

Example 19. For $\varepsilon \in [0,1]$ the rotation-invariant t-subnorm, which is dual to $S_{\mathbf{L}_\varepsilon}$ is defined by $T_{\mathbf{L}_\varepsilon} = \min(0, x+y-1-\varepsilon)$. Let $x' = 1-x$, $d = \frac{2}{3}$. Two operators, which are results of rotation-annihilation with M_1 being a uninorm are in Figure 27.

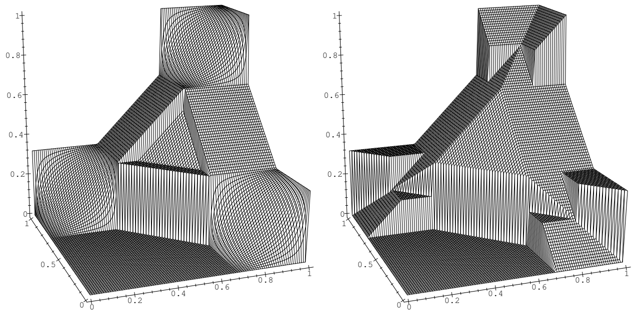

Figure 27: Rotation-annihilations of "Three Pi" and $T_{\mathbf{L}_{0.3}}$ (left), U (defined in (10)) and $T_{\mathbf{L}}$ (right), see Example 19

Remark 20. From Figures 21, 22 and 26 one may have the intuition that the obtained operations do have neutral elements, and thus they are uninorms. Indeed, the element t (which is $\frac{1}{2}$ in Figures 21 and 22) seems to be neutral when rotating t-conorms; and the element d (which is $\frac{2}{3}$ in Figure 26) seems to be neutral when applying the rotation-annihilation construction with a t-conorm and M_2. But taking into account that the obtained operations are always left-continuous (or by checking this conjecture in formulas (1) and (9)), one immediately see that it is not the case. However, denoting by ε a small positive real number, $t+\varepsilon$ ($d+\varepsilon$, respectively) behave *almost like neutral elements*, as it is easy to see. The smaller ε is the more the element $t+\varepsilon$ ($d+\varepsilon$, respectively) behave as neutral elements do. That is, the obtained operations have an "*almost* neutral" element, which may be interesting from the application viewpoint.

Example 21. Finally, we present an example in order to emphasize that the here-introduced methods can be used iteratively. Consider the operation in the rightmost operation of Figure 24. On the left of Figure 28 the rotation of it can be seen. Consider the operation, which is on the left-hand side of Figure 27. Use it as a summand on $[0.05, 1]$ in an ordinal sum (in order to obtain an operation without zero divisors). Its rotation can be seen on right of Figure 28. On the bottom of Figure 28 the rotation of the rotation of the Three Pi operation is presented.

Keywords: Aggregation, Compensation, Associativity, Uninorm, Rotation, Rotation-annihilation

Acknowledgements

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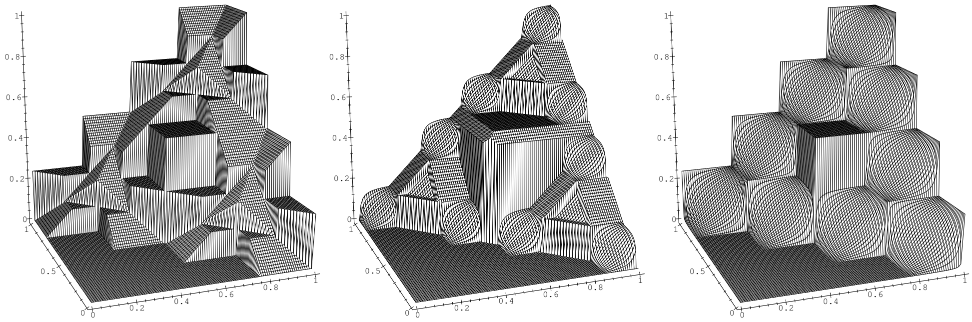

Figure 28: The operations of Example 21

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Special operators dominating continuous Archimedean t-Norms

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1 Introduction

The concept of domination has been introduced in the framework of probabilistic metric spaces [8, 7] when constructing Cartesian products of such spaces. In the framework of t-norms, domination is also needed when building fuzzy equivalence (ordering) relations from already given corresponding fuzzy relations. The crucial point during this process is the preservation of the T-transitivity of the underlying given fuzzy relations. Note that related problems of preserving special properties were also investigated in the framework of pseudo-additive measures ([5, 4]).

Standard aggregation of fuzzy equivalence (ordering) relations preserving T-transitivity is done either be means of T or $T_{\mathbf{M}}(x,y) = \min(x,y)$. Both of them, i.e. T itself and $T_{\mathbf{M}}$, trivially dominate the considered t-norm T. Staying in the framework of t-norms, in fact any t-norm T^* dominating T can be applied to preserve T-transitivity, i.e. if R_1, R_2 are two T-transitive, binary relations on a universe X, then also $T^*(R_1, R_2)$ has this property (see [3, 1]).

In several applications, other types of aggregation processes preserving T-transitivity are required (e.g. [2]). Especially different weights (degrees of importance) of input fuzzy equivalence (ordering) relations cannot be properly modelled by the aggregation with t-norms, because of their commutativity. Therefore, general T-transitivity-preserving aggregation operators have to be considered and the concept of domination in the framework of aggregation operators had to be introduced (see [6]). We will briefly recall the definition of domination of aggregation operators and some basic results.

2 Domination of aggregation operators

Definition 1. Consider an n-ary aggregation operator $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ and an m-ary aggregation operator $\mathbf{B}_{(m)} : [0,1]^m \to [0,1]$. We say that $\mathbf{A}_{(n)}$ *dominates* $\mathbf{B}_{(m)}$ ($\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$) if, for all $x_{ij} \in [0,1]$ with $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, the following property holds:

$$\begin{aligned}\mathbf{B}_{(m)}\big(\mathbf{A}_{(n)}(x_{11},\ldots,x_{1n}),\ldots,\mathbf{A}_{(n)}(x_{m1},\ldots,x_{mn})\big) \\ \leq \mathbf{A}_{(n)}\big(\mathbf{B}_{(m)}(x_{11},\ldots,x_{m1}),\ldots,\mathbf{B}_{(m)}(x_{1n},\ldots,x_{mn})\big).\end{aligned} \tag{1}$$

Note that if either n or m or both are equal to 1, because of the boundary condition of aggregation operators, $\mathbf{A}_{(n)} \gg \mathbf{B}_{(m)}$ is trivially fulfilled for any two aggregation operators $\mathbf{A}, \mathbf{B}$.

Definition 2. Let $\mathbf{A}$ and $\mathbf{B}$ be aggregation operators. We say that $\mathbf{A}$ *dominates* $\mathbf{B}$ ($\mathbf{A} \gg \mathbf{B}$), if $\mathbf{A}_{(n)}$ dominates $\mathbf{B}_{(m)}$ for all $n, m \in \mathbb{N}$.

Note that, if two aggregation operators $\mathbf{A}$ and $\mathcal{B}$ are both acting on some closed interval $I = [a,b] \subseteq [-\infty, \infty]$, then the property of domination can be easily adapted by requiring that the Inequality (1) must hold for all arguments x_{ij} from the interval I and for all $n, m \in \mathbb{N}$.

We will briefly mention some basic results concerning isomorphic aggregation operators and aggregation operators which are associative.

Consider an aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}} [a,b]^n \to [a,b]$ on $[a,b]$ and a monotone bijection $\varphi : [c,d] \to [a,b]$. The operator $\mathbf{A}_\varphi : \bigcup_{n\in\mathbb{N}} [c,d]^n \to [c,d]$ defined by

$$\mathbf{A}_\varphi(x_1, \ldots, x_n) = \varphi^{-1}\big(\mathbf{A}(\varphi(x_1), \ldots, \varphi(x_n))\big)$$

is an aggregation operator on $[c,d]$, which is isomorphic to $\mathbf{A}$.

Proposition 3. *Consider two aggregation operators* $\mathbf{A}$ *and* $\mathcal{B}$ *both acting on* $[a,b]$.

(i) $\mathbf{A} \gg \mathcal{B}$ *if and only if* $\mathbf{A}_\varphi \gg \mathcal{B}_\varphi$ *for all non-decreasing bijections* $\varphi : [c,d] \to [a,b]$.

(ii) $\mathbf{A} \gg \mathcal{B}$ *if and only if* $\mathcal{B}_\varphi \gg \mathbf{A}_\varphi$ *for all non-increasing bijections* $\varphi : [c,d] \to [a,b]$.

Proposition 4. *Let* $\mathbf{A}, \mathbf{B}$ *be two aggregation operators. Then the following holds:*

(i) If $\mathbf{B}$ *is associative and* $\mathbf{A}_{(n)} \gg \mathbf{B}_{(2)}$ *for all* $n \in \mathbb{N}$, *then* $\mathbf{A} \gg \mathbf{B}$.

(ii) If $\mathbf{A}$ *is associative and* $\mathbf{A}_{(2)} \gg \mathbf{B}_{(m)}$ *for all* $m \in \mathbb{N}$, *then* $\mathbf{A} \gg \mathbf{B}$.

3 Domination of continuous Archimedean t-norms

Next we concentrate on the domination of an aggregation operator over a continuous Archimedean t-norm, which turns out to be closely related to subadditive aggregation operators ([6], compare also [5]).

Definition 5. A function $F : [0,c]^n \to [0,c]$ is *subadditive* on $[0,c]$, if the following inequality holds for all $x_i, y_i \in [0,c]$ with $x_i + y_i \in [0,c]$:

$$F(x_1 + y_1, \ldots, x_n + y_n) \le F(x_1, \ldots, x_n) + F(y_1, \ldots, y_n).$$

An aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}} [0,c]^n \to [0,c]$ acting on $[0,c]$ is *subadditive*, if all n-ary operations $\mathbf{A}_{(n)} : [0,c]^n \to [0,c]$ are subadditive on $[0,c]$.

If we want to show that an aggregation operator $\mathbf{A}$ dominates the Łukasiewicz t-norm $T_{\mathbf{L}}$ ($\mathbf{A} \gg T_{\mathbf{L}}$) it is equivalent to prove that the Łukasiewicz t-conorm $S_{\mathbf{L}}$ dominates the dual aggregation operator $\mathbf{A}^d$ ($S_{\mathbf{L}} \gg \mathbf{A}^d$) because of the isomorphism property (see Proposition 3), i.e., for arbitrary $x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$, the following inequality must hold

$$S_{\mathbf{L}}(\mathbf{A}^d(x_1, \ldots, x_n), \mathbf{A}^d(y_1, \ldots, y_n)) \ge \mathbf{A}^d(S_{\mathbf{L}}(x_1, y_1), \ldots, S_{\mathbf{L}}(x_n, y_n))$$

being equivalent to

$$\min(\mathbf{A}^d(x_1,\dots,x_n)+\mathbf{A}^d(y_1,\dots,y_n),1) \geq \mathbf{A}^d(\min(x_1+y_1,1),\dots,\min(x_n+y_n,1)).$$

Furthermore, the last inequality can be rewritten in the following form

$$\mathbf{A}^d(x_1,\dots,x_n)+\mathbf{A}^d(y_1,\dots,y_n) \geq \mathbf{A}^d(\min(x_1+y_1,1),\dots,\min(x_n+y_n,1)).$$

If $x_i+y_i \leq 1$ for all $i \in \{1,\dots,n\}$, then we can derive that

$$\mathbf{A}^d(x_1,\dots,x_n)+\mathbf{A}^d(y_1,\dots,y_n) \geq \mathbf{A}^d(x_1+y_1,\dots,x_n+y_n)$$

expressing that $\mathbf{A}^d$ is a subadditive function on $[0,1]$. The sufficiency of the subadditivity of $\mathbf{A}^d$ to ensure $S_{\mathbf{L}} \gg \mathbf{A}^d$ follows easily from the monotonicity of $\mathbf{A}^d$.

If we are looking for some aggregation operator $\mathbf{A} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ which dominates the product t-norm $T_{\mathbf{P}}$ we can apply once again Proposition 3, i.e., $\mathbf{A} \gg T_{\mathbf{P}}$ and therefore $\mathbf{A}_\varphi \ll (T_{\mathbf{P}})_\varphi$ for some strictly decreasing bijection $\varphi : [0,\infty] \to [0,1]$. If we choose the bijection φ by

$$\varphi : [0,\infty] \to [0,1], \varphi(x) = \exp(-x),$$

we get that

$$(T_{\mathbf{P}})_\varphi(x,y) = \varphi^{-1}(\varphi(x)\cdot\varphi(y)) = -\log(\exp(-x)\cdot\exp(-y)) = x+y$$

such that an aggregation operator $\mathbf{A}$ dominates $T_{\mathbf{P}}$ if and only if its isomorphic transformation $\mathbf{A}_\varphi$ is dominated by the sum, which means in fact that the isomorphic aggregation operator $\mathbf{A}_\varphi$ is subadditive on $[0,\infty]$ (and thus concave).

In order to get an impression which aggregation operators are possible candidates for dominating a continuous Archimedean t-norm and therefore whose isomorphic transformations are subadditive we will consider certain types of aggregation operators, e.g. aggregation operators with neutral element 0 and OWA operators generated by some quantifier function.

Acknowledgements

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t-norms on countable bounded chains

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In this paper we consider t-norms on countable bounded chains. After some general preliminaries, we focus our attention on t-norms defined on C = {0,1,2,...,n}, C = {0,1,2,...,n,...,+∞} and C = {-∞,...,-n,...,-1,0,1,...,n,...,+∞} respectively. Representation theorems for divisible t-norms on C = {0,1,2,...,n,...,+∞} and C = {-∞,...,-n,...,-1,0,1,...,n,...,+∞} are obtained.

After preliminaries, some of the main results are described below

1.- Preliminaries

A *t-norm* T on a bounded chain (C , ≤ , 0 , 1) (a linear ordered set with minimum 0 and maximum 1), is a binary operation on C such that for all x,y,z∈C the following axioms are satisfied:

(T1) T(x,y) = T(y,x)
(T2) T(T(x,y),z) = T(x,T(y,z))
(T3) T(x,y) ≤ T(x,z) whenever y ≤ z
(T4) T(x,1) = x

A t-norm T on a bounded chain C is *divisible* if the following condition holds:

(DIV) For all x,y∈C with x ≤ y there is z∈C such that x = T(y,z)

A t-norm T on a bounded chain C is *archimedean* if the following condition holds:

(AR) For all x,y∈C – {0,1} there exist m∈N such that $x^{(m)} < y$

Similarly, the concept of t-conorm can be introduced in the usual way. For a t-conorm S, the DIV and AR conditions are:

(DIV) For all x,y∈C with x ≤ y there is z∈C such that y = S(x,z)

(AR) For all x,y∈C – {0,1} there exist m∈N such that $x^{(m)} > y$

2.- t-norms on C = {0,1,2,...,n-1,n}

A t-norm T on a finite chain C = {0,1,2,...,n-1,n} is called *discrete* (see [1], [3], [4]). In this case, the divisibility condition can be characterized by means of the Lipschitz property:
$T(x,y) – T(z,y) \leq x – z$ whenever $x \geq z$.
The class of divisible discrete t-norms has been characterized by Mayor and Torrens 1993 ([5]). In this paper they prove that there is a unique archimedean divisible t-norm: $T_L(x,y) = \max(x + y – n , 0)$, and any not archimedean divisible t-norm is an (non trivial) ordinal sum of archimedean divisible t-norms. More precisely:

Theorem 1
Let $n \in N$ and C = {0,1,2,...,n-1,n} be a finite chain with n+1 elements. A t-norm T on C is divisible if and only if there exists a set $I = \{0 = a_0 < a_1 < \ ... < a_p < a_{p+1} = n\} \subset C$ with $p \geq 0$ such that

$$T(x,y) = \begin{cases} a_i + max(x+y-(a_i+a_{i+1}),0) & if\ (x,y) \in [a_i, a_{i+1}]^2 \quad for\ some\ i: 0,1,...,p \\ min(x,y) & otherwise \end{cases}$$

Remark 1
Let us denote by T^I the t-norm described in this theorem. Observe that I is the set of idempotent elements of T^I
In case p = 0 that is I = {0,n}, then $T^I = T_L$
In case p = n – 1 that is I = C, then $T^I = T_M$, $T_M(x,y) = \min(x,y)$

Corollary 1
The correspondance $I \longrightarrow T^I$ is a bijection. There are 2^{n-1} divisible discrete t-norms on a finite chain of n + 1 elements.

Let N be the only *strong negation* (non-increasing and involutive function) on C = {0,1,2,...,n-1,n}, that is N(x) = n – x for all x in C. For any t-norm T on C we consider the so-called N-dual of T: T*(x,y) = N(T(N(x),N(y))) for all x,y in C. T* is a t-conorm. Analogously, given any t-conorm S on C its N-dual is defined by S*(x,y) = N(S(N(x),N(y))). S* is a t-norm on C. Obviously, (T*)* = T and (S*)* = S. Given a pair (T,S) where T is a t-norm and S a t-conorm, we call this pair a *dual pair* when T* = S (or S* = T).
Observe that given a dual pair (T,S) then we have: T is divisible (archimedean) if and only if S is divisible (archimedean).

Corollary 2
Let $n \in N$ and $C = \{0,1,2,...,n-1,n\}$ be a finite chain with n+1 elements. A t-conorm S on C is divisible if and only if there exists a set $I = \{0 = a_0 < a_1 < \; ... < a_p < a_{p+1} = n\} \subset C$ with $p \geq 0$ such that

$$S(x,y) = \begin{cases} a_i + min(x+y-2a_i, a_{i+1}-a_i) & if \ (x,y) \in [a_i, a_{i+1}]^2 \quad for\ some\ i: 0,1,...,p \\ max(x,y) & otherwise \end{cases}$$

Remark 2
Let us denote by S^I the t-norm described in this theorem. Observe that I is the set of idempotent elements of S^I
In case p = 0 that is $I = \{0,n\}$, then $S^I = S_L$: $S_L(x,y) = min(x+y, n)$. This t-conorm is the only one which is divisible and archimedean.
In case p = n – 1 that is I = C, then $S^I = S_M$: $S_M(x,y) = max(x,y)$
There are 2^{n-1} divisible discrete t-conorms on a finite chain of n + 1 elements.

A nice relation between Frank's equation ([2]) and the condition of divisibility for discrete t-norms and t-conorms is given by the following

Theorem 2
A pair (T,S) where T is a t-norm and S a t-conorm on $C = \{0,1,2,...,n-1,n\}$ is a solution of the functional equation $T(x,y) + S(x,y) = x + y$, $x,y \in C$ if and only if T and S are divisible with the same set of idempotent elements.

Remark 3
a) The number of solutions (T,S) of the Frank's equation related to $C = \{0,1,2,...,n-1,n\}$ is 2^{n-1}
b) A solution (T^I, S^I) of the Frank's equation is a dual pair if and only if N(I) = I. There is $2^{[n/2]}$ dual pairs which are solutions of this equation.

3.- t-norms on C = {0,1,2, ... , n , ... , +∞}

Theorem 3
a) There does not exist any divisible archimedean t-norm on C
b) $S(x,y) = x + y$ is the only divisible archimedean t-conorm on C

Theorem 4

a) A t-norm T on C = {0,1,2, ... , n , ... , +∞} is divisible if and only if there exist an infinite set I = {0 = $a_0 < a_1 < a_2 < ... < +\infty$} of elements of C such that

$$T(x,y) = \begin{cases} a_i + max(x+y-(a_i+a_{i+1}),0) & if\ (x,y) \in [a_i, a_{i+1}]^2\ for\ some\ i: 0,1,2,... \\ min(x,y) & otherwise \end{cases}$$

b) A t-conorm S on C = {0,1,2, ... , n , ... , +∞} is divisible if and only if one of the following conditions hold:

b.1) there exists an infinite set I = {0 = $a_0 < a_1 < a_2 < ... < +\infty$}} of elements of C such that

$$S(x,y) = \begin{cases} a_i + min(x+y-2a_i\ , a_{i+1}-a_i) & if\ (x,y) \in [a_i, a_{i+1}]^2\ for\ some\ i: 0,1,... \\ max(x,y) & otherwise \end{cases}$$

or

b.2) there exists an finite set I = {0 = $a_0 < a_1 < a_2 < ... < a_p < +\infty$} of elements of C such that

$$S(x,y) = \begin{cases} a_i + min(x+y-2a_i\ , a_{i+1}-a_i) & if\ (x,y) \in [a_i, a_{i+1}]^2\ for\ some\ i: 0,1,..., p-1 \\ x+y-a_p & if\ \ x,y \geq a_p \\ max(x,y) & otherwise \end{cases}$$

Remark 4.

a) Let us denote by T^I and S^I the t-norm and t-conorm described as in theorem 4. Observe that I is the set of idempotent elements of T^I and S^I. In case I = C, $T^I = T_M$ and $S^I = S_M$. In case I = {0,+∞}, $S^I(x,y) = x + y$ the only archimedean divisible t-conorm on C.

b) There are uncountably many divisible t-norms and t-conorms on C = {0,1,2, ... , n , ... , +∞}

c) There are no dual pairs (T,S) on C (there is no any strong negation on C).

4.- t-norms on C = {-∞,...,-n,...,-1,0,1,...,n,...,+∞}

Theorem 5

a) There does not exist any divisible archimedean t-norm on C

b) There does not exist any divisible archimedean t-conorm on C

Similar representation theorem for this case can be also stated

Theorem 6

a) A t-norm T on C = {-∞,...,-n,...,-1,0,1, ... , n , ... , +∞} is divisible if and only if one of the following conditions hold:

a.1) There exists an infinite set I = { $-\infty < a_1 < a_2 < \ldots < +\infty$ } of elements of C such that

$$T(x,y) = \begin{cases} x + y - a_1 & \text{if } x, y \le a_1 \\ a_i + max(x + y - (a_i + a_{i+1}), 0) & \text{if } (x,y) \in [a_i, a_{i+1}]^2 \text{ for some } i : 1,2,\ldots \\ min(x,y) & \text{otherwise} \end{cases}$$

or

a.2) There exists an infinite set I = { $-\infty < \ldots a_{-1} < a_0 < a_1 < a_2 < \ldots < +\infty$ } of elements of C such that

$$T(x,y) = \begin{cases} a_i + max(x + y - (a_i + a_{i+1}), 0) & \text{if } (x,y) \in [a_i, a_{i+1}]^2 \text{ for some } i : \ldots,-1,0,1,\ldots \\ min(x,y) & \text{otherwise} \end{cases}$$

b) A t-conorm S on C = {-∞,...,-n,...,-1,0,1, ... , n , ... , +∞} is divisible if and only if one of the following conditions hold:

b.1) there exists an infinite set I = { $-\infty < \ldots a_{-1} < a_0 < a_1 < \ldots < +\infty$ } of elements of C such that

$$S(x,y) = \begin{cases} a_i + min(x + y - 2a_i , a_{i+1} - a_i) & \text{if } (x,y) \in [a_i, a_{i+1}]^2 \text{ for some } i : \ldots,-1,0,1,\ldots \\ max(x,y) & \text{otherwise} \end{cases}$$

or

b.2) there exists an infinite set I = {-∞ < ... $a_{-2} < a_{-1} < a_0 < +\infty$} of elements of C such that

$$S(x,y) = \begin{cases} a_i + min(x+y-2a_i\ ,\ a_{i+1} - a_i) & if\ (x,y) \in [a_i, a_{i+1}]^2\ \ for\ some\ i \neq 0 \\ x+y-a_0 & if\ \ x,y \geq a_0 \\ max(x,y) & otherwise \end{cases}$$

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Copulas: an introduction to their properties and applications

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Copulas are functions which join or couple multivariate distribution functions to their one-dimensional margins. In the bivariate case, they share properties with triangular norms, e.g., they map $[0,1]^2$ to $[0,1]$, satisfy certain boundary conditions, are increasing in each place, etc.

Their importance in statistical modeling is primarily a consequence of Sklar's Theorem (1959): Let H be a two-dimensional distribution function with marginal distribution functions F and G. Then there exists a copula C such that $H(x,y) = C(F(x),G(y))$. Conversely, for any distribution functions F and G and any copula C, the function H defined above is a two-dimensional distribution function with margins F and G. Furthermore, if F and G are continuous, C is unique.

In this talk we present an overview of some of the most important properties and applications of copulas. Of particular interest will be the class of Archimedean copulas, which are also triangular norms. As we shall illustrate, it is easy to construct a great variety of such copulas, and members of the class have pleasing statistical properties.

In statistical modeling, dependence is often of more interest than independence, and many descriptions and measures of dependence are "distribution-free" or "scale-invariant," in that they remain unchanged under strictly increasing transformations of random variables. As Schweizer and Wolff (1981) noted, "…it is precisely the copula which captures [such] properties of joint distributions." Consequently, many scale-invariant (i.e., nonparametric) properties and measures of association are expressible in terms of copulas.

With the aid of copulas, we shall explore the relationships among dependence concepts such as concordance, quadrant dependence, and likelihood ratio dependence, and measures of association such as the population versions of Spearman's rho, Kendall's tau, and Gini's gamma. The problem of finding best-possible bounds on certain sets of copulas leads to *quasi-copulas*, and we shall consider briefly some of their properties and applications, including some recent results on the class of multivariate Archimedean quasi-copulas.

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Copulas: uniform approximation, invariance, and applications in aggregation

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1 Introduction

We will discuss some properties of and some relationships between important classes of copulas. First, we show that the both the strict and the non-strict Archimedean copulas form dense subclasses of the class of associative copulas. Next, we characterize copulas, which are invariant under the construction of survival copulas, and some related classes of copulas. Finally, we present an application of copulas in aggregation theory. Full details of these results can be found in [9, 10, 11], for basic references about copulas see [14, 17].

2 Uniform approximation of associative copulas

The set $\mathcal{X} = [0,1]^{[0,1]^2}$ of all functions from the unit square $[0,1]^2$ into the unit interval $[0,1]$, will be equipped with the topology $\mathcal{T}_\infty$ induced by the metric $d_\infty\colon \mathcal{X}^2 \longrightarrow [0,\infty]$ given by $d_\infty(f,g) = \sup\left\{|f(x,y)-g(x,y)| \;\middle|\; (x,y)\in[0,1]^2\right\}$ (corresponding to the uniform convergence).

The class of associative copulas, i.e., of all 1-Lipschitz t-norms [8, 13] is a compact subset of $\mathcal{X}$ (observe that this is not true for the class of all continuous t-norms).

The main result of this part can be formulated as follows (for the proof and more details see [9]):

Theorem 1. *The set $\mathcal{C}_{\mathbf{a}}$ of all associative copulas is the closure of both the set $\mathcal{C}_{\mathbf{s}}$ of all strict copulas and the set $\mathcal{C}_{\mathbf{ns}}$ of all non-strict Archimedean copulas.*

This means in particular that each associative copula can be approximated with arbitrary precision by some strict as well as by some non-strict Archimedean copula. Notice that $\mathcal{C}_{\mathbf{s}}$ and $\mathcal{C}_{\mathbf{ns}}$ are disjoint sets whose union, i.e., the set of Archimedean copulas, is a proper subset of $\mathcal{C}_{\mathbf{a}}$.

Taking into account the results of [8, Section 8.2] (compare also [7]), the convergence of Archimedean copulas is strongly related to the convergence of their corresponding generators. To be precise, a sequence $(C_n)_{n\in\mathcal{N}}$ of Archimedean copulas with generators $(\varphi_n)_{n\in\mathcal{N}}$ converges to an Archimedean copula C with generator φ if and only if there is a sequence of positive constants $(c_n)_{n\in\mathcal{N}}$ such that for each $x \in]0,1]$ we have $\lim_{n\to\infty} c_n \cdot \varphi_n(x) = \varphi(x)$.

Given two copulas C and D, consider their $*$-*product* $C * D\colon [0,1]^2 \longrightarrow [0,1]$ introduced in [2] by

$$C * D(x,y) = \int_0^1 \frac{\partial C(x,t)}{\partial t} \cdot \frac{\partial D(t,y)}{\partial t}\, dt.$$

The function $C * D$ is well-defined since the partial derivatives exist almost everywhere, and it is always a copula, i.e., the $*$-product is an operation on the set $\mathcal{C}$ of all copulas. Moreover, $(\mathcal{C},*)$ is a non-commutative semigroup whose annihilator is the product $T_{\mathbf{P}}$ and whose neutral element is the minimum $T_{\mathbf{M}}$ [12].

As a consequence of Theorem 1 and [2, Theorem 2.3], for each associative copula C and for each copula D there are sequences of Archimedean and strict and non-strict Archimedean copulas $(C_n)_{n\in\mathcal{N}}$, respectively, such that the sequences $(C_n)_{n\in\mathcal{N}}$ and $(C_n * D)_{n\in\mathcal{N}}$ converge uniformly to C and $C * D$, respectively.

3 Invariant copulas

For a given copula C, the corresponding survival copula (which has natural applications in reliability theory) is given by

$$\hat{C}(x,y) = x + y - 1 + C(1-x, 1-y). \tag{1}$$

It is straightforward that the operator ˆ is involutive. Two other important involutive operators on the class of all copulas correspond to $C_{0,1}$ and $C_{1,0}$ (see also [3, 8]) given by, respectively,

$$C_{0,1}(x,y) = x - C(x, 1-y), \tag{2}$$

$$C_{1,0}(x,y) = y - C(1-x, y). \tag{3}$$

We also shall write $C_{0,0} = C$ and $C_{1,1} = \hat{C}$ for each copula C.

Denote by $\mathcal{C}$ the class of all copulas, by $\mathcal{T}$ the class of all associative copulas (i.e., the class of all copulas which are also t-norms), and by $\mathcal{S}$ the class of all commutative (i.e., symmetric) copulas. Moreover, let $\overline{\mathcal{T}}$ be the convex hull of $\mathcal{T}$. Then obviously the following strict inclusions hold:

$$\mathcal{T} \subset \overline{\mathcal{T}} \subset \mathcal{S} \subset \mathcal{C}.$$

Furthermore, for each pair $(i,j) \in \{0,1\}^2$ let $\mathcal{C}_{i,j}$ be the class of all copulas which are invariant under the corresponding involutive transformation, i.e.,

$$\mathcal{C}_{i,j} = \{C \in \mathcal{C} \mid C_{i,j} = C\}.$$

It is trivial that $\mathcal{C}_{0,0} = \mathcal{C}$.

Theorem 2. *Let $C \in \mathcal{C}$ be a copula and $(i,j) \in \{0,1\}^2$. Then we have $C \in \mathcal{C}_{i,j}$ if and only if there is a $D \in \mathcal{C}$ such that $D_{(i,j)} = C$, where*

$$D_{(i,j)} = \frac{D + D_{i,j}}{2}.$$

Denote by $\mathcal{C}^*$ the set of all copulas which are invariant under (1)–(3), i.e.,

$$\mathcal{C}^* = \mathcal{C}_{0,1} \cap \mathcal{C}_{1,0} \cap \mathcal{C}_{1,1}.$$

Theorem 3. *Let C be a copula. Then we have $C \in \mathcal{C}^*$ if and only if there is a copula $D \in \mathcal{C}$ such that $D^* = C$, where*

$$D^* = \frac{D + D_{0,1} + D_{1,0} + D_{1,1}}{4}.$$

Two prominent members of $\mathcal{C}^*$ are the product $T_{\mathbf{P}}$ and the copula K given by $K = \frac{T_{\mathbf{M}}+T_{\mathbf{L}}}{2}$. The importance of Frank t-norms [4] is also exemplified by the following result concerning associative survival copulas.

Proposition 4. *Let C be an associative copula. Then we have $C \in \mathcal{C}_{1,1}$ if and only if there is a $\lambda \in [0,\infty]$ such that $C = T^{\mathbf{F}}_{\lambda}$ or if C is an ordinal sum of Frank t-norms of the form*

$$C = (\langle a_k, b_k, T^{\mathbf{F}}_{\lambda_k} \rangle)_{k \in K},$$

where for each $k \in K$ there is a $k' \in K$ such that $\lambda_k = \lambda_{k'}$ and $a_k + b_{k'} = b_k + a_{k'} = 1$.

However, the only associative copula which is invariant under (2) or (3) is the product $T_{\mathbf{P}}$.

Full details and proofs of the results in this section are contained in [10].

4 Aggregation based on copulas

Let X be a non-empty index set and $f\colon X \longrightarrow [0,1]$ the input system to be aggregated. Let $(X, \mathcal{A}, m)$ be a fuzzy measure space, i.e., $\mathcal{A}$ is a σ-algebra of subsets of X (in the case of a finite set X we usually take $\mathcal{A} = 2^X$), and $m\colon \mathcal{A} \longrightarrow [0,1]$ a fuzzy measure as introduced in [18], thus satisfying $m(\emptyset) = 0$, $m(X) = 1$ and $m(A) \leq m(B)$ whenever $A \subseteq B$. Denote by $\mathcal{L}(\mathcal{A})$ the set of all $\mathcal{A}$-measurable functions from X to $[0,1]$.

Definition 5. Consider two fuzzy measure spaces $(X, \mathcal{A}, m)$ and $(]0,1[^2, \mathcal{B}(]0,1[^2), \mu)$. The functional $M_{m,\mu}\colon \mathcal{L}(\mathcal{A}) \longrightarrow [0,1]$ given by

$$M_{m,\mu}(f) = \mu(D_{m,f}),$$

will be called (m,μ)-*aggregation operator*, where

$$D_{m,f} = \{(x,y) \in \,]0,1[^2 \mid y < m(\{f \geq x\})\}.$$

Special fuzzy measures μ imply reasonable properties of the (m,μ)-aggregation operator $M_{m,\mu}$:

Proposition 6. *Let $C\colon [0,1]^2 \longrightarrow [0,1]$ be a copula and denote by μ_C the unique probability measure on $(]0,1[^2, \mathcal{B}(]0,1[^2))$ with $\mu_C(]0,x[\times]0,y[) = C(x,y)$ for all $(x,y) \in \,]0,1[^2$. Then, for each fuzzy measure space $(X, \mathcal{A}, m)$, the (m,μ_C)-aggregation operator M_{m,μ_C} is an idempotent aggregation operator and we have $M_{m,\mu_C}(\mathbb{1}_A) = m(A)$ for all $A \in \mathcal{A}$.*

Note that such a copula-based approach to aggregation was originally proposed in [5] (see also [6]) for the Frank family of t-norms (see, e.g., [4, 8]). Depending on the choice of the copula C, we obtain some well-known types of integrals.

Example 7. Keeping the notations of Proposition 6, we obtain the following special cases:

(i) If C equals the standard product $T_{\mathbf{P}}$, i.e., $\mu_{T_{\mathbf{P}}}$ is the Lebesgue measure on $\mathcal{B}(]0,1[^2)$, then $M_{m,\mu_{T_{\mathbf{P}}}}$ is just the Choquet integral with respect to m (see [1, 15]).

If, in addition, m is a σ-additive measure on $(X,\mathcal{A})$, then $M_{m,\mu_{T_{\mathbf{P}}}}$ coincides with the classical Lebesgue integral with respect to m, and for $X = \{1,2,\dots,n\}$ we obtain a weighted mean.

If $X = \{1,2,\dots,n\}$ and if m is a symmetric fuzzy measure on $(X,2^X)$ then $M_{m,\mu_{T_{\mathbf{P}}}}$ is an OWA operator [19].

(ii) If C equals the minimum $T_{\mathbf{M}}$ then

$$\mu_{T_{\mathbf{M}}}(A) = \lambda(\{x \in]0,1[\mid (x,x) \in A\}),$$

and $M_{m,\mu_{T_{\mathbf{M}}}}$ equals the Sugeno integral (see [18] and also [15]).

If $X = \{1,2,\dots,n\}$ and if m is a symmetric fuzzy measure on $(X,2^X)$ then $M_{m,\mu_{T_{\mathbf{M}}}}$ is an WOWM (weighted ordered weighted maximum) operator [16].

(iii) If C equals the Łukasiewicz t-norm $T_{\mathbf{L}}$ then

$$\mu_{T_{\mathbf{L}}}(A) = \lambda(\{x \in]0,1[\mid (x,1-x) \in A\}),$$

and if the index set X is finite, then $M_{m,\mu_{T_{\mathbf{L}}}}$ is the so-called opposite Sugeno integral [5].

Concerning dual aggregation operators we obtain the following result:

Proposition 8. *Let X be a finite set. Keeping the notations and hypotheses of Proposition 6, we have*

$$M^d_{m,\mu_C} = M_{m^d,\mu_{\hat{C}}}. \tag{4}$$

Observe that if a copula C coincides with its survival copula $\hat{C}$, then a special form of (4) holds, namely, $M^d_{m,\mu_C} = M_{m^d,\mu_C}$. All copulas with the property $C = \hat{C}$ were characterized in [10]. In particular, because of Proposition 4 (see also [4]) an associative copula C coincides with its survival copula $\hat{C}$ if and only if C is either a member of the family of Frank t-norms $(T^{\mathbf{F}}_\lambda)_{\lambda\in[0,\infty]}$ or if C is a symmetric ordinal sum of Frank t-norms [8, 10]. Because of $T^{\mathbf{F}}_0 = T_{\mathbf{M}}$, $T^{\mathbf{F}}_1 = T_{\mathbf{P}}$, and $T^{\mathbf{F}}_\infty = T_{\mathbf{L}}$, for all Sugeno, Choquet and opposite Sugeno integrals we have (for X finite)

$$\left(\int_X f\,dm\right)^d = \int_X f\,dm^d.$$

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T-norms and copulas in fuzzy preference modelling

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This contribution is organized in two major parts. The aim of the first part is to revise the axiomatic construction of (additive) fuzzy preference structures and is the result of a joint collaboration with J. Fodor. We first introduce the notion of a generator triplet consisting of a preference, indifference and incomparability generator, suitable for constructing fuzzy preference structures from a given fuzzy preference relation. We then show that such a triplet is uniquely determined by a symmetric indifference generator i located between the Łukasiewicz t-norm and the minimum operator. The main results concern the link with the axiomatic framework of Fodor and Roubens. We introduce the notion of a monotone generator triplet and show that such a triplet is characterized by an increasing 1-Lipschitz indifference generator (such as a commutative copula, for instance). Further characterizations concern that case that i is an ordinal sum of Frank t-norms, and finally, the case that i is a Frank t-norm, which corresponds to the fact that the generator triplet is determined by t-norms only (in fact, by two Frank t-norms with reciprocal parameters).

The second part consists of a study of the transitivity of a reciprocal representation of fuzzy preference structures without incomparability and is the result of a joint collaboration with H. De Meyer and S. Jenei. For a reciprocal relation Q on a set of alternatives A, we introduce the concept of cycle-transitivity which is based upon the ordering of the degrees $Q(a,b)$, $Q(b,c)$ and $Q(c,a)$, for all triplets $(a,b,c) \in A^3$. Each type of cycle-transitivity is determined by an upper bound U; there is also an associated dual lower bound. We investigate suitable upper bounds and introduce the notion of a self-dual upper bound. We show that cycle-transitivity generalizes stochastic transitivity. Also, we show that under very mild conditions, fuzzy transitivity (i.e. C-transitivity, with C a conjunctor) can be translated into cycle-transitivity. For a commutative copula C, for instance, C-transitivity is equivalent to cycle-transitivity with as upper bound the dual of C and as lower bound the corresponding survival copula. In the more familiar context of t-norms, this means for instance that T-transitivity with T a Frank t-norm, is equivalent to cycle-transitivity with as upper bound its dual t-conorm and as lower bound the Frank t-norm T itself.

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The role of copulas in discrete and continuous dice models

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We first introduce the notion of a discrete dice model as a framework for describing a class of probabilistic relations (or equivalently, a class of reciprocal relations). The transitivity of the probabilistic relation generated by such a dice model is a special type of cycle-transitivity that is situated between moderate stochastic transitivity or product-transitivity on the one side, and Łukasiewicz-transitivity on the other side, and which we call dice-transitivity.

The discrete dice model can be regarded as a consistent way of mutually comparing random variables from a given collection of independent discrete random variables that are uniformly distributed on discrete number sets. This interpretation allows to extend the dice model so that arbitrary, not necessarily independent, discrete or absolutely continuous random variables can be compared. It is shown that the n-copula expressing the joint cumulative distribution (c.d.f.) of the collection of random variables (generalized dice) as a function of the univariate marginal c.d.f.'s, plays a key role in the determination of the transitivity of the probabilistic relation generated by the collection. When the copula is the product copula (P-copula), the random variables are independent and for arbitrary marginal c.d.f.'s, the transitivity of the generated probabilistic relation is at least dice-transitive. When the copula is the min-copula (M-copula), the generated probabilistic relation is at least Łukasiewicz-transitive, and when the copula for bivariate marginals is the Łukasiewicz copula (W-copula), then the generated probabilistic relation is at least moderately stochastic transitive.

Moreover, if the marginal distributions are restricted to normal distributions, then the W-copula and the P-copula yield probabilistic relations that are moderately stochastic transitive, whereas the M-copula yields probabilistic relations that are weakly stochastic transitive. This is also the type transitivity obtained when the joint c.d.f. is the standard multivariate normal distribution with covariance matrix Σ.

Finally, we discuss some interesting features of the discrete models obtained by considering respectively the M-copula or W-copula in combination with discrete uniform marginal c.d.f.'s (the P-copula combined with discrete uniform marginal c.d.f.'s yields the classical dice model).

Keywords: copulas, dice model, probabilistic relation, stochastic transitivity, T-transitivity.

On fuzzy type theory

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Abstract

In the paper, the formal type theory is generalized to fuzzy one. The structure of truth values is assumed to be the Łukasiewicz algebra since the formulation of FTT based on it can be done in a most elegant way. Some properties of theories of fuzzy type theory are demonstrated and the completeness saying that each consistent theory has a frame model is proved. We will follow the way of the development of the classical type theory as elaborated especially by A. Church and L. Henkin.

1 Syntax and Semantics of Fuzzy Type Theory

In this paper, we present the formal system of FTT. Because of the limited space and complicated technical character, we have omitted most proofs. The complete paper with full proofs can be obtained from the author upon request.

1.1 Basic syntactical elements

1.1.1 Types

Let ε, o be distinct objects. The set of types is the smallest set *Types* satisfying:

(i) $\varepsilon, o \in Types$,

(ii) If $\alpha, \beta \in Types$ then $(\alpha\beta) \in Types$.

The type ε represents elements and o truth values.

1.1.2 Primitive symbols

(i) Variables $x_\alpha, \dots$ where $\alpha \in Types$.

(ii) Special constants $c_\alpha, \dots$ where $\alpha \in Types$. We will consider the following concrete special constants: $\mathbf{E}_{(o\alpha)\alpha}$ for every $\alpha \in Types$ and $\mathbf{C}_{(oo)o}$.

(iii) Auxiliary symbols: λ, brackets.

1.1.3 Formulas

The set $Form_\alpha$ is a set of formulas of type $\alpha \in Types$, which is the smallest set satisfying:

(i) $x_\alpha \in Form_\alpha$ and $c_\alpha \in Form_\alpha$,

(ii) if $B \in Form_{\beta\alpha}$ and $A \in Form_\alpha$ then $(BA) \in Form_\beta$,

(iii) if $A \in Form_\beta$ then $\lambda x_\alpha A \in Form_{\beta\alpha}$,

If $A \in Form_\alpha$ is a formula of the type $\alpha \in Types$ then we will write A_α.

1.2 Semantics

1.2.1 Truth values

We will work with the structure of truth values forming the Łukasiewicz MV-algebra

$$\mathcal{L}_{Ł} = \langle [0,1], \otimes, \oplus, \neg, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

where $a \otimes b = 0 \vee (a + b - 1)$ is Łukasiewicz conjunction, $a \oplus b = 1 \wedge (a + b)$ is Łukasiewicz disjunction, $a \to b = \neg a \oplus b = 1 \wedge (1 - a + b)$ is implication and $\neg a = 1 - a$ is negation $(a, b \in [0,1])$. We, furthermore, work with the *biresiduation* operation $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

1.2.2 Frame

Let D be a set of objects and L be a set of truth values. A *frame* based on D, L is a family $(M_\alpha)_{\alpha \in Types}$ of sets where

(i) $M_\varepsilon = D$ is a set of objects and $M_o = L$ is a set of truth values,

(ii) For each type $\gamma = \beta\alpha$, M_γ is a set of functions $M_\gamma \subseteq M_\beta^{M_\alpha}$ specified below.

1.2.3 Fuzzy equality

The *fuzzy equality* on M_α is a binary fuzzy relation $=_\alpha \underset{\sim}{\subset} M_\alpha \times M_\alpha$, i.e. a function

$$=_\alpha : M_\alpha \times M_\alpha \to L.$$

To stress that $m_\alpha =_\alpha m'_\alpha$ holds in some degree $c \in L$ we will write $[m_\alpha =_\alpha m'_\alpha]$.

The fuzzy equality is supposed to be reflexive $[m_\alpha =_\alpha m_\alpha] = \mathbf{1}$, symmetric $[m_\alpha =_\alpha m'_\alpha] = [m'_\alpha =_\alpha m_\alpha]$, and $\otimes$-transitive

$$[m_\alpha =_\alpha m'_\alpha] \otimes [m'_\alpha =_\alpha m''_\alpha] \leq [m_\alpha =_\alpha m''_\alpha], \qquad m_\alpha, m'_\alpha, m''_\alpha \in M_\alpha.$$

1.2.4 Extensional functions

Let $F : M_{\alpha_1} \times \cdots \times M_{\alpha_n} \rightarrow M_\beta$ be a function. We say that it is extensional w.r.t fuzzy equalities $=_{\alpha_1}, \ldots, =_{\alpha_n}, =_\beta$ if there are natural numbers $q_1, \ldots, q_n \geq 1$ such that

$$[m_{\alpha_1} \ =_{\alpha_1} \ m'_{\alpha_1}]^{q_1} \otimes \cdots \otimes [m_{\alpha_n} \ =_{\alpha_n} \ m'_{\alpha_n}]^{q_n} \leq [F(m_{\alpha_1}, \ldots, m_{\alpha_n}) \ =_\beta \ F(m'_{\alpha_1}, \ldots, m'_{\alpha_n})] \quad (2)$$

holds for all $m_{\alpha_i}, m'_{\alpha_i} \in M_{\alpha_i}$, $i = 1, \ldots, n$.

Lemma 1. *Let $=_\beta$ be an extensional fuzzy equality. Then the function $=_{\beta\alpha}: M_\beta^{M_\alpha} \times M_\beta^{M_\alpha} \rightarrow L$ defined by*

$$[m_{\beta\alpha} =_{\beta\alpha} m'_{\beta\alpha}] = \bigwedge_{m_\alpha \in M_\alpha} [m_{\beta\alpha}(m_\alpha) =_\beta m'_{\beta\alpha}(m'_\alpha)] \quad (3)$$

for every $m_{\beta\alpha}, m'_{\beta\alpha} \in M_\beta^{M_\alpha}$ is an extensional fuzzy equality.

1.2.5 Frame model

Let $(M_\alpha)_{\alpha \in Types}$ be a frame. Then the frame model is a tuple

$$I = \langle (M_\alpha, =_\alpha)_{\alpha \in Types}, \mathcal{L} \rangle \quad (4)$$

where:

(i) The $\mathcal{L}$ is the Łukasiewicz MV-algebra, where its support $L = M_o$.

(ii) The $=_\alpha$ is a fuzzy equality on M_α where $=_o$ is $\leftrightarrow$, $=_\varepsilon \subsetapprox M_\varepsilon \times M_\varepsilon$ is an extensional fuzzy equality on M_ε and otherwise $=_\alpha$ is the fuzzy equality (3).

(iii) If $\alpha \neq o, \varepsilon$ then each function $F \in M_\alpha$ is extensional.

1.2.6 Basic definitions

(a) Equivalence $\equiv \ := \lambda x_\alpha (\lambda y_\alpha \mathbf{E}_{(o\alpha)\alpha} y_\alpha) x_\alpha$.

(b) Conjunction $\wedge := \lambda x_o (\lambda y_o \mathbf{C}_{(oo)o} y_o) x_o$.

1.2.7 Interpretation

Given a frame model I, the interpretation I of all formulas is the assignment of meaning to them.

An assignment p to the variables over I is a function on variables such that $p(x_\alpha) \in M_\alpha$ for every type $\alpha \in Types$. The set of all assignments over I be denoted by $\mathrm{Asg}(I)$.

(i) If x_α is a variable then $I_p(x_\alpha) = p(x_\alpha)$.

(ii) If c_α is a constant then $I_p(c_\alpha)$ is some element from M_α. If $\alpha \neq o, \varepsilon$ then $p(c_\alpha)$ is an extensional function. As a special case $I_p(\mathbf{E}_{(o\alpha)\alpha})(m')(m) = [m =_\alpha m'] \in L$ and $I_p(\mathbf{C}_{(oo)o})(a)(b) = a \wedge b$ for all $a, b \in L$.

(iii) The interpretation of the formula $B_{\beta\alpha} A_\alpha$ of type β is $I_p(B_{\beta\alpha} A_\alpha) = I_p(B_{\beta\alpha})(I_p(A_\alpha))$.

(iv) The interpretation of the formula $\lambda x_\alpha A_\beta$ of type $\beta\alpha$ is the function

$$I_p(\lambda x_\alpha A_\beta) = F : M_\alpha \to \{I_{p'}(A_\beta) \mid p' \in \mathrm{Asg}(I)\}$$

such that $F(m_\alpha) = I_{p'}(A_\beta)$ for some assignment p' such that $p'(x_\alpha) = m_\alpha$ and $p'(y_\gamma) = p(y_\gamma)$ for all $y_\gamma \neq x_\alpha$ (i.e. p' differs from p only in the variable x_α) and the function F is extensional w.r.t "$=_\alpha$" and "$=_\beta$".

Let us denote the set of assignments p' due to Item (iv) by $\mathrm{Asg}(I_p)$.

Lemma 2. *For every $\alpha \in Types$ and every assignment p, $I_p(A_\alpha) \in M_\alpha$ holds true.*

1.2.8 Further definitions

(a) Representation of truth $\top := (\lambda x_o x_o \equiv \lambda x_o x_o)$ and falsity $\bot := (\lambda x_o x_o \equiv \lambda x_o \top)$.

(b) Negation $\neg := \lambda x_o(\bot \equiv x_o)$.

(c) Implication $\Rightarrow\ := \lambda x_o(\lambda y_o((x_o \wedge y_o) \equiv x_o))$.

(d) Special connectives: $\vee := \lambda x_o(\lambda y_o((x_o \Rightarrow y_o) \Rightarrow y_o))$, (*disjunction*), $\mathbf{\&} := \lambda x_o(\lambda y_o(\neg(x_o \Rightarrow \neg y_o)))$, (*strong (Łukasiewicz) conjunction*), $\nabla := \lambda x_o(\lambda y_o(\neg(\neg A_o \mathbf{\&} \neg B_o)))$ (*strong (Łukasiewicz) disjunction*).

(e) Quantifiers: $(\forall x_\alpha)A_o := (\lambda x_\alpha A_o \equiv \lambda x_\alpha \top)$ and $(\exists x_\alpha)A_o := \neg(\forall x_\alpha)\neg A_o$.

As a special case, $A^n := \underbrace{A \mathbf{\&} \cdots \mathbf{\&} A}_{n-\text{times}}$.

Lemma 3. *Let $A_o, B_o \in Form_o$. Then for every assignment $p \in Asg(I)$*

(a) $I_p(\top) = \mathbf{1}$, $I_p(\bot) = \mathbf{0}$.

(b) $I_p(\neg A_o) = I_p(A_o) \to \mathbf{0}$

(c) $I_p(A_o \vee B_o) = I_p(A_o) \vee I_p(B_o)$

(d) $I_p(A_o \Rightarrow B_o) = I_p(A_o) \to I_p(B_o)$

(e) $I_p(A_o \mathbf{\&} B_o) = I_p(A_o) \otimes I_p(B_o)$

(f) $I_p(A_o \nabla B_o) = I_p(A_o) \oplus I_p(B_o)$

(g) $I_p((\forall x_\alpha)A_o) = \bigwedge_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' \in Asg(I_p)}} I_{p'}(A_o)$

(h) $I_p((\exists x_\alpha)A_o) = \bigvee_{\substack{m_\alpha = p'(x_\alpha) \in M_\alpha \\ p' \in Asg(I_p)}} I_{p'}(A_o)$

1.2.9 Axioms

(FT1). $(x_\alpha \equiv y_\alpha)^q \Rightarrow (f_{\beta\alpha} x_\alpha \equiv f_{\beta\alpha} y_\alpha)$ for some $q \geq 1$.

(FT2). $(\forall x_\alpha)(f_{\beta\alpha} x_\alpha \equiv g_{\beta\alpha} x_\alpha) \equiv (f_{\beta\alpha} \equiv g_{\beta\alpha})$

(FT3). $(\lambda x_\alpha B_\beta)A_\alpha \equiv C_\beta$

where C_β is obtained from B_β by replacing all substitutable occurrences of x_α in it by A_α.

(FT4). $(A_o \equiv \top) \equiv A_o$

(FT5). $(A_o \vee B_o) \equiv (B_o \vee A_o)$

(FT6). $A_o \wedge B_o \equiv B_o \wedge A_o$

(FT7). $A_o \wedge \top \equiv A_o$

(FT8). $(A_o \wedge B_o) \wedge C_o \equiv A_o \wedge (B_o \wedge C_o)$

(FT9). $(A_o \Rightarrow (B_o \Rightarrow C_o)) \Rightarrow (B_o \Rightarrow (A_o \Rightarrow C_o)$

(FT10). $(\neg B_o \Rightarrow \neg A_o) \equiv (A_o \Rightarrow B_o)$

(FT11). $(\forall x_\alpha)(A_o \Rightarrow B_o) \Rightarrow (A_o \Rightarrow (\forall x_\alpha)B_o)$

1.2.10 Inference rule and provability

The following is an inference rule in FTT.

(R) *Let $A_\alpha \equiv A'_\alpha$ and $B \in Form_o$. Then we infer B' where B' comes form B by replacing one occurrence of A_α, which is not preceded by λ, by A'_α.*

The concept of provability and proof are defined in the same way as in classical logic. A theory T over FTT is a set of formulas of type o, i.e. $T \subset Form_o$. If $A \in Form_o$ and it is provable in T then we write $T \vdash A$, as usual.

Lemma 4. *(a) For every interpretation $\mathcal{I}$ and assignment p, $\mathcal{I}_p(FTi) = \mathbf{1}$ where $i = 1, \ldots, 11$.*

(b) The inference rule (R) is sound, i.e. $\mathcal{I}_p(A_\alpha \equiv A'_\alpha) \otimes \mathcal{I}_p(B) \leq \mathcal{I}_p(B')$.

Corollary 5 (Soundness). *The fuzzy type theory is sound, i.e. the following holds for every theory T: If $T \vdash A_o$ then $\mathcal{I}_p(A_o) = \mathbf{1}$ holds for every assignment $p \in Asg$ and every frame model $\mathcal{I}$.*

2 Special properties of FFT

Theorem 6. *The following is provable in FTT.*

(a) *If* $\vdash A_o$ *and* $\vdash A_o \equiv B_o$ *then* $\vdash B_o$.

(b) $\vdash A_\alpha \equiv A_\alpha$, $\alpha \in Types$.

(c) $\vdash \top$.

(d) *If* $\vdash A_\alpha \equiv B_\alpha$ *then* $\vdash B_\alpha \equiv A_\alpha$.

(e) *If* $\vdash A_\alpha \equiv B_\alpha$ *and* $\vdash B_\alpha \equiv C_\alpha$ *then* $\vdash A_\alpha \equiv C_\alpha$.

(f) $\vdash A_o$ *iff* $\vdash A_o \equiv \top$.

Theorem 7 (Logical rules). *(a)* *If* $\vdash A_o$ *and* $\vdash A_o \Rightarrow B_o$ *then* $\vdash B_o$.

(b) *If* $\vdash A_o$ *then* $\vdash (\forall x_\alpha)A_o$.

Theorem 8. *(a)* $\vdash (A_o \Rightarrow B_o) \Rightarrow ((B_o \Rightarrow C_o) \Rightarrow (A_o \Rightarrow C_o))$

(b) $\vdash A_o \wedge A_o \equiv A_o$

(c) $\vdash A_o \Rightarrow (B_o \Rightarrow A_o)$

(d) $\vdash (\neg A_o) \equiv (A_o \Rightarrow \bot)$,

(e) $\vdash \neg\neg A_o \equiv A_o$.

(f) $\vdash (x_\beta \equiv y_\beta)^{q_1} \Rightarrow ((f_{\alpha\beta} \equiv g_{\alpha\beta})^{q_2} \Rightarrow (f_{\alpha\beta} x_\beta \equiv g_{\alpha\beta} y_\beta))$ *for some* $q_1, q_2 \geq 1$.

Theorem 9 (Substitution axioms). *(a)* $\vdash (\forall x_\alpha) B_o \Rightarrow C_o$,

(b) $\vdash C_o \Rightarrow (\exists x_\alpha) B_o$.

where C_o *is obtained from* B_o *by substitution of some formula* A_α *substitutable to it for all free occurrences of* x_α.

It follows from the previous presentation that FTT contains the formal system of predicate Łukasiewicz logic and hence, all its theorems also provable in FTT.

3 Theories in FTT

If T be a theory and $A \in Form_o$ a formula the by $T \cup \{A\}$ is a theory whose set of special axioms is extended by A.

Theorem 10 (Deduction theorem). *Let* T *be a theory,* $A \in Form_o$ *a formula. Then* $T \cup \{A\} \vdash B$ *iff there is* $n \geq 1$ *such that* $T \vdash A^n \Rightarrow B$ *holds for every formula* $B \in Form_o$.

A theory T is *contradictory* if $T \vdash \bot$. Otherwise it is *consistent*. A theory T is *complete* if for every two formulas A_o, B_o either $T \vdash A_o \Rightarrow B_o$ or $T \vdash B_o \Rightarrow A_o$. A theory T is *maximal consistent* if each its extension T', $T' \supset T$ is inconsistent.

Theorem 11. *Every consistent theory* T *can be extended to a maximal consistent theory* $\overline{T}$ *which is complete.*

3.1 Syntactic model of FTT and completeness

Let T be a consistent complete theory. We define the equivalence on the set of all formulas as follows:

$$A_\alpha \sim B_\alpha \Leftrightarrow T \vdash A_\alpha \equiv B_\alpha.$$

The equivalence class of a formula A_α of type α will be denoted by $|A_\alpha|$. Furthermore, we put $M_\alpha = \{|A_\alpha| \mid A_\alpha \in Form_\alpha\}$, for all $\alpha \in Types$. If $\alpha \neq o, \varepsilon$ then

$$M_{\beta\alpha} = \{m_{\beta\alpha} \mid m_{\beta\alpha} : M_\alpha \to M_\beta\}$$

where $m_{\beta\alpha} = |A_{\beta\alpha}|$ for some $A_{\beta\alpha} \in Form_{\beta\alpha}$ and $m_{\beta\alpha}(|B_\alpha|) = |A_{\beta\alpha} B_\alpha|$ for every $B_\alpha \in Form_\alpha$. We may define the operations on the set M_o using logical connectives as usual. Then we obtain the following theorem.

Theorem 12. *The algebra*

$$\mathcal{L}_T = \langle M_o, \otimes, \oplus, \neg, \mathbf{1}, \mathbf{0} \rangle \tag{5}$$

is a locally finite, linearly ordered MV-algebra.

Now we will consider the embedding

$$h : \mathcal{L}_T \to \mathcal{L}_{\text{Ł}}. \tag{6}$$

Recall that h preserves all suprema and infima existing in $\mathcal{L}_T$ (see also [4], Lemma 5.4.23).

To define fuzzy equality, we put $[|A_\alpha| =_\alpha |B_\alpha|] = h(|A_\alpha \equiv B_\alpha|)$ for all $\alpha \in \mathit{Types}$ where h is the embedding. It can be proved that this is an extensional fuzzy equality on M_α and it has the properties discussed above.

The syntactic frame model is the tuple

$$I^S = \langle (M_\alpha, =_\alpha)_{\alpha \in \mathit{Types}}, \mathcal{L}_{\text{Ł}} \rangle \tag{7}$$

where $M_o = [0,1]$ and for all $\alpha \in \mathit{Types} - \{o\}$, M_α are the sets.

The assignment p of elements to variables is the following: $p(x_o) = h(|A_o|)$ and $p(x_\alpha) = |A_\alpha|$ for $\alpha \neq o$ where $|A_\alpha| \in M_\alpha$. We put:

(i) If x_α is a variable then $I_p^S(x_\alpha) = p(x_\alpha)$.

(ii) If c_α, $\alpha \neq o$ is a constant then $I_p^S(c_\alpha)$ is some element from M_α. As a special case, $I_p^S(c_o)$ is element from $h(\mathcal{L}_T)$. The interpretation $I_p^S(\mathbf{E}_{(o\alpha)\alpha})$ is the fuzzy equality depending on the type α and $I_p^S(\mathbf{C}_{(oo)o})$ is the meet operation $\wedge$ on $h(\mathcal{L}_T)$.

(iii) Interpretation of the formula $B_{\beta\alpha}A_\alpha$ is $I_p^S(B_{\beta\alpha}A_\alpha) = I_p^S(B_{\beta\alpha})(I_p(A_\alpha))$.

(iv) The interpretation of the formula $\lambda x_\alpha A_\beta$ of type $\beta\alpha$ is the function

$$I_p^S(\lambda x_\alpha A_\beta) = F : M_\alpha \to \{I_{p'}^S(A_\beta) \mid p' \in \mathrm{Asg}(I^S)\}$$

such that $F(|A_\alpha|) = I_{p'}^S(A_\beta) = |(\lambda x_\alpha A_\beta)A_\alpha|$ for some assignment p' which differs from p only in the variable x_α.

Lemma 13. *For all* $\alpha \in \mathit{Types}$

$$[|A_\alpha| =_\alpha |B_\alpha|] = h(|A_\alpha \equiv B_\alpha|) = I_p^S(A_\alpha \equiv B_\alpha). \tag{8}$$

Lemma 14. *Each function* $m_{\beta\alpha} = |C_{\beta\alpha}| \in M_{\beta\alpha}$ *is extensional w.r.t.* $=_\alpha$ *and* $=_\beta$.

Theorem 15. *A theory* T *is consistent iff it has a model* I.

4 Conclusion

This paper is focused on further development of the ideas of fuzzy logic towards more general framework, which is the type theory. Our motivation stems especially from linguistics since fuzzy set theory presents itself first of all as mathematical theory enabling to master parts of natural language semantics, namely when vagueness is prevailing. Since natural language is much more complex phenomenon than predicate first-order logic (classical or fuzzy), we are convinced that higher order logical calculus is necessary.

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Varieties generated by t-norms

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We are concerned with the variety $\mathcal{T}$ of algebras of type $(2,2,2,0,0)$ generated by the algebra $(\mathbb{I},\circ)$, where $\mathbb{I} = ([0,1],\wedge,\vee,0,1)$ is the unit interval with minimum and maximum determined by the usual order and.$\circ \neq \wedge$ is a continuous t-norm. We have shown that a strict t-norm and a nilpotent t-norm, and in fact any continuous t-norm except minimum, generate the same variety. Moreover, this variety is not generated by any finite algebra [1,2]. However, we have not determined whether or not there is a finite set of equations that determines this variety.

In an attempt to answer this question, we consider the variety $\mathcal{E}$ of algebras of type $(2,2,2,0,0)$ consisting of all commutative, lattice-ordered monoids $(\mathbb{L},\circ)$. By this we mean

- $L = (L,\wedge,\vee,0,1)$ is a bounded, distributive lattice
- $(L,\circ,1)$ is a commutative semigroup with identity
- The semigroup operation $\circ$ distributes over both meet and join.

The variety $\mathcal{E}$ is determined by a finite set of equations—namely, the equations that define a bounded, distributive lattice, together with the equations that define a commutative semigroup with identity and the equations that say $\circ$ distributes over both meet and join. Clearly $\mathcal{E}$ contains the variety generated by an algebra $(\mathbb{I},\circ)$ for any t-norm $\circ$, in particular, $T \subseteq \mathcal{E}$.

An algebra is subdirectly irreducible if for every subdirect product embedding $\mathbb{A} \subseteq \prod_i \mathbb{A}_i$, at least one of the projections is one-to-one, hence an isomorphism. An equivalent condition is that there is a pair of elements (a,b) with $a \neq b$ that are not separated by any homomorphism that is not an embedding, that is, every homomorphism f from $\mathbb{A}$ to another algebra is either one-to-one or satisfies $f(a) = f(b)$. Another way to say this is that (a,b) belongs to every nontrivial congruence of $\mathbb{A}$. A variety is generated by its subdirectly irreducible algebras, and identifying these subdirectly irreducible algebras is key to understanding the variety.

Proposition 1. *A subdirectly irreducible algebra $(\mathbb{L},\circ)$ in $\mathcal{E}$ has a unique atom a that lies beneath every nonzero element of $\mathbb{L}$, and the pair $(0,a)$ belongs to every nontrivial congruence.*

A nonempty subset I of L is an **ideal** of $\mathbb{L}$ if for every $x \in I$, $y \in L$, $y \leq x$ implies $y \in I$, and for every $x,y \in I$, $x \vee y \in I$. An ideal $\mathbb{I}$ of $\mathbb{L}$ is **prime** if $x \wedge y \in I$ implies $x \in I$ or $y \in I$. For $x \in L$, $\mathbb{I}$ an ideal of $\mathbb{L}$, $(\mathbb{I} : x) = \{y \in L : xy \in I\}$. For $\mathbb{I} = \{0\}$, this is called the **annihilator** of x, and we write $(\{0\} : x) = (0 : x)$.

Lemma 2. *The following hold for elements $x,y \in L$ and ideals $\mathbb{I} \subseteq \mathbb{L}$.*

1. $(\mathbb{I} : x \vee y) = (\mathbb{I} : x) \cap (\mathbb{I} : y)$
2. $(\mathbb{I} : x \wedge y) = (\mathbb{I} : x) \cup (\mathbb{I} : y)$ *if* $\mathbb{I}$ *is prime.*
3. $y\,(\mathbb{I} : xy) \subseteq (\mathbb{I} : x) \subseteq (\mathbb{I} : x \circ y)$

From this it follows quickly that if $x \leq y$ then $(\mathbb{I} : x) \supseteq (\mathbb{I} : y)$.

Proposition 3. *Let* $\mathbb{I}$ *be a prime ideal of* $\mathbb{L}$. *The relation on* $(\mathbb{L}, \circ)$ *defined by*

$$x \cong y \text{ if and only if } (\mathbb{I} : x) = (\mathbb{I} : y)$$

is a congruence.

To prove this proposition we need to show that if $(\mathbb{I} : x) = (\mathbb{I} : y)$, then for any $z \in L$,

$$(\mathbb{I} : (x \vee z)) = (\mathbb{I} : (y \vee z))$$
$$(\mathbb{I} : (x \wedge z)) = (\mathbb{I} : (y \wedge z))$$
$$(\mathbb{I} : z \circ x) = (\mathbb{I} : z \circ y)$$

which is straight forward. The following theorem gives a useful characterization of the subdirectly irreducible algebras in $\mathcal{E}$.

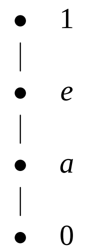
Theorem 4. *An algebra* $(\mathbb{L}, \circ)$ *is subdirectly irreducible if and only if* $(\mathbb{L}, \circ)$ *has a unique atom that lies below every nonzero element of* $\mathbb{L}$ *and the annihilators* $\{(0 : x) : x \in \mathbb{L}\}$ *are distinct.*

Theorem 5. *If an algebra* $(\mathbb{L}, \circ)$ *in* $\mathcal{E}$ *is subdirectly irreducible, then* $(\mathbb{L}, \circ)$ *is a chain. A finite chain is subdirectly irreducible in* $\mathcal{E}$ *if and only if the residual* $\eta(x) = \bigvee \{y \in L : y \circ x = 0\}$ *is an involution.*

Every subvariety of $\mathcal{E}$ is generated by its finite members. The problem of showing that $\mathcal{T} = \mathcal{E}$ (or $\mathcal{T} \neq \mathcal{E}$) is thus reduced to identifying which finite chains are subdirectly irreducible in $\mathcal{E}$, and then showing whether or not these subdirectly irreducibles are generated by $(\mathbb{I}, \circ)$.

Every finite cyclic algebra in $\mathcal{E}$ is subdirectly irreducible. These can be realized as subalgebras of the Łukasiewicz (bounded product) t-norm, hence belong to $\mathcal{T}$.

Another example of a subdirectly irreducible algebra in $\mathcal{E}$ is the four element chain



with the multiplication $e \circ e = e$ and $e \circ a = a \circ a = 0$. This algebra can be obtained as a homomorphic image of an ultrapower of $(\mathbb{I}, \circ)$, with $\circ$ a nilpotent t-norm, thus also belongs to T.

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Additive refinements of qualitative decision criteria
Part I: possibilistic preference functionals

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Information about preference and uncertainty in decision problems cannot always be quantified in a simple way, but only qualitative evaluations can sometimes be attained. As a consequence, the topic of qualitative decision theory is a natural one to consider: can we make efficient decision on the basis of qualitative information?

Giving up the quantification of utility and uncertainty has lead to give up expected utility (EU) criterion as well — the principe of qualitative decision [3, 2] making is to model uncertainty by an *ordinal* plausibility relation on events and preference by a *weak order* on consequences of decisions. In [3] two qualitative criteria based on possibility theory, an optimistic and a pessimistic one, whose definitions only require a (finite) completely ordered scale for utility and uncertainty are proposed. Let S be a set of states, X a set of consequence and X^S the set of possible acts (in decision under uncertainty, an act is a function $f : S \mapsto X$):

Definition 1 (Possibilistic utilities). Let $L = [0_L, 1_L]$ be a finite ordinal scale , $n : L \to L$ the order reversing function of L, $\pi : S \to L$ a possibility distribution on S and $\mu : X \to L$ a utility function on X.

- $< S, X, L, \pi, \mu >$ will be called a qualitative possibisitic utility model (QPU-model)
- the optimistic possibilistic utility of f is:
  $U_{OPT,\pi,\mu}(f) = \max_{s \in S} \min(\pi(s), \mu(f(s)))$
- pessimisitic utility of f is : $U_{PES,\pi,\mu}(f) = \min_{s \in S} \max(n(\pi(s)), \mu(f(s)))$
- $\succeq_{OPT,\pi,\mu}$ and $\succeq_{PES,\pi,\mu}$ are classically defined from $U_{OPT,\pi,\mu}$ and $U_{PES,\pi,\mu}$

These criteria proved to be not efficient enough, in the sense that they fail to satisfy the principle of strict Pareto dominance: $\forall s, \mu(f(s)) \geq \mu(g(s))$ and $\exists s^*, \pi(s^*) > 0$ and $\mu(f(s^*)) > \mu(g(s^*))$ does not imply $f \succ_{OPT,\pi,\mu} g$ nor $f \succ_{PES,\pi,\mu} g$

This drawback is not observed within expected utility theory since the following *Sure-Thing Principle* (STP) [5] insures that identical consequences do not influence the relative preference between two events.

$$\text{STP:} \forall f, g, h, h', fAh \succeq gAh \Leftrightarrow fAh' \succeq gAh'$$

So, the question is whether it is possible or not to reconciliate possibilistic criteria and efficiency. The answer seems to be no: in [4] it is shown that the possibilistic criteria cannot obey the STP, except in a very particular case: when the actual state of the world is known, i.e. when there is no uncertainty at all! So, we cannot both stay in the pure QPU framework and satisfy the Pareto principle. The idea is then to try to cope with this problem by proposing *refinements* of the possibilistic criteria that obey the Sure Thing Principle. Formally:

Definition 2 (Refinement). $\succeq'$ refines $\succeq$ iff $\forall f,g \in X^S$, $f \succ g \Rightarrow f \succ' g$.

Since we are looking for weak orders it is natural to think of refinements based on expected utility. Concerning the optimistic utility criterion, we obtain the following result:

Theorem 3. *Let* $< S,X,L,\pi,\mu >$ *be a possibilistic model based on a scale* $L = (\alpha_0 = 0_L < \alpha_1 < \ldots < \alpha_k = 1_L)$ *. The function* $\chi : L \rightarrow [0,1]$ *defined by:*

- $\chi(0_L) = 0, \chi(\alpha_i) = \frac{v}{N^{2^{k-i}}}, i = 1,\ldots,k$
- $v = (\sum_{i=1,\ldots,k} \frac{n_i}{N^{2^{k-i}}})^{-1}$

is such that:

- $\chi \circ \pi$ *is a probability distribution*
- $\succeq_{EU,\chi\circ\pi,\chi\circ\mu}$ *refines* $\succeq_{OPT,\pi,\mu}$
- $\chi \circ \pi$ *(resp.* $\chi \circ \mu$ *) and* π *(resp.* μ*) are ordinaly equivalent*

So for any $< S,X,L,\pi,\mu >$ we are able to propose an EU model that refines the former. This model is thus perfectly compatible with the optimistic qualitative utility and more decisive than it. Moreover, since it is based on expected utility it satisfies the Sure Thing Principle as well as Pareto dominance and does not use other information than the original one - it is unbiased. Moreover, it can be shown that, if we do not accept to introduce a bias in the EU-refinement, it is unique, up to an isomorphism.

When considering the pessimistic qualitative model, the same kind of result can be obtained. First of all, notice that $\succeq_{PES,\pi,\mu}$ and $\succeq_{OPT,\pi,\mu}$ are dual relations:

Proposition 4. *Let* $< S,X,L,\pi,\mu >$ *be a QPU model. It holds that:*
$\forall f,g \in X^S, f \succeq_{PES,\pi,\mu} g \Leftrightarrow g \succeq_{OPT,\pi,\mu'} f$*, where* $\mu' = n \circ \mu$

This gives rise to the following definition of pessimistic EU-refinement:

Theorem 5. *Let* $< S,X,L,\pi,\mu >$ *be a QPU model and* $\chi : L \rightarrow [0,1]$ *be the transformation of L w.r.t.* π *identified Theorem 3. Let* $p = \chi \circ \pi$ *and* $u' = \chi(1_L) - \chi \circ n \circ \mu$*; it holds that:*

- $\succeq_{EU,p,u'}$ *is a refinement of* $\succeq_{PES,\pi,\mu}$
- p *(resp.* u' *) and* π *(resp.* μ*) are ordinaly equivalent*
- *any unbiased EU-refinement of* $\succeq_{PES,\pi,\mu}$ *is ordinally equivalent to* $\succeq_{EU,p,u'}$

So, if $< S, X, L, \pi, \mu >$ a QPU model, it is always possible to build a probabilistic transformation χ using Theorem 3, and thus a probability $p = \chi \circ \pi$ and two utility functions $u = \chi \circ \mu$ and $u' = \chi(1) - \chi \circ n \circ \mu$ that define the unbiased EU-refinements of the optimistic and pessimistic utility criteria respectively.

This proves an important result for bridging qualitative possibilistic decision theory and expected utility theory: we have shown than any optimistic or pessimistic QPU model can be refined by a EU model. So, (i) possibilistic decision criteria are compatible with the classical expected utility criterion and (ii) choosing a EU model is advantageous, since it leads to a EU-refinement of the original rule (thus, a more decisive criterion) and it allows to satisfy the STP and the principle of Pareto.

But this does not mean that qualitativeness and ordinality are given up. For instance, in both cases, the probability measures are "big-stepped probabilities", i.e. satisfy [1] :

$$\forall s \in S, P(\{s\}) > P(\{s', P(\{s'\}) < P(\{s\})\})$$

States are clustered in ordinal classes and any state of one class is more plausible that any event built on the lower classes. Although probabilistic and based on additive manipulations of utilities, these new criteria remain ordinal (it is actually possible to show they generalize well known ordinal weighted means, namely the leximin and leximax procedures.) And this is very natural: since we come from an ordinal model and do not accept any bias, we go to another (probabilistic but) ordinal model, in which the numbers only encode orders of magnitude.

The result of the present research can be viewed in a more general perspective: the optimistic and pessimistic utilities are not limited to decision under uncertainty and can be view as general maximin and minimax procedures (used for instance in multi criteria decision making, voting theory, etc) : we have shown that they can be refined by a classical weighted sum, when the strict Pareto principle is required. This raises a new question: can we extend this principle to any other instance of Sugeno integral [7] ? this is the topic of the second part of the present presentation.

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Additive refinements of qualitative decision criteria. Part II: Sugeno integral

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Part I has shown that prioritized maximin and minimax aggregations can be refined by a classical weighted sum, as soon as the strict consistency with the Pareto principle is required. It can thus be asked if the same question can be solved for discrete Sugeno integrals [3] since prioritized minimum and maximum are special cases of fuzzy integrals.

The first result is negative. One basic reason why prioritized maximin and minimax aggregations can be refined by a weighted average with fixed weights is that these operations do not violate independence (the sure thing principle) in a drastic way. Indeed the ordering relations induced by $U_{OPT,\pi,\mu}(f)$ and $U_{PESS,\pi,\mu}(f)$ satisfy a weaker independence condition:

$$\text{WSTP:} \forall f,g,h,h', fAh \succ gAh \Rightarrow fAh' \succeq gAh'.$$

So modifying two acts by altering their common consequences never results in a strong preference reversal. On the contrary such a preference reversal is clearly possible for Sugeno integral because for a fuzzy measure γ and three sets A, B, C, where C is disjoint from both A and B, one may have $\gamma(A) > \gamma(B)$ and $\gamma(B \cup C) > \gamma(A \cup C)$. This feature makes it impossible to refine rankings of acts induced by Sugeno integrals by means of another functional which satisfies the sure thing principle.In particular, a Sugeno integral with respect to a given fuzzy measure cannot be presented by an expected utility with respect to a single probability distribution.

However it makes sense to try and refine a Sugeno integral-based ordering by means of a Choquet integral[2][1]. Indeed the expression of a Sugeno integral and of a discrete Choquet integral are similar. Moreover while Choquet integrals are additive for comonotonic acts, Sugeno integrals are both maxitive and minitive for comonotonic acts — recall that two acts f, g are comonotonic iff there exists a single permutation σ on the states of S that rearrange the values of both $\mu(f)$ and $\mu(g)$ in non decreasing order, i.e. such that:

$$\mu(f(s_{\sigma(1)})) \leq \mu(f(s_{\sigma(2)})) \leq \cdots \leq \mu(f(s_{\sigma(n)}))$$

and

$$\mu(g(s_{\sigma(1)})) \leq \mu(g(s_{\sigma(2)})) \leq \cdots \leq \mu(g(s_{\sigma(n)}))$$

A Sugeno integral serving as a preference functional to evaluate act f is of the form:

$$SUG_{\gamma,\mu}(f) = \max_{i=1,n} \min(\gamma(A_i^\sigma), \mu(f(s_{\sigma(i)})))$$

where γ is a monotonic set function ranging on a finite chain L (a qualitative fuzzy measure), μ a utility function taking its values on the same L, σ is a permutation rearranging the values $\mu(f(s))$ in non-decreasing order, and $A_i^\sigma = \{s_{\sigma(i)}, \ldots, s_{\sigma(n)}\}$

Similarly a Choquet integral reads:

$$Ch_{\nu,\mu}(f) = \sum_{s \in S} (\nu(A_i^\sigma) - \nu(A_{i+1}^\sigma)) \times u(f(s_{\sigma(i)}))$$

where ν is a numerical fuzzy measure and u a numerical utility function.

Now, consider a set of acts F_σ that share the same permutation σ (i.e. a set of comonotonic acts). For any of these acts, the expression of the Sugeno integral comes down to a prioritized maximum (an optimistic utility) with respect to a possibility distribution $\pi_\sigma(s_{\sigma(i)}) = \gamma(A_i^\sigma)$

$$\forall f \in F_\sigma : SUG_{\gamma,\mu}(f) = \max_{s \in S} \min(\pi_\sigma(s), \mu(f(s)))$$

So the results of Part I apply when restricted to comonotonic acts : the restriction of $\succeq_{SUG_{\gamma,\mu}}$ to any F_σ can be refined without bias by an expected utility based on a big-stepped probability p_σ and a big-stepped utility function u:

$$\forall f \in F_\sigma : EU_{p_\sigma,u}(f) = \sum_{s \in S} p_\sigma(s) \times u(f(s))$$

The point is that one will get different probability and utility measures for the different F_σ [2] The idea is then to consider that the different p_σ are the projections of a common "big-stepped fuzzy measure" ν such that:

$$\forall F_\sigma, p_\sigma(s_{\sigma(i)}) = \nu(A_i^\sigma) - \nu(A_{i+1}^\sigma)$$

In this context, $EU_{p_\sigma,u}$ is the restriction to F_σ of the Choquet integral $Ch_{\nu,u}(f)$. We have shown that the previous system of equation is always consistent. Moreover, according to Part I, we know that: whatever F_σ, $Ch_{\nu,u} = EU(p_\sigma, u)$ defines an unbiased refinement of $SUG_{\gamma,\mu} = U_{OPT,\gamma,\mu}$. This suggests that, for any $\gamma : 2^S \to L$, $\mu : X \to L$, there exists a fuzzy measure ν on 2^S and a utility function u in X such that, whatever $f, g \in X^S$:

$$SUG_{\gamma,\mu}(f) > SUG_{\gamma,\mu}(g) \Rightarrow Ch_{\nu,\mu'}(f) > Ch_{\nu,\mu'}(g)$$

Moreover, for any permutation σ of the elements in S $p_\sigma(s_{\sigma(i)})$ must be a big-stepped probability. As

[2] But all the F_σ can share the same u, which depend on L but not on σ.

a consequence, when all states have distinct confidence values $\nu(A)$, the big-stepped fuzzy measure is such that:

$$\forall A \subseteq S, \nu(A) > 2 \times \nu(B)$$

for all proper subsets B of A. A general definition of such measure by a necessary and sufficient condition is a topic for further research.

Finally, we would like to suggest that an alternative approach to refine the Sugeno integral by a Choquet integral may start from the expression of Sugeno integral involving all subsets of S:

$$SUG_{\gamma,\mu}(f) = \max_{A \subseteq S} \min(\gamma(A), \min_{s \in A} \mu(f(s)))$$

and the corresponding expression of the Choquet integral in terms of the Moebius transform m_γ of γ:

$$Ch_{\gamma,\mu}(f) = \sum_{A \subseteq S} m_\gamma(A) \times \min_{s \in A} \mu(f(s))$$

Further research shall also include a deeper exploration of this alternative refinement and the exploration of the relationship between the two approaches, in particular of the the relationships between big-stepped fuzzy measures and (big-stepped) belief functions (i.e. Choquet utilities that rely on a positive and big-stepped m) ?

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Applications of fuzzy orderings: an overview

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There are two most fundamental relational concepts in mathematics which accompany mathematicians as well as computer scientists and engineers throughout their life in science—*equivalence relations* (reflexive, symmetric, and transitive relations) and *(partial) orderings* (reflexive, antisymmetric, and transitive relations).

It is not surprising that, within the early gold rush of fuzzification of virtually any classical mathematical concept, these two fundamental types of relations did not have to await the introduction of their fuzzy counterparts for a long time [22].

Fuzzy equivalence relations are now well-accepted concepts for expressing equivalence/equality in vague environments [8, 13, 16, 18, 20, 21] (in contrast to Zadeh's original definition, now with the additional degree of freedom that the conjunction in transitivity may be modeled by an arbitrary triangular norm [15]).

In the meantime, fuzzy equivalence relations have turned out to be helpful tools in various disciplines, in particular, as soon as the interpretation of fuzzy sets, partitions, and controllers [16, 21, 10, 14] is concerned. More direct practical applications have emerged in flexible query systems [12, 17] and fuzzy databases in general [19].

Fuzzy (partial) orderings have been introduced more or less in parallel with fuzzy equivalence relations [22], however, they have never played a significant role in real-world applications.

This paper advocates a "similarity-based" generalization of fuzzy orderings, however, not from the pure mathematical viewpoint of logic or algebra (for what we would like to refer to the extensive studies in [2, 3, 5, 11]). Instead, we attempt to demonstrate the potential for applications by means of considering comprehensive overviews of four case studies. Those are flexible query systems [7], ordering-based modifiers [1, 9], and orderings of fuzzy sets [4]. Finally, we also discuss the interpretability property, for which orderings of fuzzy sets are of fundamental importance [6].

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Fuzzy function as a solution to a system of fuzzy relation equations

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Abstract

We give here a discussion of a fuzzy function which is given by a system of fuzzy relation equations. We demonstrate, how problems of interpolation and approximation of fuzzy functions are connected with solvability of systems of fuzzy relation equations. First we explain the general framework, and later on we prove some particular results related to the problem of the best approximation.
Key words: system of fuzzy relation equations, solvability and approximate solvability of a fuzzy relation equation system, fuzzy function, interpolation and approximation of fuzzy functions

1 Introduction

We will concern with a problem of fuzzy functions representation by a solution to a system of fuzzy relation equations. In order to introduce this stuff we need an algebra of fuzzy logic operations. We choose a BL-algebra which has been introduced by Hájek in [5] and which in a certain sense generalizes boolean algebra. This appears in the extension of the set of boolean operations by two semigroup operations which constitute so called adjoined couple. The following definition summarizes definitions originally introduced in [5].

Definition 1. A *BL-algebra* is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

with four binary operations and two constants such that

(i) $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a lattice with $\mathbf{0}$ and $\mathbf{1}$ as the least and greatest elements w.r.t. the lattice ordering,

(ii) $(L, *, \mathbf{1})$ is a commutative semigroup with unit $\mathbf{1}$, such that the multiplication $*$ is associative, commutative and $1 * x = x$ for all $x \in L$,

(iii) $*$ and $\rightarrow$ form an adjoint pair, i.e.
$z \leq (x \rightarrow y)$ iff $x * z \leq y$ for all $x, y, z \in L$,

(iv) and moreover, for all $x,y \in L$
$x*(x \to y) = x \wedge y$,
$(x \to y) \vee (y \to x) = \mathbf{1}$.

Another two operations of $\mathcal{L}$: unary $\neg$ and binary $\leftrightarrow$ can be defined by

$$\neg x = x \to \mathbf{0},$$
$$x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

The following properties will be widely used in the sequel:

$$x \le y \Leftrightarrow (x \to y) = \mathbf{1},$$
$$x \leftrightarrow y = \mathbf{1} \Leftrightarrow x = y.$$

Note that if a lattice $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is given, then BL-algebra is completely defined by the choice of multiplication operation $*$. In particular, $L = [0,1]$ and $*$ is known as a t-norm.

Let us fix some BL-algebra $\mathcal{L}$ with a support L and take X and Y as arbitrary universes. Denote $\mathcal{F}(\mathbf{X})$ a set of all fuzzy subsets of X, i.e. a set of all functions $\{A : X \to L\}$. A system of fuzzy relation equations

$$A_i \circ R = B_i, \quad 1 \le i \le n, \tag{1}$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ and '$\circ$' is the sup-*-composition, is considered with respect to unknown fuzzy relation R. Very often system (1) is connected with applications like fuzzy control, identification of fuzzy systems, prediction of fuzzy systems, decision-making, etc. Such systems arise in the process of formalization of some list of linguistic IF–THEN rules, which well recommends itself as an approximating instrument for continuous dependencies. Because a solution of (1) may not exist in general, the problem to investigate necessary and sufficient, or also only sufficient conditions for solvability arises. This problem has been widely studied in the literature, and some nice theoretical results have been obtained. Let us point out some of them: Sanchez [12], Perf-Tonis [11], Gavalec [1] with necessary and sufficient conditions, Gottwald [2], Klawonn [8] with sufficient conditions.

Of course, all of these results have practical importance only in the the case when the universes of discourse X and Y are finite. In the case when these universes of discourse are infinite, however, those results can be systematized and considered in the light of a new topic which is *fuzzy functions and their representations*.

In the present paper we will introduce the problem of solvability of fuzzy relation equations in a new framework as the problem of interpolation and approximation of a fuzzy function.

2 Interpolation and approximation of a fuzzy function

The notion of a fuzzy function is not well established in the literature. Imprecisely, it has been used to mean the often so called fuzzy systems. Precisely this notion was defined e.g. in Klawonn's paper [8] where it has been introduced with respect to two similarity relations on the universes for the independent and the dependent variables.

Trying to be as much as possible close to the classical case we give the following definition.

Definition 2. Let $\mathcal{F}(\mathbf{X}), \mathcal{F}(\mathbf{Y})$ be the classes of all fuzzy subsets of universes of discourse X and Y. A (perhaps multivalued) mapping f from $\mathcal{F}(\mathbf{X})$ into $\mathcal{F}(\mathbf{Y})$ is called a *fuzzy function* if for any fuzzy subsets $A, A' \in \mathcal{F}(\mathbf{X})$ and for fuzzy subsets $B, B' \in \mathcal{F}(\mathbf{Y})$ which are f-related with A, A', respectively, the following holds true

$$A = A' \to B = B'. \tag{2}$$

Example 3. Any fuzzy relation $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ determines via sup-*-composition a fuzzy function, defined as the mapping f_R from $\mathcal{F}(\mathbf{X})$ to $\mathcal{F}(\mathbf{Y})$ which is described by

$$f_R(A)(y) = (A \circ R)(y) = \bigvee_{x \in X} (A(x) * R(x,y)).$$

In this example, fuzzy set $f_R(A) = A \circ R$ is the value of fuzzy function f_R determined by R in the "fuzzy point" determined by A.

Nor Definition 2, neither the above given Example do not provide us with a way, how a fuzzy function can be constructed, and that is why, the problem of construction (e.g. representation by a formula) is of a primary importance.

Very often a fuzzy function is described partially by a list of fuzzy IF–THEN rules

$$\text{IF } x \text{ is } A_i \qquad \text{THEN } y \text{ is } B_i, \qquad i = 1, \dots, n,$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$. This description gives only a partially fixed mapping procedure $A_i \to B_i$. Thus the problem of the completion for the "missing points" appears. The natural requirement for such a completion is that it should agree with the original data.

This leads us to the problem known as *interpolation* problem.

Definition 4. Let a list of original data, consisting of ordered pairs of fuzzy sets (A_i, B_i), $i = 1, \dots, n$, be given. A fuzzy function f defined on $\mathcal{F}(\mathbf{X})$ *interpolates* these data if

$$f(A_i) = B_i, \qquad i = 1, \dots, n. \tag{3}$$

We will also call f an interpolating fuzzy function.

As a side remark we mention that, even supposing the existence of an interpolating fuzzy function, it is usually not unique. The solution of the interpolation problem without reference to any directly specified class of interpolating functions is essentially arbitrary, even in a classical case. That is the reason why in classical mathematics the interpolation problem is solved usually in a predetermined class of "simple" functions, e.g. in the (or: some) class of polynomials.

We will consider a solution to the fuzzy interpolation problem in the class of fuzzy functions $\mathbf{F}_R$ represented by fuzzy relations. It is easy to see that a fuzzy relation R represents an interpolating fuzzy function with respect to the given data (A_i, B_i), $i = 1, \dots, n$, if and only if R is a solution of the corresponding system (1) of relation equations.

2.1 Approximate solutions and their approximation quality

The restriction of interpolating functions to the class $\mathbf{F}_R$ may, however, create a new problem: that of the existence of an interpolating function within this restricted class. Then the problem of interpolation becomes intertwined with the problem of approximation. And this means here to find inside $\mathbf{F}_R$ a function which "suitably approximates" the fuzzy function one intended to interpolate.

Besides a set of approximating objects one needs to estimate their *quality*, and to rank the approximating objects accordingly. One possibility is measuring some kind of "distance" or "similarity" between an object from $\mathbf{F}_R$ and the particular object which is to be approximated.

As we have mentioned at the beginning, a system (1) of relation equations is not always solvable. In this situation, being again interested in a completing of a partial function given by pairs (A_i,B_i), we have to break the requirement of agreement with the original data. This leads us to the definition of the notion of an approximate solution to the system (1). We also consider this approximate solution as an approximating fuzzy function with respect to the given data (A_i,B_i), $i = 1,\dots,n$.

Two things have to be specified for this approximation problem: an approximating space and a quality of approximation.

Let us fix the original set of argument-value pairs (A_i,B_i), $i = 1,\dots,n$, and consider the following approximating space of all fuzzy relations on $X \times Y$

$$\mathcal{R} = \{R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y}). \tag{4}$$

An evaluation of a quality of approximation come from a comparison of the intended values B_i and those ones realized by R, i.e. from an index

$$\delta(R) = \bigwedge_{i=1}^{n} \bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A_i \circ R)(y)). \tag{5}$$

Let us remark that $\delta(R)$ is essentially the solution degree introduced by Gottwald, cf. [3].

Being equipped with this measure $\delta(R)$ for the quality of an approximation R we may compare two different approximate solution saying that $R_1 \in \mathcal{R}$ is better than $R_2 \in \mathcal{R}$ if

$$\delta(R_2) \leq \delta(R_1).$$

It is easy to see that in this way we have introduced a preorder on the set $\mathcal{R}$ defined in (4).

In the previous studies on systems of fuzzy relation equations, two types of approximate solutions have played a prominent role, without having been tied with a clearly chosen approximation space: the MA-relation $R_{\mathrm{MA}} = \bigcup_{i=1}^{n}(A_i \times B_i)$ of Mamdani/Assilian, and the S-relation $\widehat{R} = \bigcap_{i=1}^{n}(A_i \rhd B_i)$ first considered by Sanchez. In forming these relations two particular types $A \times B, A \rhd B$ of fuzzy relations, each determined by a pair (A,B) of fuzzy sets, are used which are defined by the membership functions

$$A \times B(x,y) = A(x) * B(y), \tag{6}$$
$$A \rhd B(x,y) = A(x) \rightarrow B(y). \tag{7}$$

We called these fuzzy relations *pseudo-solutions* in IPMU02 and shall follow this usage here.

However, in this paper we will consider two other, more specified approximation spaces which are subspaces of $\mathcal{R}$:

$$\mathcal{R}_l = \{R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y}) : A_i \circ R = C_i, \quad 1 \leq i \leq n, \quad \text{for some} \quad C_1,\dots,C_n \in \mathcal{F}(\mathbf{Y}) \quad \text{such that} \quad C_i \subseteq B_i\} \tag{8}$$

and

$$\mathcal{R}_u = \{R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y}) : A_i \circ R = C_i, \quad 1 \leq i \leq n, \\ \text{for some} \quad C_1, \ldots, C_n \in \mathcal{F}(\mathbf{Y}) \quad \text{such that} \quad C_i \supseteq B_i\}. \quad (9)$$

In discussions later on which use these approximation spaces we will not only refer to their elements, we will also refer to the (solvable) systems of relation equations which determine these elements. Then we will denote the systems which determine the elements of $\mathcal{R}_l$ as **l-approximating systems*, and those which determine the elements of $\mathcal{R}_u$ as **u-approximating systems*.

In the literature on fuzzy relation equations the following rankings for approximation quality have been used:

- the solution degrees $\delta(R)$ of Gottwald (or the difference between these solution degrees and the solvability degree);
- the preordering between solutions R' of systems $A_i \circ R = B_i'$ which satisfy $B_i' \subseteq B_i$ for all $1 \leq i \leq n$, given by
$$R' \leqq_W R'' \quad \text{iff} \quad B_i' \subseteq B_i'' \subseteq B_i, \quad 1 \leq i \leq n, \quad (10)$$
which was implicitly used by Wu [13] and later on by Klir/Yuan [6, 7].

Of course, this last mentioned preordering could, and should be defined more general e.g. w.r.t. a similarity degree E for fuzzy sets as

$$R' \leqslant_E R'' \quad \text{iff} \quad E(B_i', B_i) \leq E(B_i'', B_i), \quad 1 \leq i \leq n, \quad (11)$$

or in a similar way w.r.t. a metric in the class of all fuzzy sets.

2.2 Optimal approximations

Having some "quality index" available to evaluate the quality of particular approximations allows to (somehow) compare different approximations.

This, however, is usually not sufficient. One likes to know more, viz. something like *best possible* approximations. And this can be understood as the search for (suitably) extremal elements among the approximating objects, of course extremal w.r.t. some ranking induced by the previously mentioned quality indices.

Looking again at our standard examples the situation was that

- in Gottwald's approach through solution and solvability degrees the best possible approximations had not been discussed explicitly;
- in Wu's approach only the best possible approximations in sense of the preordering (10) have been considered.

For the general situation we shall use the following terminology.

Definition 5. Suppose that a ranking ρ is given for the approximating objects. An approximating object $\varphi \in \mathcal{R}$ is ρ-*optimal* iff there does not exists in $\mathcal{R}$ an approximating object which is ranked higher than φ.

3 Some optimality considerations

The problem arises immediately whether the two standard pseudo-solutions $\widehat{R}$ and R_{MA} are optimal approximate solutions for suitable approximation spaces. For the S-pseudo-solution $\widehat{R}$ such an optimality was shown in [6, 7] w.r.t. the approximation set $\mathcal{R}_l$ and a ranking similar to (10).

We show that $\widehat{R}$ is even an optimal approximate solution in the approximation set $\mathcal{R}_l$ equipped with the ranking (11).

Proposition 6. *The fuzzy relation $\widehat{R}$ is always an optimal approximate solution in $\mathcal{R}_l$ under the ranking (11).*

Proof. We know, e.g. from [4], that one always has $A_i \circ \widehat{R} \subseteq B_i$ for the fuzzy relation

$$\widehat{R} = \bigcap_{i=1}^{n} A_i \rhd B_i \,.$$

Now consider a family of fuzzy sets C_i with $A_i \circ \widehat{R} \subseteq C_i \subseteq B_i$ for all $i = 1, \ldots, n$ and such that the system

$$A_i \circ R = C_i \tag{12}$$

of relation equations is solvable. Let $\widehat{S}$ be its maximal solution

$$\widehat{S} = \bigcap_{i=1}^{n} A_i \rhd C_i \,.$$

From $C_i \subseteq B_i$ we have immediately $A_i \rhd C_i \subseteq A_i \rhd B_i$ and thus $\widehat{S} \subseteq \widehat{R}$. This gives

$$C_i = A_i \circ \widehat{S} \subseteq A_i \circ \widehat{R} \subseteq C_i$$

and thus

$$A_i \circ \widehat{R} = C_i \,.$$

That means that no system (12) with $A_i \circ \widehat{R} \subset C_i \subseteq B_i$ for some $1 \leq i \leq n$ is solvable, i.e. $\widehat{R}$ is an optimal approximate solution. □

For the MA-pseudo-solution the situation is different.

Proposition 7. *There exist systems* (1) *of relation equations for which their MA-pseudo-solution R_{MA} is an approximate solution in the approximation set $\mathcal{R}_u$ which, however, is not optimal in this set under the ranking (11).*

Proof. Let us consider the following system of relation equations with input-output data

$$A_1 = (.5, 1, .5, 0)\,, \qquad B_1 = (.5, 1, 0, 0)\,,$$
$$A_2 = (0, .5, 1, .5)\,, \qquad B_2 = (0, 0, .5, 1)\,.$$

Then we have immediately

$$A_1 \times B_1 = \begin{pmatrix} .5 & .5 & 0 & 0 \\ .5 & 1 & 0 & 0 \\ .5 & .5 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_2 \times B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & .5 & .5 \\ 0 & 0 & .5 & 1 \\ 0 & 0 & .5 & .5 \end{pmatrix}$$

and therefore

$$R_{\mathrm{MA}} = (A_1 \times B_1) \cup (A_2 \times B_2) = \begin{pmatrix} .5 & .5 & 0 & 0 \\ .5 & 1 & .5 & .5 \\ .5 & .5 & .5 & 1 \\ 0 & 0 & .5 & .5 \end{pmatrix}$$

This gives

$$A_1 \circ R_{\mathrm{MA}} = (.5, 1, .5, .5)\,, \qquad\qquad A_2 \circ R_{\mathrm{MA}} = (.5, .5, .5, 1)\,.$$

To see the non-optimality of R_{MA} consider the following modification T of R_{MA} given by

$$T = \begin{pmatrix} .5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & .5 & 0 \end{pmatrix}$$

Then we find

$$A_1 \circ T = (.5, 1, 0, .5) \qquad \text{and} \qquad A_2 \circ T = (0, .5, .5, 1)\,,$$

and hence see that the fuzzy relation T solves the system

$$\begin{aligned} A_1 \circ R &= (.5, 1, 0, .5)\,, \\ A_2 \circ R &= (0, .5, .5, 1) \end{aligned}$$

of fuzzy relation equations. And this system is a strongly better *u-approximating system w.r.t. the initial system as is the *u-approximating system

$$\begin{aligned} A_1 \circ R &= (.5, 1, .5, .5)\,, \\ A_2 \circ R &= (.5, .5, .5, 1) \end{aligned}$$

which has R_{MA} as its solution. □

A closer inspection of the proof of Proposition 6 shows that the crucial difference of the previous optimality result for $\widehat{R}$ to the present situation of R_{MA} is that in the former case the solvable approximating system has its own (largest) solution $\widehat{S}$. But in the present situation a solvable approximating system may fail to have his MA-pseudo-solution R_{MA} as a solution.

However, this remark leads us to a partial optimality result w.r.t. the MA-pseudo-solution.

Definition 8. Let us call a system (1) of relation equations *MA-solvable* iff its MA-pseudo-solution R_{MA} is a solution of this system.

Then we have the following result.

Proposition 9. *If a system* (1) *of relation equations has an MA-solvable *u-approximating system*

$$R''A_i = B_i^*\,, \qquad i = 1, \dots, n \tag{13}$$

such that for the MA-pseudo-solution R_{MA} *of* (1) *one has*

$$B_i \subseteqq B_i^* \subseteqq A_i \circ R_{\mathrm{MA}}\,, \qquad i = 1, \dots, n\,,$$

then one has

$$B_i^* = A_i \circ R_{\mathrm{MA}} \quad \textit{for all} \quad i = 1, \dots, n\,.$$

Proof. Let $R^*_{\mathrm{MA}} = \bigcup_{i=1}^n A_i \times B_i^*$ be the MA-(pseudo-)solution of (13). Then one has because of the monotonic dependency of the MA-pseudo-solution from the (input and) output data

$$R_{\mathrm{MA}} \subseteq R^*_{\mathrm{MA}}$$

and therefore for each $i = 1, \ldots, n$

$$B_i^* \subseteq A_i \circ R_{\mathrm{MA}} \subseteq A_i \circ R^*_{\mathrm{MA}} = B_i^* \,,$$

which just means $B_i^* = A_i \circ R_{\mathrm{MA}}$. □

Corollary 10. *If all input sets of* (1) *are normal then the system*

$$A_i \circ R = A_i \circ R_{\mathrm{MA}} \,, \qquad i = 1, \ldots, n \,, \tag{14}$$

*is the smallest MA-solvable *u-supersystem for* (1).

Proof. From the normality of the input sets one has $B_i \subseteq A_i \circ R_{\mathrm{MA}}$ for all $i = 1, \ldots, n$. So a smaller MA-solvable *u-supersystem (13) would have to satisfy $B_i \subseteqq B_i^* \subseteqq A_i \circ R$ for all $i = 1, \ldots, n$. But then it coincides with (14). □

Corollary 11. *Let $\widehat{R}$ be the S-pseudo-solution of* (1) *and suppose that the modified system*

$$A_i \circ R = A_i \circ \widehat{R}, \qquad i = 1, \ldots, n \,, \tag{15}$$

is MA-solvable. Then the iterated pseudo-solution $R_{\mathrm{MA}}[\widehat{R}[B_k]''A_k]$, introduced in [4]*, is an optimal *l-approximate solution of* (1).

Proof. Assume that (15) is MA-solvable. Then its MA-solution is by construction of the system (16) exactly the iterated pseudo-solution $R_{\mathrm{MA}}[\widehat{R}[B_k]''A_k]$ of (1).

Therefore one has

$$A_i \circ R_{\mathrm{MA}}[\widehat{R}[B_k]''A_k] = A_i \circ \widehat{R} \subseteqq B_i \,, \qquad i = 1, \ldots, n \,.$$

Now Proposition 6, i.e. the optimality of $\widehat{R}$ as a *u-approximate solution yields immediately the optimality of $R_{\mathrm{MA}}[\widehat{R}[B_k]''A_k]$. □

This last Proposition can be further generalized. To do this assume that $\mathbb{S}$ is some *pseudo-solution strategy*, i.e. some mapping from the class of families $(A_i, B_i)_{1 \le i \le n}$ of input-output data pairs into the class of fuzzy relations, which yields for any given system (1) of relation equations an $\mathbb{S}$-pseudo-solution $R_{\mathbb{S}}$. Of course the system (1) will be called $\mathbb{S}$-*solvable* iff $R_{\mathbb{S}}$ is a solution of the system (1).

Definition 12. We shall say that the $\mathbb{S}$-pseudo-solution $R_{\mathbb{S}}$ *depends isotonically* (w.r.t. inclusion) on the output data of the system (1) of relation equations iff the condition

$$\text{if} \quad B_i \subseteqq B_i' \ \text{for all} \ i = 1, \ldots, n \quad \text{then} \quad R_{\mathbb{S}} \subseteqq R_{\mathbb{S}}'.$$

holds true for the $\mathbb{S}$-pseudo-solutions $R_{\mathbb{S}}$ of the system (1) and $R'_{\mathbb{S}}$ of an "output-modified" system $R''A_i = A_i \circ R = B_i'$, $i = 1, \ldots, n$.

Definition 13. Furthermore we understand by an $\mathbb{S}$-optimal *u-approximate solution of the system (1) the $\mathbb{S}$-pseudo-solution of an $\mathbb{S}$-solvable *u-approximating system of (1) which has the additional property that no strongly better *u-approximating system of (1) is $\mathbb{S}$-solvable.

Proposition 14. *Suppose that the $\mathbb{S}$-pseudo-solution depends isotonically $($w.r.t. inclusion$)$ on the output data of the systems of relation equations. Assume furthermore that for the $\mathbb{S}$-pseudo-solution $R_{\mathbb{S}}$ of (1) one always has $B_i \subseteqq A_i \circ R_{\mathbb{S}}$ $($or that one always has $A_i \circ R_{\mathbb{S}} \subseteqq B_i)$ for all $i = 1, \ldots, n$. Then the $\mathbb{S}$-pseudo-solution $R_{\mathbb{S}}$ of (1) is an $\mathbb{S}$-optimal *u-approximate $($or: *l-approximate$)$ solution of the system (1).*

Proof. We discuss only the *u-approximating case, the other one is treated similarly.

Consider an $\mathbb{S}$-solvable system

$$A_i \circ R = B_i^*, \qquad i = 1, \ldots, n \tag{16}$$

with $\mathbb{S}$-pseudo-solution $R_{\mathbb{S}}^*$ which satisfies for the $\mathbb{S}$-pseudo-solution R_S of (1) the inclusion relations

$$B_i \subseteqq B_i^* \subseteqq A_i \circ R_{\mathbb{S}}, \qquad i = 1, \ldots, n.$$

Then one has

$$A_i \circ R_{\mathbb{S}}^* = B_i^* \subseteqq A_i \circ R_{\mathbb{S}} \subseteqq A_i \circ R_{\mathbb{S}}^*, \qquad i = 1, \ldots, n,$$

and hence the relationship

$$B_i^* = A_i \circ R_{\mathbb{S}} \qquad \text{for all} \quad i = 1, \ldots, n.$$

□

It is immediately clear that Corollary 10 is the particular case of the MA-pseudo-solution strategy. But also Proposition 6 is a particular case of this Proposition: the case of the S-pseudo-solution strategy (having in mind that S-solvability and solvability are equivalent notions).

4 Concluding remarks

A notion of a fuzzy function as a mapping between universes of fuzzy sets with a uniqueness property has been introduced. In this setting, a precise and approximate solutions to a system of fuzzy relation equations are considered as the interpolating and approximating fuzzy functions with respect to a given data. We recall the necessary and sufficient conditions of solvability of a system of fuzzy relation equations and concentrated on a problem of approximate solvability. First we explained the general framework, and later on we proved some particular results related to the problem of the best approximation in different approximation spaces.

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Compatible extensions of fuzzy relations

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In this paper we introduce the notion of the compatible extension of a fuzzy relation and we prove an extension theorem for fuzzy relations. Our result generalizes to fuzzy set theory an extension theorem proved by Duggan for crisp relations. We also obtain fuzzy versions of some theorems of Szpilrajn, Hansson and Suzumura. A classical Szpilrajn theorem asserts that any strict partial order is a subrelation of a strict linear order. Later this result lead to a wide range of extension theorems. Hansson proved that every preorder can be extended to a total preorder. Suzumura refined Hansson's result by proving that a relation has a total and transitive compatible extension if and only if it is transitive-consistent. A very general extension theorem was proved by Duggan. Duggan's result generalizes all the known extension theorems and some new interesting follow from it. Zadeh proved a fuzzy form of the Szpilrajn's theorem. This paper is another contribution to this problem following Duggan's trend. Let X be a non-empty set. A *fuzzy relation* on X is a function $R : X^2 \to [0,1]$. If R,Q are two fuzzy relations on X, then $R \subseteq Q$ means that $R(x,y) \leq Q(x,y)$ for any $x,y \in X$; in this case Q is called an *extension* of R. A fuzzy relation R is *transitive* if $R(x,y) \wedge R(y,z) \leq R(x,z)$ for all $x,y,z \in X$. The *transitive closure* $T(R)$ of a fuzzy relation R is the intersection of all transitive fuzzy relations including R. For any fuzzy relation R let us define the fuzzy relation P_R by $P_R(x,y) = R(x,y) \wedge \neg R(y,x)$. Let R,Q be two fuzzy relations on X. Q is said to be a *compatible extension* of R if $R \subseteq Q$ and $P_R \subseteq P_Q$. A class $\mathcal{R}$ of fuzzy relations on X is *closed upward* if for any totally ordered family $\{R_i\}_{i \in I}$ of fuzzy relations in $\mathcal{R}$, we have $\bigcup_{i \in I} R_i \in \mathcal{R}$. A fuzzy relation R is *total* if for any $x \neq y$ we have $R(x,y) \vee R(y,x) > 0$. A class $\mathcal{R}$ of fuzzy relations on X is *arc-receptive* if for any $x \neq y$ and for any transitive fuzzy relation $R \in \mathcal{R}$, $R(y,x) = 0$ implies $T(R[x,y]) \in \mathcal{R}$.

The following result is a generalization of Duggans's extension theorem:

Theorem 1. *Let $\mathcal{R}$ be a closed upward and arc-receptive class of fuzzy relations on X. For any transitive fuzzy relation $R \in \mathcal{R}$ there exists a total and transitive fuzzy relation $R^* \in \mathcal{R}$ such that R^* is a compatible extension of R.*

A relation R is *transitive-consistent* (consistent in Suzumura terminology) if for any integer $n \geq 2$ and for any $z_1, \ldots, z_n \in X$, $(z_1,z_2) \in P_R$ and $(z_2,z_3), \ldots,$ $(z_{n-1},z_n) \in R$ implies $(z_n,z_1) \notin R$. In [1] it was proved that R is transitive-consistent if and only if $P_R \subseteq P_{T(R)}$. Any transitive relation is transitive-consistent. Suzumura's theorem [2] asserts that a crisp relation has a total and transitive compatible extension if and only if it is transitive-consistent. We give a fuzzy generalization of this result.

Theorem 2. *For a fuzzy relation R on X the following are equivalent:*
(i) R has a total and transitive compatible extension Q; (ii) R has a transitive compatible extension Q; (iii) R is transitive-consistent.

Keywords: Fuzzy relation; Compatible extension; Transitive-consistent

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Triangular norms: some open problems

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1 Introduction

Triangular norms are, on the one hand, special semigroups and, on the other hand, solutions of some functional equations [1, 8, 15, 16]. This mixture quite often requires new approaches to answer questions about the nature of triangular norms. There are some problems which were stated some time ago and remained unsolved for years. An example for this is the question whether the domination is a transitive relation on the class of t-norms (this problem was posed by B. Schweizer and A. Sklar [16]). Recall that a t-norm T_1 dominates a t-norm T_2 (in symbols $T_1 \gg T_2$) if for all $x, y, u, v \in [0,1]$

$$T_1(T_2(x,y), T_2(u,v)) \geq T_2(T_1(x,u), T_1(y,v)). \tag{1}$$

Obviously, we have $T_{\mathbf{M}} \gg T$ and $T \gg T$ for each t-norm T, and that $T_1 \gg T_2$ implies $T_1 \geq T_2$ (therefore the relation $\gg$ is reflexive and antisymmetric).

The aim of this note is to recall some well-known problems of the past (which have been solved meanwhile) and to state several problems which are open and have not been posed so far for a wider audience. Some of the solved problems were already mentioned in the monographs [8, 16] and in a special note devoted to open problems [11].

2 Some solved problems

Solved Problem 1. Suppose that an Archimedean t-norm T has a continuous diagonal. Is T necessarily continuous?

This problem goes back to [16], and it can be easily transformed to the case of an arbitrary t-norm with continuous diagonal. A negative answer was given by G. M. Krause [10], for a more detailed discussion of this topic see [8, Appendix B] and [18, 12].

Solved Problem 2. Let T be a cancellative t-norm which is continuous in the point $(1,1)$. Is T necessarily continuous?

This problem was posed by E. Pap in [11]. A negative answer was given by M. Budinčević and M. S. Kurilić [2]. Moreover, there are non-continuous cancellative t-norms which are left-continuous [17], see also [8, Example 2.29(ii)]. On the other hand, for an Archimedean t-norm its left-continuity is equivalent to its continuity [9], and for a cancellative Archimedean t-norm its continuity is equivalent with its continuity in the point $(1,1)$ [5]. Therefore all counterexamples regarding this problem are necessarily non-Archimedean.

Solved Problem 3. Can each (continuous) function in

$$\mathcal{D} = \{\delta \in [0,1]^{[0,1]} \mid \delta \text{ is non-decreasing and } \delta \leq \mathrm{id}_{[0,1]} \text{ and } \delta(1) = 1\}.$$

be extended to a t-norm, i.e., do we have $\mathcal{D} = \{\delta_T \mid T \text{ is a t-norm}\}$ (compare [7])?

Here the function $\delta_T\colon [0,1] \longrightarrow [0,1]$ denotes the diagonal section of a t-norm T given by $\delta_T(x) = T(x,x)$.

This problem was stated in [8, Remark 7.20], and a negative answer was given by A. Mesiarová [13] showing that the function $\delta\colon [0,1] \longrightarrow [0,1]$ given by

$$\delta(x) = \begin{cases} \frac{x}{2} & \text{if } x \in [0,0.5]\,, \\ 0.25 & \text{if } x \in \,]0.5,0.75]\,, \\ 3x-2 & \text{otherwise,} \end{cases}$$

cannot be the diagonal of a t-norm, although we have $\delta \in \mathcal{D}$.

Solved Problem 4. Let T be a continuous t-norm on $[0,1]^2$ (i.e., a an Abelian semigroup operation $T\colon [0,1]^2 \times [0,1]^2 \longrightarrow [0,1]^2$ with neutral element $(1,1)$ which is non-decreasing with respect to the product order on $[0,1]^2$). Is T necessarily the Cartesian product of two t-norms on $[0,1]$?

This problem was stated in [3], and a counterexample was provided by S. Jenei and B. De Baets [6].

3 Some open problems

Open Problem 5. Let T be a continuous Archimedean t-norm with additive generator $t\colon [0,1] \longrightarrow [0,\infty]$ and $a \in \,]0,0.5[$. Prove or disprove that

$$T(\max(x-a,0),\min(x+a,1)) \leq T(x,x)$$

holds for all $x \in [0,1]$ if and only if t is convex.

This problem has been posed by J. Fodor. Note that a positive solution of this problem would induce a new characterization of associative copulas.

Open Problem 6. Let T be a conditionally cancellative (left-continuous) t-norm which is continuous in the point $(1,1)$. Is T necessarily continuous?

A t-norm T satisfies the conditional cancellation law if $T(x,y) = T(x,z) > 0$ implies $y = z$ [8, Definition 2.9]. Note that, for t-norms without zero divisors, this is exactly the solved problem 2.

Open Problem 7. Characterize all continuous (Archimedean) t-norms T such that the restriction of T to $\mathbb{Q}^2$ is a binary operation on $[0,1] \cap \mathbb{Q}$.

This problem was inspired by some work of S. Jenei and F. Montagna on the extension of t-norms.

Open Problem 8. (i) Characterize all strictly decreasing functions $t\colon [0,1] \longrightarrow [0,\infty]$ with $t(1) = 0$ such that the operation $T\colon [0,1]^2 \longrightarrow [0,1]$ given by

$$T(x,y) = t^{(-1)}(t(x) + t(y)) \tag{2}$$

is a t-norm, where the pseudo-inverse $t^{(-1)}\colon [0,\infty] \longrightarrow [0,1]$ is given by

$$t^{(-1)}(u) = \sup\{x \in [0,1] \mid t(x) > u\}.$$

(ii) Characterize all strictly decreasing functions $t\colon [0,1] \longrightarrow [0,\infty]$ with $t(1) = 0$ such that the operation $T\colon [0,1]^2 \longrightarrow [0,1]$ given by (2) is a t-norm and such that for all $n > 2$ and for all $x_1, x_2, \ldots, x_n \in [0,1]$ we have

$$T(x_1, x_2, \ldots, x_n) = t^{(-1)}\Big(\sum_{i=1}^{n} t(x_i)\Big). \tag{3}$$

Note that each t-norm T induced by some function t satisfying (2) and (3) is necessarily Archimedean. However, there are non-Archimedean t-norms T induced by functions satisfying (2) only [19].

Open Problem 9. For a given pair of a t-norm T and its dual t-conorm S, characterize all binary aggregation operators $\mathbf{A}\colon [0,1]^2 \longrightarrow [0,1]$ such that $\mathbf{A} \gg T$ and $S \gg \mathbf{A}$, where the domination relation $\gg$ is given by (1).

Recall that a function $\mathbf{A}\colon [0,1]^2 \longrightarrow [0,1]$ is called a (binary) aggregation operator if it is nondecreasing and satisfies $\mathbf{A}(0,0) = 0$ and $\mathbf{A}(1,1) = 1$ (for details concerning domination see [14]).

Also the dual problem of characterizing, for a given binary aggregation operator $\mathbf{A}$, all t-norms T such that $\mathbf{A} \gg T$ and $S \gg \mathbf{A}$ holds, where S is the t-conorm dual to T, is of interest.

Open Problem 10. Given a binary aggregation operator $\mathbf{A}\colon [0,1]^2 \longrightarrow [0,1]$, characterize all pairs (T,S) of a t-norm T and a t-conorm S such that for all $(x,y) \in [0,1]^2$ we have

$$\mathbf{A}(T(x,y), S(x,y)) = \mathbf{A}(x,y). \tag{4}$$

Clearly, in the case where $\mathbf{A}$ equals the arithmetic mean, (4) is just the Frank functional equation which was completely solved in [4]. In the case $\mathbf{A} = T_{\mathbf{P}}$ this problem was recently solved by G. Mayor.

Another modification of the Frank functional equation was proposed by J. Fodor: For a given t-norm T_0 and a given t-conorm S_0, characterize all t-norms T and t-conorms S such that for all $(x,y) \in [0,1]^2$ we have

$$T(x,y) + S(x,y) = T_0(x,y) + S_0(x,y).$$

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Aggregation on bounded bipolar scales

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1 Introduction

It is often the case in practice that one has to deal with bounded bipolar scales instead of the usual $[0,1]$ interval. Bipolar scales are symmetric around a central point, the *neutral value*, usually denoted by 0, and can be either of numerical nature (an interval of $\mathbb{R}$ containing 0), or of ordinal nature (only order matters): see a construction of an ordinal bipolar scale in [1, 3, 2]. Bipolar scales are bounded if there exist a least and a greatest element (denoted for example -1 and 1). Values above the neutral value 0 (positive values) are considered as attractive, good, while those under 0 are considered as repulsive, bad, etc.

There are psychological evidences that human behaviour reflects bipolarity, and behaviours in e.g. decision differ when utilities or scores are positive or negative. The well-known Cumulative Prospect Theory (CPT) model [9] is a powerful bipolar model, where a Choquet integral is used to aggregate the positive utilities and negative utilities separately, and two capacities ν_1, ν_2 are used, one for the positive part, the other for the negative part.

The question is now to produce a panoply of aggregation operators for the bipolar case, extending those already known for the unipolar case $[0,1]$, while possibly imposing some structural properties. We address here 3 topics, described below. This can be seen as first steps in this direction.

2 Symmetric pseudo-additions and multiplications

The aim is to define pseudo-additions and pseudo-multiplications, say on $[-1,1]$, so as to get a structure close to a ring, or an Abelian group, if only one operation is considered. A natural starting point seems to take t-norms and t-conorms on $[0,1]$, and get them symmetrized.

We show that if the t-conorm is nilpotent, then there is no way to build even a group. If the t-conorm is strict, then a group can be obtained, and in this case the symmetrized t-conorm corresponds to a uninorm rescaled on $[-1,1]$. However, one cannot obtain a ring anyway (at least with our assumptions) [4].

We show that these results are closely related to the theory of ordered Abelian groups and Hölder theorem [8].

3 Symmetric operations on symmetric ordinal scales

The aim is the same than in Section 2, and can be seen as the case where the t-conorm and t-norm is the max and min operators. We define a symmetric maximum $\varovee$, and a symmetric minimum $\varowedge$, in such a way that the structure is as close as possible to a ring. In fact, it is shown that imposing a symmetric element prevents the symmetric maximum to be associative. In order to cope with non associativity, we propose various rules of computation, which define unambiguously the value of expressions such that $\varovee_{i\in I} a_i$, and study in detail the properties of the rules [2].

4 Bi-capacities

It is known that, with discrete universes, the Choquet integral can be seen as a general family of aggregation operators. If the underlying scale is bipolar, the Choquet integral extends usually in 2 ways: the symmetric integral (or Šipoš integral), and the asymmetric integral (see [7] for propserties). A more general way is to consider the CPT model. Yet, more general extensions can be done, where there is a real interconnection between positive and negative parts. This is achieved through the concept of *bi-capacities* [6, 5], which code the value taken by the integral for functions being 1 on some subset A, and -1 on some subset B. The Choquet integral w.r.t a bi-capacity generalizes the CPT model, and can be interpreted, as the Choquet integral, with the help of the Shapley value, and interaction indices.

Key words: t-norm, uninorm, ordered group, bipolar scale, capacity, Choquet integral

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Addition, Multiplication and Distributivity

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The motivation for our investigations is coming from integration theory.
The definition for the Lebesgue-Integral of simple functions uses an addition, a multiplication, a distributive law and a measure.
To avoid the additivity of the measure the Choquet-Integral requires only an isotone set function which disappears at the empty set (a so-called fuzzy measure), but a difference is needed now.

To define a more general integral it thus seems naturally to consider a fuzzy measure and three generalized functions defined on an arbitrary interval
$[A,B] \quad , \quad -\infty \leq A < B \leq \infty$:

a pseudo-addition, a pseudo-multiplication and a pseudo-difference,
which are connected by an appropriate distributive law so that the
three operations are fitting.

So there are two steps :

First the interaction of fitting pseudo-additions and pseudo-multiplications connected by a distributivity law has to be investigated.

Then - in a second step - one has to choose an appropriate pseudo-difference to define an integral satisfying desirable properties.

Let us start with some remarks :

We assume that a pseudoaddition is essentially a t-conorm on $[A,B]$ but a pseudo-multiplication is only an isotone function which is continuous on $(A,B]^2$ (neither associativity nor commutativity is required).

The usual one-sided distributivity is a rather strong property so that the class of fitting pseudo-multiplications is very restricted (for example, the usual (bounded) addition and (bounded) multiplication are not fitting operations).

We can offer a weak distributive law for which extensive classes of pseudo-additions and pseudo-multiplications are fitting. Actually, we use 2 different pseudo-additions to introduce one-sided distributivity laws (here we only give the left-distributivity law) :

Definition 1. Let $\Delta, \blacksquare : [A,B]^2 \to [A,B]$ be pseudo-additions with generator sets $\{k_m : [a_m^{\Delta}, b_m^{\Delta}] \to [0,\infty] \mid m \in K_{\Delta}\}$ and $\{h_l : [a_l^{\blacksquare}, b_l^{\blacksquare}] \to [0,\infty] \mid l \in K_{\blacksquare}\}$.

Moreover, let $\mathcal{D}_{\Delta} := \{b_m^{\Delta} : m \in K_{\Delta}\}$ and let $\diamond : [A,B]^2 \to [A,B]$ be pseudo-multiplicaton.

Then $\diamond$ satisfies the weak left-distributivity law with respect to $(\Delta, \blacksquare)$ iff $a \Delta b \notin \mathcal{D}_{\Delta}$ implies $(a \Delta b) \diamond x = (a \diamond x) \blacksquare (b \diamond x)$ for all $a,b,x \in (A,B]$.

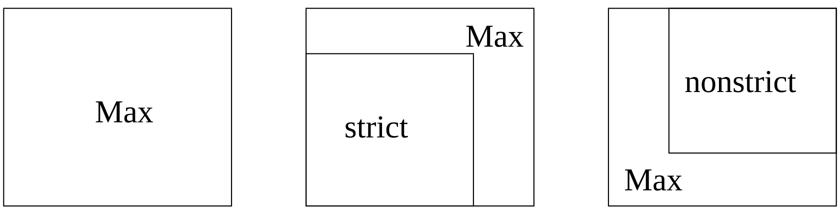
This means that the usual distributivity law holds if $a \Delta b \notin \mathcal{D}_{\Delta}$ is no right endpoint of an "archimedean" interval $[a_m^{\Delta}, b_m^{\Delta}]$.

(By "usual left- (or right-) distributivity" we mean the equations $(a \Delta b) \diamond x = (a \diamond x) \blacksquare (b \diamond x)$ (or $a \diamond (x \Delta y) = (a \diamond x) \blacksquare (a \diamond y)$) are satisfied for all $x,y,a,b \in (A,B]$).

Using additional axioms like the existence of a one-sided unit we investigate the structure of the pseudo-multiplication (and its influence on the ordinal-sum-structure of the pseudo-addition) and the possibility of a representation of the pseudo-multiplication by generators of the pseudoaddition.

We present the following special case of a more general result (Suprisingly a similar result holds in the case of the validity of a one-sided "usual" left distributivity law).

Theorem 2. Let $\Delta, \blacksquare : [A,B]^2 \to [A,B]$ be pseudo-additions,
let $\diamond : [A,B]^2 \to [A,B]$ be a pseudo-multiplication which satisfies the weak left-distributivity law with respect to $(\Delta, \blacksquare)$ and let $\diamond$ have a right unit
(that is, there is an $e \in (A,b]$ such that $a \diamond e = a$ for all $a \in (A,B]$).
(I) Then we have
(a) $\Delta = \blacksquare$.
(b) If $\diamond$ has a left unit
(that is, there is an $\tilde{e} \in (A,b]$ such that $\tilde{e} \diamond a = a$ for all $a \in (A,B]$)
then Δ has one of the following structures



(II) For all $m \in K_\Delta, l \in K_\blacksquare$ there is a monotonic increasing, continuous function $g_{m,l} : (A,b] \to [0,\infty]$ with the property

$$\bigwedge_{a \in (a_m^\Delta, b_m^\Delta]} \bigwedge_{x \in (A,B]} [\quad a \diamond x \in (a_l^\blacksquare, b_l^\blacksquare] \to a \diamond x = h_l^{(-1)}(k_m(a) \cdot g_{m,l}(x)) \quad].$$

Theorem 2 has rather weak assumptions. In the literature rather often $\diamond$ is assumed to be a uni-norm or a t-norm, but the above result shows that neither associativity nor commutativity is needed. In contrary, in many cases we get that $\diamond$ is automatically associative and commutative.

Note that the structure of the pseudo-addition in Theorem 2 reduces to an ordinal sum with at most 2 "archimedean intervals". This explains why in all existing examples in the literature at most 2 "archimedean intervals" were chosen (see [1]).

Moreover, we can give representations of fitting pseudo-multiplications in all "archimedean intervals" using the generators of the pseudo-addition.
>From this result we get - for example - very easily a recent result of Klement, Mesiar and Pap concerning t-norms and t-conorms which satisfy a restricted distributive law (see [2]).

Concerning the second step we define - like proposed by Murofushi and Sugeno (see [4]) - a mapping $-_\Delta : [A,B]^2 \to [A,B]$ to be pseudo-difference with respect to a pseudo-addition Δ iff $a -_\Delta b := inf\{c \in [A,B] : b\Delta c \geq a\}$.

But here we have no restriction to archimedean pseudo-additions.
Fortunately this pseudo-difference is very compatible with the weak left- and right-distributivity law.

Our integral definition for measurable functions f is based on the fuzzy-t-conorm integral of Murofushi and Sugeno (see [4]). To define an integral we need only two continuous t-conorms, a fitting pseudo-multiplication and an arbitrary fuzzy-measure μ.
But of course, if we want to prove the theorem of monotone convergence, we need that μ is continuous from below.
The usual results concerning integrals like monotonicity and commonotone additivity are presented (rather often the proof for the monotonicity of a "fuzzy" integral has gaps, we will point out that the proof for the monotonicity is not trivial).

Further the fuzzy-measure can be decomposed into "fuzzy-measure components" fitting to the ordinal structure of the pseudo-addition and pseudo-multiplication, so that we have in each "archimedean interval" a nice representation with the generators of the pseudo-addition and pseudo-multiplication.

Finally a characterization result for the integral can be represented which is similar to a result of Benvenuti and Mesiar (see [1]).

Theorem 3. Let $(X, \mathcal{A})$ be a measurable space.
Further, let $\Delta, \blacksquare : [0,B]^2 \to [0,B]$ be pseudo-additions and
let $\diamond : [0,B]^2 \to [0,B]$ be a pseudo-multiplication satisfying the usual left- and right-distributivity and

having a left-unit $\widetilde{e}$ and having 0 as neutral element
(that is $a=0$ or $x=0$ implies $a \diamond x = 0$ for all $a,x \in [0,B]$).

Moreover let $\mathcal{F} : \{ f : X \to [0,B] \mid f$ measurable $\}$ and
let $I : \mathcal{F} \to [0,B]$ be a function.

Then there exists a $\blacksquare$ -decomposable fuzzy measure $\mu : \mathcal{A} \to [0,B]$
(that is, $U \cap V = \emptyset$ implies $\mu(U \cup V) = \mu(U) \blacksquare \mu(V)$)
which is continuous from below satisfying

$$\bigwedge_{f \in \mathcal{F}} I(f) = \int f d\mu$$

(where $\int$ is our fuzzy integral) iff

1. I is monotonic increasing,

2. I is continuous from below,

3. I is decomposable $(U \cap V = \emptyset$ implies $I(\widetilde{e}\mathbf{1}_{U \cup V}) = I(\widetilde{e}\mathbf{1}_U) \blacksquare I(\mathbf{1}_V))$,

4. I is weakly homogeneous $(I(a\mathbf{1}_M) = a \diamond I(\widetilde{e}\mathbf{1}_M))$,

5. I is additive for commonotone functions f,g

(if $f(x) < f(x)$ implies $g(x) \leq g(y)$ then $I(f \Delta g) = I(f) \blacksquare I(g)$).

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Decision-making in fuzzy logic control with the degree of coincidence

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In control theory and also in theory of approximate reasoning, introduced by Zadeh in [14], much of the knowledge of system behavior and system control can be stated in the form of if-then rules. The Fuzzy Logic Control, FLC has been carried out searching for different mathematical models in order to supply these rules. In most sources it was suggested to represent an

if x is A then y is B

rule in the form of fuzzy implication (shortly $Imp(A,B)$, relation (shortly $R(A,B)$), or simply as a connection (for example as a t-norm, $T(A,B)$) between the so called rule premise: x is A, and rule consequence: y is B. Let x be from universe X, y from universe Y, and let x and y be linguistic variables. Fuzzy subset A of X is characterized by its membership function $\mu_A : X \to [0,1]$. The most significant differences between the models of FLC-s lie in the definition of this connection, relation or implication.

The other important part of the FLC is the inference mechanism. One of the widely used methods is the Generalized Modus Ponens (GMP), in which the main point is, that the inference y is B' is obtained when the propositions are:

- the ith rule from the rule system of n rules: *if* x *is* A_i *then* y *is* B_i,
- and the system input x is A'.

GMP represents the real influences of the implication or connection choice on the inference mechanisms in fuzzy systems [4], [13]. Usually the general rule consequence for one rule from a rule system is obtained by

$$B'(y) = \sup_{x\in X}(T(A'(x), Imp(A(x), B(y))).$$

In this field we can find the new results for left-continuous t-norms in [1]. The connection $Imp(A,B)$ is generally defined, and specially it can be some t-norm, too.

In engineering applications the Mamdani implication is widely used. The Mamdani GMP with Mamdani implication inference rule says, that the membership function of the consequence B' is defined by

$$B'(y) = \sup_{x\in X}(\min(A'(x), \min(A(x), B(y)))$$

or generally

$$B'(y) = \sup_{x\in X}(T(A'(x), T(A(x), B(y))), \tag{1}$$

where T is a left-continuous t-norm. Thus we obtain from (1)

$$B'(y) = T(\sup_{x \in X}(T(A'(x), A(x)), B(y)).$$

Generally speaking, the consequence (rule output) is given with a fuzzy set $B'(y)$, which is derived from rule consequence $B(y)$, as a cut of the $B(y)$. This cut, $\sup_{x \in X} T(A'(x), A(x))$, is the generalized degree of firing level of the rule [13], considering actual rule base input $A'(x)$, and usually depends on the covering over $A(x)$ and $A'(x)$. But first of all it depends on the sup of the membership function of $T(A'(x), A(x))$.

The FLC rule base output is constructed as a crisp value calculated with a defuzzification model, from rule base output. Rule base output is an aggregation of all rule consequences $B'_i(y)$ in rule base. A t-conorm S is usually used as an aggregation operator

$$y_{out} = S(B'_n, S(B'_{n-1}, S(\ldots, S(B'_2, B'_1)))).$$

In system control, however, intuitively one would expect: let's make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The family of evolutionary operators ([9]), and the family of distance-based operators ([8]), satisfy that properties, but the covering over $A(x)$ and $A'(x)$ is not really reflected by the sup of the membership function of the $T_e^{\max}(A'(x), A(x))$ ($T_e^{\max}$is the maximum distance based operator). Hence, and because of the non-continuity of the distance-based operators, it was unreasonable to use the classical degree of firing, to give expression for the coincidence of the rule premise (fuzzy set A), and system input (fuzzy set A'). Therefore a Degree of Coincidence (*Doc*) for those fuzzy sets has been initiated. It is nothing else, but the proportion of area under membership function of the distance-based intersection of those fuzzy sets, and the area under membership function of their union (using max as the fuzzy union)

$$Doc = \frac{\int_X T_e^{\max}(A'(x), A(x))dx}{\int_X \max(A'(x), A(x))dx}.$$

This definition has two advantages:

- it considers the width of coincidence of A and A', and not only the "height", the sup, and
- the rule output is weighted with a measure of coincidence of A and A' in each rule ([10]).

The rule output fuzzy set B' is achieved as a cut of rule consequence B with *Doc*

$$B'(y) = T_e^{\min}(B(y), Doc) \qquad or \qquad B'(y) = T_e^{\max}(B(y), Doc).$$

It is easy to prove that $Doc \in [0,1]$, and $Doc = 1$ if A and A' cover each other, which implies $B'(y) = B(y)$, and $Doc = 0$ if A and A' have no point of contact, which implies $B'(y) = 0$.

The FLC rule base output is constructed as above explained. The output is constructed as a crisp value calculated from rule base output, which is an aggregation of all rule consequences $B'_i(y)$ in rule base. For aggregation, distance based operators $S_e^{\min}$ or $S_e^{\max}$ can be used.

We can see the justification for this line of reasoning in the simulations of a simple dynamic system, using distance based operator-pairs $T_e^{\min}, S_e^{\min}$ or $T_e^{\max}, S_e^{\max}$ [12].

An additional possibility is if the cut $B_i'(y)$ of the rule consequence $B_i(y)$ is calculated from the expression ([11])

$$Doc = \frac{\int_Y B'(y)\,dy}{\int_Y B(y)\,dy}.$$

Based on this fact, we have for triangular membership functions $A(x), A'(x)$, $B(y)$ that

$$B'(y) = \max(B(y), 1 - \sqrt{1 - Doc}).$$

The $B'(y)$ is obtained as a weighted fuzzy set, and the weight parameter (Doc) depends on $\int_X T_e^{\max}(A'(x), A(x))\,dx$. It is a measure related to the area under membership function $T_e^{\max}(A'(x), A(x))$, and it is a non-additive measure related to t-norm and t-conorm (in the domain and in the range) in the spirit as it can be found in [2] and [5],[6],[7]. Using this fact a connection between *Doc* type of inference mechanism and generalized fuzzy measures and integrals has been investigated.

The further steps are the investigation of measure-properties of different degrees of firing types used by FLC, and the use of the other types of fuzzy integrals from the pseudo-analysis in decision-making by FLC.

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Convex structure of the space of fuzzy measures

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Abstract

Fuzzy measures (T-measures) on T-tribes are a fuzzification of measures on σ-algebras. They were characterized recently in [4]. Here we investigate the convex structure of probability T-measures (T-states).

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Let T be a fixed strict triangular norm (t-norm), i.e., a binary operation $T\colon [0,1]^2 \to [0,1]$ which is commutative, associative, continuous, strictly increasing (except for the boundary of the domain) and satisfies the boundary condition $T(a,1) = a$ for all $a \in [0,1]$ (see [9, 16]). A T-*tribe* is a collection $\mathcal{T}$ of fuzzy subsets which contains the empty set and which is closed under the standard fuzzy complement and (the pointwise application of) the triangular norm T (extended to countably many arguments). The notion of a T-tribe was introduced by Butnariu and Klement [5, 6] as a generalization of a σ-algebra of subsets of a set. Further, they introduced the notion of T-measure as a generalization of a σ-additive measure on a σ-algebra (here $S\colon [0,1] \to [0,1]$ denotes the triangular conorm dual to T, i.e., $S(a,b) = 1 - T(1-a, 1-b)$): A function $m\colon \mathcal{T} \to R$ is called a T-*measure* iff it satisfies the following axioms:

$$m(0) = 0,$$
$$m(T(A,B)) + m(S(A,B)) = m(A) + m(B),$$
$$A_n \nearrow A \Longrightarrow m(A_n) \to m(A),$$

where the symbol $\nearrow$ denotes monotone increasing convergence. The notion of T-measure is not only a natural generalization of a classical measure. It is also the base of successful applications in game theory. Many deep mathematical results, including a generalization of Liapunoff Theorem, were proved in [1, 2, 6]. An overview of fuzzy measures can be found in [7].

The *strict Frank triangular norms* T_s, $s \in (0,\infty)$ (see [8]) are t-norms of the form

$$T_s(a,b) = \log_s \left(1 + \frac{(s^a - 1)(s^b - 1)}{s - 1} \right)$$

for $s \in (0,\infty) \setminus \{1\}$ or

$$T_1(a,b) = a \cdot b$$

(the product t-norm) for $s = 1$.

A characterization of monotonic T-measures for a Frank triangular norm T has been presented in [6] and completed in [13]:

Theorem 1. *Let T be a strict Frank triangular norm and let $\mathcal{T}$ be a T-tribe. Then the set $C(\mathcal{T})$ of all* crisp *elements of $\mathcal{T}$ (i.e., those attaining only values $0,1$) is a σ-algebra. Each T-measure μ on $\mathcal{T}$ is of the form*

$$\mu(A) = \nu(A^{-1}((0,1])) + \int A \,\mathrm{d}\lambda,$$

where ν, λ are (classical) measures on $C(\mathcal{T})$ (up to the standard identification of sets with their characteristic functions).

In [15], the latter theorem was extended to a more general case of so-called *nearly Frank t-norms* (see [12, 14]). On the other hand, for all other t-norms T the general form of a T-measure reduces to

$$\mu(A) = \nu(A^{-1}((0,1])),$$

where ν is a (classical) measure on $C(\mathcal{T})$. These results were obtained under an additional assumption of monotonicity of the T-measure in question; nevertheless, an independent generalization in [3] (for Frank t-norms) and [4] (for the general case) show that the characterization of nonmonotonic T-measures remains essentially the same.

We tried to compare probability measures (called also *states*) on σ-algebras and *probability T-measures* (called also *T-states*) on T-tribes. They form convex sets. In many cases the space of all T-states is affinely homeomorphic to the state space of some σ-algebra. This correspondence is canonical in the case when the t-norm T is not nearly Frank; then T-states on a T-tribe $\mathcal{T}$ are uniquely determined by their restriction to the σ-algebra $C(\mathcal{T})$. A less trivial correspondence is obtained for Frank and nearly Frank t-norms where the above restriction is not injective. Still in many cases (e.g., for *semigenerated tribes* introduced in [11]) the T-state space is affinely homeomorphic to the state space of some σ-algebra (different from $C(\mathcal{T})$).

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