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25th Linz Seminar on Fuzzy Set Theory

Abstracts

Mathematics of Fuzzy Systems

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Erich Peter Klement, Endre Pap
Editors

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MATHEMATICS OF FUZZY SYSTEMS

ABSTRACTS

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Since their inception in 1979 the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The seminar is deliberately kept small and intimate so that informal critical discussion remains central. There are no parallel sessions and during the week there are several round tables to discuss open problems and promising directions for further work.

LINZ 2004 will be already the 25$^{\text{th}}$ seminar carrying on this tradition. It is therefore a good opportunity to review the most important mathematical aspects of fuzzy systems. As usual, the aim of the seminar is an intermediate and interactive exchange of surveys and recent results. We expect that the presented talks will provide a comprehensive mathematical framework for the theory and application of fuzzy systems.

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Contents

Fuzzy random variables: development and state of the art

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1 Introduction

The concept of random variable is clearly fundamental to the fields of Probability and Statistics.

A random experiment is a process in which the result or outcome is not known with certainty before the experiment is performed. A (classical) random variable is a measurable function defined on the sample space of the random experiment which converts each particular experimental outcome into a real or vectorial value. Measurability is supposed to guarantee that many useful probabilities can be computed.

In addition to randomness, a certain imprecision can arise either in perceiving or reporting existing real/vectorial values, or in identifying existing values which are essentially imprecise. Fuzzy random variables have been introduced to model imprecisely-valued measurable functions where imprecision is formalized in terms of fuzzy sets.

2 Fuzzy random variables as a model for fuzzy perceptions/observations of existing real-valued random mechanisms

Kwakernaak (1978, 1979), and later Kruse and Meyer (1987) in a more elaborated way, introduced fuzzy random variables as a model for the situations in which fuzzy imprecision arises either in the perception or in the report of values of a real-valued random variable (referred to as the 'original').

Let $\mathcal{F}_c(\mathbb{R})$ denote the class of the normal convex fuzzy subsets of the Euclidean space $\mathbb{R}$ having compact α-levels for $\alpha \in [0,1]$, that is, the class of mappings $U : \mathbb{R} \longrightarrow [0,1]$ such that $U_\alpha = \{x \in \mathbb{R} \,|\, U(x) \geq \alpha\}$ if $\alpha \in (0,1]$, $= \mathrm{cl}(\mathrm{supp}\, U)$ if $\alpha = 0$, are nonempty compact intervals. Then,

Definition 1. (Kruse and Meyer, 1987) Let $(\Omega, \mathcal{A}, P)$ be a probability space. A **fuzzy random variable** is a mapping $\mathcal{X} : \Omega \longrightarrow \mathcal{F}_c(\mathbb{R})$ such that for any $\alpha \in [0,1]$ the real-valued mappings $\inf \mathcal{X}_\alpha : \Omega \longrightarrow \mathbb{R}$, $\sup \mathcal{X}_\alpha : \Omega \longrightarrow \mathbb{R}$ (with $\inf \mathcal{X}_\alpha(\omega) = \inf \big(\mathcal{X}(\omega)\big)_\alpha$, $\sup \mathcal{X}_\alpha(\omega) = \sup \big(\mathcal{X}(\omega)\big)_\alpha$, for all $\omega \in \Omega$) are real-valued random variables (i.e., Borel-measurable real-valued functions).

In this approach when one refers to *parameters* associated with a fuzzy random variable, one is considering either real/vectorial-valued parameters of the probability distribution of the original random variable or fuzzy-valued parameters defined on the basis of Zadeh's extension principle (see

Kruse and Meyer, 1987). Thus, if $\theta(X)$ is a parameter of a real-valued random variable X, and $\mathcal{E}(\Omega,\mathcal{A},P)$ is the class of possible originals of $\mathcal{X}$, the associated ***fuzzy parameter*** of variable $\mathcal{X}$ corresponds to $\theta(\mathcal{X}) : \mathbb{R} \longrightarrow [0,1]$ such that

$$\theta(\mathcal{X})(t) = \sup_{X\in\mathcal{E}(\Omega,\mathcal{A},P),\,\theta(X)=t} \ \inf_{\omega\in\Omega} \left\{\mathcal{X}(\omega)\big(X(\omega)\big)\right\} \quad \text{for all} \quad t \in \mathbb{R}.$$

In particular, when $\theta(X) = E(X|P)$ is the population expected value of X, then the *population fuzzy expected value* $\theta(\mathcal{X})$ is the fuzzy set in $\mathcal{F}_c(\mathbb{R})$ such that $\big(\theta(\mathcal{X})\big)_\alpha = \big[E(\inf \mathcal{X}_\alpha|P), E(\sup \mathcal{X}_\alpha|P)\big]$ for all $\alpha \in [0,1]$.

3 Fuzzy random variables as a model for existing fuzzy-valued random mechanisms

A second approach to fuzzy random variables conceives them as a model for the situations in which fuzzy imprecision arises in the definition of the values of the random mechanism or variable. More precisely, a fuzzy random variable is intended to be a measurable function defined on the sample space of the random experiment and converting each particular experimental outcome into a fuzzy subset of a separable Banach space (often a Euclidean one).

Let $(B, |\cdot|)$ be a separable Banach space, and let $\mathcal{F}(B) = \{U : B \longrightarrow [0,1] \,|\, U_\alpha \in \mathcal{K}(B)$ for all $\alpha \in [0,1]\}$, with $U_\alpha = \{x \in B \,|\, U(x) \geq \alpha\}$ for $\alpha \in (0,1]$, $=$ cl(supp$U$) if $\alpha = 0$, and $\mathcal{K}(B) = \{$nonempty bounded and closed subsets of $B\}$. In other words, $\mathcal{F}(B)$ is the class of the normal upper semicontinuous $[0,1]$-valued functions defined on B with bounded closure of the support.

Puri and Ralescu (1986) formalized fuzzy random variables (also called *random fuzzy sets*) as an extension of random sets as follows:

Definition 2. (Puri and Ralescu, 1986) Let $(\Omega,\mathcal{A},P)$ be a probability space. A **fuzzy random variable** is a mapping $\mathcal{X} : \Omega \longrightarrow \mathcal{F}(B)$ such that for any $\alpha \in [0,1]$ the set-valued mapping $\mathcal{X}_\alpha : \Omega \longrightarrow \mathcal{K}(B)$ (with $\mathcal{X}_\alpha(\omega) = \big(\mathcal{X}(\omega)\big)_\alpha$ for all $\omega \in \Omega$) is a compact random set, that is, it is Borel-measurable with the Borel σ-field generated by the topology associated with the well-known Hausdorff metric on $\mathcal{K}(B)$,

$$d_H(K,K') = \max\left\{\sup_{k\in K}\inf_{k'\in K'}|k-k'|,\ \sup_{k'\in K'}\inf_{k\in K}|k-k'|\right\}.$$

Recently (see Colubi *et al.*, 2001, 2002), Definition 3.1 has been proven to be equivalent to the one formalizing fuzzy random variables as $\mathcal{F}(B)$-valued random elements (that is, Borel-measurable $\mathcal{F}(B)$-valued functions) when $\mathcal{F}(B)$ is equipped with the Skorohod metric

$$d_S(U,U') = \inf_{\lambda\in\Lambda}\max\left\{\sup_\alpha |\lambda(\alpha)-\alpha|\,,\ \sup_\alpha d_H(U_\alpha, U'_{\lambda(\alpha)})\right\},$$

where $\Lambda = \{\lambda : [0,1] \longrightarrow [0,1] \,|$ strict increasing function with $\lambda(0) = 0, \lambda(1) = 1\}$ for $U,U' \in \mathcal{F}(B)$.

Furthermore (see also Colubi *et al.*, 2001, 2002), the measurability condition in Definition 3.1 has been proven to be equivalent to that (cf. Diamond and Kloeden, 1994) based on the d_q metrics on $\mathcal{F}(B)$ by Klement *et al.* (1986), for all $q \in [1,\infty)$, where $d_q(U,U') = \left(\int_{[0,1]} \left[d_H(U_\alpha, U'_\alpha)\right]^q d\alpha\right)^{1/q}$.

On the other hand, Klement *et al.* (1986) have introduced fuzzy random variables as $\mathcal{F}(B)$-valued random elements when $\mathcal{F}(B)$ is equipped with the sup-metric, that is,

Definition 3. (Klement, Puri and Ralescu, 1986) Let $(\Omega,\mathcal{A},P)$ be a probability space. A **fuzzy random variable** is a mapping $\mathcal{X}:\Omega\longrightarrow\mathcal{F}(B)$ which is Borel-measurable with the Borel σ-field generated by the topology associated with the metric

$$d_\infty(U,U')=\sup_{\alpha\in[0,1]} d_H(U_\alpha,U'_\alpha)$$

for all $U,U'\in\mathcal{F}(B)$.

The connections between notions in Definitions 3.1 and 3.2 are the following ones (see Colubi *et al.*, 2001, 2002):

Proposition 4. *If $\mathcal{X}:\Omega\longrightarrow\mathcal{F}(B)$ is Borel-measurable with the Borel σ-field generated by the topology associated with d_∞, then it is Borel-measurable with the Borel σ-field generated by the topology associated with the d_S.*

However, the converse implication fails, since the requirements for fuzzy random variables in Definition 3.2 are too rectrictive. An illustrative counterexample for this assertion can be found in Colubi *et al.* (2002).

Moreover, when $B=\mathbb{R}$ and $\operatorname{Im}\mathcal{X}\subset\mathcal{F}_c(\mathbb{R})$, then Definitions 2.1 and 3.1 coincide (see, for instance, Zhong and Zhou, 1987), although they represent models for different situations in practice.

In this second approach the *parameters* associated with a fuzzy random variable $\mathcal{X}$ are usually defined on the basis of those for the corresponding parameters for compact random sets. As an example, if $E\big(d_H(\mathcal{X}_0,\{0\})\big|P\big)<\infty$, then Puri and Ralescu (1986) have defined the ***fuzzy expected value*** of $\mathcal{X}$ as the unique fuzzy set $E(\mathcal{X}|P)\in\mathcal{F}(B)$ such that $\big(E(\mathcal{X}|P)\big)_\alpha=$ Aumann's integral of the compact random set $\mathcal{X}_\alpha$ for all $\alpha\in[0,1]$ (i.e., $\big(E(\mathcal{X}|P)\big)_\alpha=\big\{E(X|P)\,|\,X:\Omega\longrightarrow B,\ X\in L^1(\Omega,\mathcal{A},P),\ X\in\mathcal{X}_\alpha\ a.s.\,[P]\big\}$).

It is convenient to remark that ranges of fuzzy random variables have been extended in some studies to include unbounded values (see, for instance, Li and Ogura, 1999).

4 Some probabilistic and statistical studies concerning fuzzy random variables

Since from a mathematical viewpoint Definition 3.1 includes 2.1 and 3.2 as special cases, from now on we will assume fuzzy random variables we deal with are random elements in the Skorohod sense.

Metric properties of the space $(\mathcal{F}(B),d_S)$ indicate (Colubi *et al.*, 2002) that it is complete and separable. In this respect, it should be pointed out that $(\mathcal{F}(B),d_\infty)$ is complete but non-separable (Puri and Ralescu, 1986, Klement *et al.*, 1986), whence handling this space would be definitely more complex than working with $(\mathcal{F}(B),d_S)$.

Fuzzy random variables can be *characterized* in general *as certain limits of sequences of elementary types* (more precisely, either simple or having simple α-levels) of these variables (see López-Díaz and Gil, 1997, 1998a).

In the set-valued case, when we work in a statistical setting the choice of the Aumann expectation (1965) of a compact random set among possible integrals (see, for instance, Molchanov, 1998), can be justified by means of the Laws of Large Numbers for random sets (see Artstein and Vitale, 1975).

In an analogous way, *Laws of Large Numbers* (like those by Klement *et al.*, 1986, Colubi *et al.*, 1999, Molchanov, 1999, Taylor *et al.*, 2001, Krätschmer, 2002, Proske and Puri, 2003, and so on) justify the choice of the fuzzy expected value in Puri and Ralescu's sense.

Some other probabilistic results concerning *differentiability* (see, for instance, Puri and Ralescu, 1983, Román-Flores and Rojas Medar, 1998, Rodríguez-Muñiz *et al.*, 2003), *integrability* (see, for instance, Gong and Wu, 2002, Rodríguez-Muñiz and López-Díaz, 2003, Krätschmer, 2004), *limit theorems* (cf. Taylor *et al.*, 2001, Li *et al.*, 2003), *reversing the order of integration* (see López-Díaz and Gil, 1998b), *fuzzy martingales* (see, Stojaković, 1994, Li and Ogura, 2001, Terán, 2003), and so on, can be found in the recent literature.

In which concern statistical developments involving fuzzy random variables, we can mention on one hand *decision problems including fuzzy-valued utilities* or *rewards* (see Gil and López-Díaz, 1996, Kurano *et al.*, 2002), *regression analysis* (see Näther and Körner, 2002, Wünsche and Näther, 2002) and, on the other hand, recent studies on *inferential techniques on either real- or fuzzy-valued parameters of a fuzzy random variable*. As an example for the last one, we can mention inferences on the fuzzy expected value of a fuzzy random variable (see Körner, 2000, Montenegro *et al.*, 2003).

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Fuzzy filter functors revisited: a 2-categorical overview

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It was noted in [1] that limits in 2-categories are not as easy to describe as limits in ordinary categories. In particular, there is a class of (2-categorical) limits called weighted or indexed limits [2-4] which can be defined as follows: take a 2-functor G: **D** $\longrightarrow$ **H**, and define also a weight on **D** to be a 2-functor F: **D** $\longrightarrow$ **CAT** (the category of categories). If, following the notation of [4], we denote by [**D**, **CAT**] the 2-category whose objects are 2-functors from **D** to **CAT**, whose 1-cells are natural transformations, and whose 2-cells are modifications, then the F-weighted limit of G: **D** $\longrightarrow$ **H** is a representing object for [**D**, **CAT]** (F, **H**(-, G)). A great advantage of a 2-categorical point-of-view and of the more general versions of limits it allows is that paths are opened to simple algebraic constructions over categories which may not be available if the scope is restricted to ordinary categories.

Eklund and Gahler [5] defined a fuzzy filter to be an element M of L^{Lx} s.t.

(1) $M(\alpha^-) = \alpha$ (α is the constant mapping of X into L with value α)

(2) f, g in L^X and $f \leq g$ imply $M(f) \leq M(g)$

(3) $\forall$f, g in L^X, $M(f \wedge g) \geq M(f) \wedge M(g)$

where X is a set and M is a meet semilattice. They then define the covariant set functor F_L related to L, which they call the fuzzy filter functor, to be the functor which assigns each set X to the set of all (L-) fuzzy filters on X. When L is {0, 1}, they call F_L the proper filter functor. As noted in [1], they show that the proper filter functor becomes a monad in certain cases, as does the fuzzy filter functor, and they point out that Eilenberg-Moore objects can be defined for the proper filter functor as monad, but they do not carry this further to F_L as monad.

But, again as discussed in [1], a great deal more can be done. In fact, it is proved in [1] that the fuzzy filter functor is (isomorphic to) a 2-functor F_S : $\mathbf{D}_{simp}$ $\longrightarrow$ **idl**, where $\mathbf{D}_{simp}$ is the category with one object (*) whose morphisms (from * to *) comprise the simplicial category of finite ordinals and order-preserving maps (see [2] or [4]) and **idl** is the 2-category whose 1-cells (morphisms) are order-ideals (relations compatible with the orders on the domain and on the codomain) and whose 2-cells are inclusions. Given this, Eilenberg-Moore objects for the fuzzy filter functor emerge naturally as they do for any other 2-functor from $\mathbf{D}_{simp}$ to any 2-category. Furthermore, it is known that Eilenberg-Moore objects are constructible from products, inserters, and equifiers (see [4], p. 44). Since Eilenberg-Moore objects are constructible from the fuzzy filter functor, it must be the case that a relation exists between the fuzzy filter functor and the more elementary constructs products, inserters, and equifiers. This means that the fuzzy filter functor can be broken down into a set of simpler functors (details and references may be found in [1]).

This paper carries these results and ideas a bit further. We describe, first of all, exactly what these constructs (products, inserters, and equifiers) which underlie the fuzzy functor would look like. Second, we show that certain already known properties of the fuzzy filter functor have very simple

2-categorical analogues. Thus, for instance, the fact the fuzzy filter monad is a submonad of the crisp filter monad ([5], Prop. 7.14, p.135) has an interesting simple expression when expressed in 2-categorical terms. We also explore the implications of this 2-categorical view for fuzzy topology.

Perhaps most importantly, this paper takes a number of finite limit types, including products, inserters, equifiers, inverters, and lax limits, and describes how requirements for their existence "constrain" the set (lattice, semilattice, ...) over which the fuzzy filters are taken and how they constrain the nature of the fuzzy filters themselves. Consider, for instance, the inserter. If I is the inserter object and A is any object, then for any pair of 1-cells a, b : A -> I, there must exist a 2-cell (the "inserted" 2-cell) θ : i • a => i • b (see, e.g., [4], p. 38). Now consider the enriched monad D of finitely generated up sets over **Pos** as described in [6] (see esp. p. 262). The internal hom-functor over its (strict) algebras is given by the equalizer

Hom((B, $\leq$), (A, $\leq$)) = all order-preserving maps in A^B ordered pointwise

which equalizes $A^B -> A^{(DB)}$ and $A^B -> DA^{(DB)} -> A^{(DB)}$. If we now "generalize" this equalizer to be an inserter, we must restrict further the set of maps in Hom((B, $\leq$), (A, $\leq$)) to ensure that the inserted 2-cells actually exist. Such restrictions and conditions have interesting implications for the nature of Eilenberg-Moore objects over fuzzy filters and for fuzzy topologies.

SOME USEFUL BACKGROUND

MONADS

Given a category C, a monad consists of an endofunctor T along with two natural transformations η: $id_C \longrightarrow$ T and $\mu : T^2 \longrightarrow$ T s.t. μ(A) • Tη(A) = id_{TA} = μ(A) • η(TA) and μ(A) • Tμ(A) = μ(A) • μ(TA). In a 2-category **C**, a monad may be defined as an object X along with an endo-1-cell S and two 2-cells, a unit 2-cell η: 1 $\longrightarrow$ S and a multiplication 2-cell μ: SS $\longrightarrow$ S [7].

LIMITS

Generally speaking, one takes limits over functors whose domains are small categories (**I**, **J**, ...) and whose codomains are locally small categories (**C**). The so-called abstract definition of a limit is the definition in terms of representations. For a functor G: **I** $\longrightarrow$ **C**, an object L in **C** is a limit for G iff there is a representation C(X, L) $\cong$ [I, C](ΔX, G), i.e., for every natural transformation from the diagonal functor to G there is a representing morphism from X to the limit object in **C** for every object X in **C**. This description converts readily to the "concrete" description in terms of cones. Note that for ordinary categories the natural isomorphism from C(X, L) (the covariant hom-functor) to [I, C](ΔX, G), two functors from C to **Set**, is straightforward.

Now consider the definition of a limit in a 2-category. We shall define such a limit, for reasons that will soon be apparent, as follows: a (2-categorical) limit L for a 2-functor G: **I** $\longrightarrow$ **C** is given by the representation C(X, L) $\cong$ [I^{op}, Cat](!, (ΔX, G)) (here **!** is the terminal 2-category - see [8]). There are two important aspects of this definition for our purposes. First, both sides of the isomorphism are categories (by assumption), so we must take account of 2-cells as well as morphisms (1-cells). Second, we expand the right side to include !; this doesn't accomplish anything in particular here but will prove useful when we turn to weighted limits below. As far as 2-cells are concerned, the universal property requires that there be an invertible 2-cell which takes each G_u ($G_i \longrightarrow G_j$) • τ_i (X $\longrightarrow G_i$) to τ_j (X $\longrightarrow G_j$).

For full generality, we need somewhat more from our notion of a limit. Suppose we replace the terminal 2-category (**!**) in our definition above by a (any) 2-functor from **I** to **Cat** (known as an "index" or "weight"). Now, since (ΔX, G) is also a functor from **I** to **Cat** (by composition of G and the hom-functor), a weighted cone over G with vertex X is given by a 2-natural transformation from F to (ΔX, G). Where an ordinary limit, then, requires only a single morphism to connect X with each vertex of G (each G(-) in the diagram below), a weighted limit requires a set of morphisms (in fact, since these are 2-categories, a category of morphisms) from X to each G_i.

$$\begin{array}{ccc} \leftarrow & X & \longrightarrow \\ \downarrow^{f_j} & & \downarrow^{f_k} \\ G(j) & \underset{G(u)}{\longrightarrow} & G(k) \end{array} \qquad u : j \longrightarrow k$$

FUZZY FILTERS AS MONADS

There are two fundamental objectives served by the construction of monads in a (2-) category **C**. One is the derivation of Eilenberg-Moore objects or algebras from the monads, and the other is the construction of a (2-) category of the monads themselves through which functors into **C** may be factored. Suppose we begin with **rel**, the 2-category whose objects are sets, whose 1-cells (morphisms) are binary relations, and whose 2-cells (morphisms of morphisms) are inclusions. We ask what sorts of objects in **rel** (i.e., which sets), if any, are monadic, that is, which objects X may be equipped with an associative multiplication χ (more specifically, an endo-1-cell x and a 2-cell χ such that χ takes xx to x and is associative), and with a unit 2-cell χ' from X (i.e., the 1-cell id_x) to x (note that we follow here, for the most part, the presentation in [9]). Monads in **rel**, then, are just pre-ordered sets $<$X, $\leq$$>$, the pre-ordering providing the obvious multiplication and unit. These pre-ordered sets can be seen to be the objects of a 2-category with morphisms (called M-modules by Koslowski in [9]) $<$X, $\leq$$>$ $\longrightarrow$ $<$Y,$< - >$ those relations r for which ($\leq$r) $\subseteq$ r $\subseteq$ (r$<$-). In other words, if there is an arrow x $\longrightarrow$ x' in X and r: x' $\longrightarrow$ y then also r: x $\longrightarrow$ y and the same mutatis mutandis for r, y, and y'. Such relations are called order-ideals, and, along with inclusions as 2-cells, they comprise the category of monads in **rel** called **idl** in [9].

We know also ([5], p. 135) that the fuzzy filter functor generates monads in (the ordinary category) **set**, i.e., that the set of all fuzzy filters on a set X is a monad in **set**. However, in a 2-categorical sense, the set of all fuzzy filters on a set X may also be thought of as a monad in **idl**, since such monads are simply closure operators (see [9], p. 198). It is this role which leads to the construction of Eilenberg-Moore objects for the fuzzy filter functor and hence to pie limits as described above.

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On a natural fuzzification of Boolean logic

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1 Introduction

In this communication we propose two logically sound fuzzification and defuzzification techniques for implementing a credibility calculus on a set of propositional expressions. Both rely on a credibility evaluation domain using the rational interval $[-1,1]$ where the sign carries a split truth/falseness denotation. The first technique implements the classic min and max operators where as the second technique implements Bochvar-like operators. Main interest in the communication is given to the concept of *natural fuzzification* of a propositional calculus. A formal definition is proposed and the demonstration that both fuzzification techniques indeed verify this definition is provided.

2 Logical fuzzification and polarization: an adjoint pair

2.1 Introducing logical fuzziness

Let P be a set of constants or ground propositions. Let $\neg$, $\vee$ and $\wedge$ denote respectively the contradiction, disjunction and conjunction operators.

The set E of all *well formulated finite expressions* will be generated inductively from the following grammar:

$$\forall p \in P \quad : \quad p \in E, \tag{1}$$

$$\forall x, y \in E \quad : \quad \neg x \mid (x) \mid x \vee y \mid x \wedge y \ \in E. \tag{2}$$

The unary *contradiction* operator $\neg$ has a higher precedence in the interpretation of a formula, but we generally use brackets to control the application range of a given operator and thus to make all formulas have unambiguous semantics. We suppose in the sequel that all other operators such as implication, equivalence, xor etc are derived with the help of these three basic operators: contradiction, conjunction and disjunction.

With these well formulated propositional expressions we associate a rational credibility evaluation $r : E \rightarrow [-1,1]$ where $\forall x, y \in E$, $r_x = 1$ means x is *certainly true*, $r_x = -1$ means that x is *certainly false* and $r_x > r_y$ (resp. $r_x < r_y$) means that propositional expression x is more (resp. less) credible than propositional expression y. Such a credibility domain is called $\mathcal{L}$, and we denote $E^{\mathcal{L}} = \{(x, r_x) \mid$

$x \in E, r_x \in [-1,1]\}$ a given set of such more or less credible propositional expressions, also called for short $\mathcal{L}$-expressions.

We implement the *contradiction* operator on $\mathcal{L}$-expressions by simply *changing the sign* of the associated credibility evaluation, i.e.

$$\forall (x, r_x) \in E^{\mathcal{L}} : \neg(r, r_x) = (\neg x, -r_x). \quad (3)$$

The sign exchange thus implements an antitone bijection on the rational interval $[-1,1]$ where the *zero* value appears as contradiction fix-point.

In classical bi-valued logic, it is usual to work syntactically only on the *truth* point of view of the logic, the *untruth* or *falseness* point of view being redundant through the coercion to the excluded middle. For instance, writing "$(a,b) \in R$" implicitly means assuming that this proposition is actually true and its contradiction false, otherwise we would write "$(a,b) \notin R$".

We will also rely syntactically on such an implicit truthfulness point of view and always denote the truthfulness possibly induced from the underlying credibility calculus through a truth projection operator[1] μ, acting as a *positive* domain and range restriction on the credibility operator r.

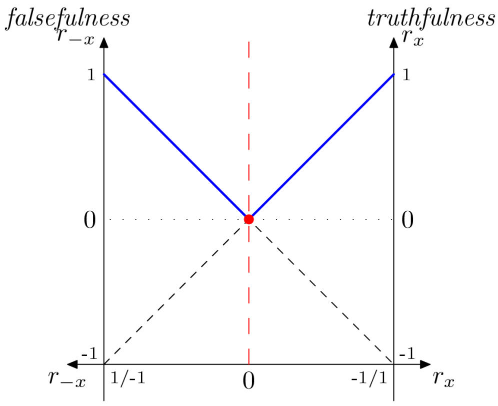

Figure 1: Split Truth/Falseness Semantics

Let $(x, r_x) \in \mathcal{E}^{\mathcal{L}}$ be an $\mathcal{L}$-expression:

$$\mu(x, r_x) = \begin{cases} (x, r_x) \text{ if } r_x \geq r_{\neg x}, \\ (\neg x, r_{\neg x}) \text{ otherwise.} \end{cases} \quad (4)$$

Truthfulness of a given expression x is thus only defined in case the expression's credibility r_x exceeds the credibility $r_{\neg x}$ of its contradiction $\neg x$, otherwise the logical point of view is switched to $\neg x$, i.e the contradicted version of the expression (see Figure 1).

As $r_x \geq r_{\neg x} \Leftrightarrow r_x \geq 0$ it follows from Equation 4 that the sign ($+$ or $-$) of r_x immediately carries the truth functional semantics of $\mathcal{L}$-expressions, in the sense that an $\mathcal{L}$-expression (x, r_x) such that $r_x \geq 0$ may be called *more or less true* ($\mathcal{L}$-true for short) and an expression (x, r_x) such that $r_x \leq 0$ may be called *more or less false* ($\mathcal{L}$-false for short).

[1] In fuzzy set theory, the μ operator generally denotes a fuzzy membership function. We here choose the same μ symbol on purpose as our main $\mathcal{L}$-valued formulas mostly concern $\mathcal{L}$-valued characteristic functions.

Only 0-valued expressions appear to be both $\mathcal{L}$-true and $\mathcal{L}$-false, therefore they are called $\mathcal{L}-$ *undetermined* [2].

To be able to compute the credibility evaluation associated with any $\mathcal{L}$-expression, we still need to implement $\mathcal{L}$-valued versions of the conjunction and disjunction operators.

The classic min and max operators may be used:
$\forall (x, r_x), (y, r_y) \in \mathcal{E}^{\mathcal{L}}$:

$$(x, r_x) \vee (y, r_y) = (x \vee y, \max(r_x, r_y)) \quad (5)$$
$$(x, r_x) \wedge (y, r_y) = (x \wedge y, \min(r_x, r_y)) \quad (6)$$

The operator triple $< -, \min, \max >$ implements on the rational interval $[-1, 1]$ an ordinal credibility calculus, denoted for short $\mathcal{L}_o$, that gives a first example of what we shall call a *natural fuzzification* of propositional calculus.

To appreciate usefulness of our split truth/falseness semantics, let us look at what happens in the $\mathcal{L}_o$-valued framework with the truthfulness of certain classical tautologies or antilogies.

For instance, truthfulness of the tautology $(x \vee \neg x)$ is always given, as $\max(r_x, -r_x)) \geq 0$ in any case. Tautological $\mathcal{L}_o$-valued propositions thus appear as being $\mathcal{L}_o$-true in any case. Therefore we call them $\mathcal{L}_o$*-tautologies*. On the other hand, truthfulness of the antilogy $(x \wedge \neg x)$ is only defined when $\min(r_x, r_{\neg x}) = 0$. More or less "untruthfulness" of such an expression is however always given. Therefore, we call such propositions $\mathcal{L}_o$*-antilogies*.

Finally, let us investigate an implicative $\mathcal{L}_o$-tautology such as the modus ponens for instance. If we take the classical negative (Kleene-Dienes) definition of the implication, i.e. falseness of the conjunction of $r(x)$ and $\neg r(y)$, we obtain

$$\min(r_x, \max(-r_x, r_y)) \geq 0 \Rightarrow r_y \geq 0,$$

i.e. the following $\mathcal{L}_o$-tautology: " (x, r_x) and $(x, r_x) \Rightarrow (y, r_y)$ being conjointly $\mathcal{L}_o$-true always implies (y, r_y) being $\mathcal{L}_o$-true ".

As a main result of our construction, we recover in this sense all classical tautologies and antilogies as particular limit case if we reduce our $\mathcal{L}_o$-valued credibility calculus to a bi-valued $\{-1, 1\}$ one.

2.2 On natural logical polarization

To explore the formal consequences of our split truth/falseness semantics, we need to formalize the logical *defuzzification* or *polarization* we implicitly operate when applying to $\mathcal{L}$-expressions an $\mathcal{L}$-true or $\mathcal{L}$-false denotation.

Unfortunately, the standard defuzzification technique, denoted in the fuzzy literature as λ-cuts (see Fodor & Roubens [4]), where $\lambda \in [-1, 1]$ represents the level of credibility r_x from which on a given $\mathcal{L}$-expression is considered to be true, is not generally consistent with our split truth/falseness semantics (see Bisdorff [2]).

[2]"*... I have long felt that it is a serious defect in existing logic that it takes no heed of the* limit *between two realms. I do not say that the Principle of Excluded Middle is downright* false*; but I* do *say that in every field of thought whatsoever there is an intermediate ground between* positive assertion *and* negative assertion *which is just as Real as they. ...* "(C. S. Peirce, Letter from February 29, 1909 to William James)

What we need is an extended three-valued cut operator (see Bisdorff & Roubens [1]). Let $E^{\mathcal{L}}$ be a set of $\mathcal{L}$-expressions and let $\mathcal{L}^3$ denote the restriction of $\mathcal{L}$ to the three credibility values $\{-1,0,1\}$. $\pi : E^{\mathcal{L}} \to E^{\mathcal{L}^3}$ represents a logical polarization operator defined as follows:
$\forall (x,r_x) \in E^{\mathcal{L}}$:

$$\pi(x,r_x) = \begin{cases} (x,1) & \Leftrightarrow \quad r_x > 0 \\ (x,-1) & \Leftrightarrow \quad r_x < 0 \\ (x,0) & \Leftrightarrow \quad r_x = 0 \end{cases}$$

That π operator indeed implements our split truth/falseness semantics may be summarized by stating the following categorical equation.

$$\mu \circ \pi \; = \; \pi \circ \mu. \tag{7}$$

and a credibility calculus $\mathcal{L}$ verifying Equation 7 is called *natural*.

For instance, we may show that $\mathcal{L}_o$ implements a such natural credibility calculus. For this we must proof that the π operation gives a natural transformation of $\mathcal{L}_o$-valued expressions. Following the general inductive construction of $E^{\mathcal{L}}$ it is sufficient to show naturality of $\mathcal{L}_o$ for each of the basic logical operators.

$\mathcal{L}_o$-valued contradiction: for any $(x,r_x) \in E^{\mathcal{L}_o}$, if $r_x > 0$, $\mu(\pi(x,r_x)) = \mu(x,1) = (x,1) = \pi(\mu(x,r_x))$; if $r_x < 0$, $\mu(\pi(x,r_x)) = \mu(x,-1) = (\neg x,1) = \pi(\neg x,-r_x) = \pi(\mu(x,r_x))$; and if $r_x = 0$, $\mu(\pi(x,r_x)) = \mu(x,0) = (x,1) == \pi(x,r_x) = \pi(\mu(x,r_x))$.

$\mathcal{L}_o$-valued disjubction: for any $(x,r_x),(y,r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ or $r_y > 0$, $\mu(\pi(x \vee y, \max(r_x,r_y))) = \mu(x \vee y,1) = (x \vee y,1) = \pi(x \vee y, \max(r_x,r_y)) = \pi(\mu(x \vee y, \max(r_x,r_y)))$; if $r_x < 0$ and $r_y < 0$, $\mu(\pi(x \vee y, \max(r_x,r_y))) = \mu(x \vee y,-1) = (\neg(x \vee y),1) = \pi(\neg(x \vee y), \min(-r_x,-r_y)) = \pi(\mu(x \vee y, \max(r_x,r_y)))$.

Finally, $\mathcal{L}_o$-valued conjunction: for any $(x,r_x),(y,r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ and $r_y > 0$, $\mu(\pi(x \wedge y, \min(r_x,r_y))) = \mu(x \wedge y,1) = (x \wedge y,1) = \pi(x \wedge y, \min(r_x,r_y)) = \pi(\mu(x \wedge y, \min(r_x,r_y)))$; if $r_x < 0$ or $r_y < 0$, $\mu(\pi(x \wedge y, \min(r_x,r_y))) = \mu(x \wedge y,-1) = (\neg(x \wedge y),1) = \pi(\neg(x \wedge y), \max(-r_x,-r_y)) = \pi(\mu(x \wedge y, \min(r_x,r_y)))$.

This completes the demonstration.

The $\mathcal{L}_o$ credibility calculus is however not the only possible natural credibility calculus we may define on E.

3 A Bochvar-like fuzzification of propositional expressions

A second example is given by a multiplicative fuzzification of the classic three-valued Bochvar logic. We shall denote $\mathcal{L}_b$ such a credibility calculus where the operator triple is denoted $< -, \curlyvee, \curlywedge >$.

We keep the traditional sign exchange as $\mathcal{L}_b$-valued contradiction.

The *multiplicative conjunction* operator $\curlywedge$ on a set $E^{\mathcal{L}}$ of $\mathcal{L}$-expressions is defined as follows:

$$\forall x,y \in E : r_{x \wedge y} = r_x \curlywedge r_y = \begin{cases} \mid r_x \times r_y \mid & \text{if } (r_x > 0) \; \wedge \; r_y > 0), \\ - \mid r_x \times r_y \mid & \text{otherwise.} \end{cases}$$

In Figure 2, we may notice that the $\curlywedge$-operator, when restricted to a $\{-1,1\}$-valued domain, is isomorphic to the classic Boolean conjunction operator.

Similarly, we define the *multiplicative disjunction* operator $\curlyvee$ as follows:

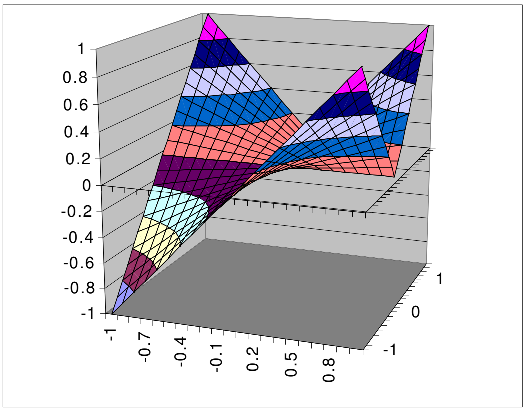

Figure 2: Graphical representation of the multiplicative conjunctive operator

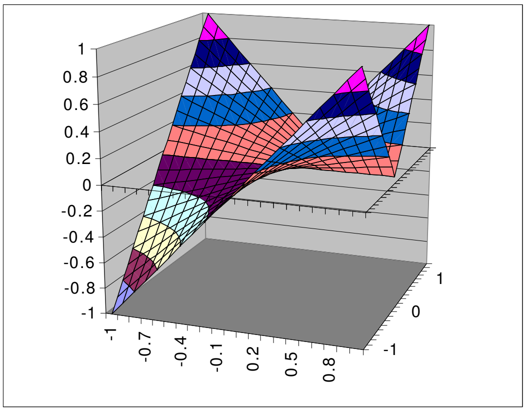

Figure 3: Graphical representation of the multiplicative disjunctive operator

$$\forall x,y \in P : r_{x \vee y} = r_x \curlyvee r_y = \begin{cases} -\mid r_x \times r_y \mid & \text{if } (r_x < 0) \wedge (r_y < 0), \\ \mid r_x \times r_y \mid & \text{otherwise.} \end{cases}$$

Again, we may notice in Figure 3 that we recover in the limit, when restricted to only $-1, 1$-valued expressions, the classic Boolean disjunction operator.

First, we may verify that the De Morgan duality properties are verified in $\mathcal{L}_b$. Indeed, we easily see that:

$$\forall (x, r_x), (y, r_y) \in E^{\mathcal{L}_b} : r_{x \wedge y} = r_{(\neg(\neg x \vee \neg y))}.$$

Indeed, if $r_x, r_y > 0$, $r_x \curlywedge r_y = r_x \times r_y$. At the same time, $r_{\neg x} \curlyvee r_{\neg y} = (r_{\neg x} \times r_{\neg y}) = -(r_x \times r_y)$. On the contrary, if $r_x, r_y < 0, r_x \curlywedge r_y = -(r_x \times r_y)$, then $r_{\neg x} \curlyvee r_{\neg y}) = (r_{\neg x} \times r_{\neg y}) = (-r_x \times -r(_y) = r_x \times r_y$. If either $r_x > 0$ and $r_y < 0$ or vice versa, the duality relation is equally verified.

It is most interesting to notice that in the case where both $\mathcal{L}_b$-expressions are $\mathcal{L}_b$-true, respectively $\mathcal{L}_b$-false, both operators $\curlywedge$ and $\curlyvee$ give the same $\mathcal{L}_b$-credibility. The operators diverge in their result only when contradictory $\mathcal{L}_b$-truth assessments are to be combined. The conjunctive operator aligns the $\mathcal{L}_b$-false part where as the disjunctive operator sustains the $\mathcal{L}_b$-true part of the pair of propositions.

We may furthermore notice that the negational fix-point, the zero value, figures as logical "*black hole*" as is usual in the three-valued Bochvar logic, absorbing all possible logical determinism through any of both binary operators.

$$\forall (x, r_x) \in E^{\mathcal{L}_b} : r_x \curlywedge 0 = r_x \curlyvee 0 = 0.$$

Let us denote $E^{\mathcal{L}_b}_{/-1;1}$ the equivalence classes of all certainly true or false $\mathcal{L}_b$-expressions. The restriction of the $\mathcal{L}_b$ credibility calculus to $E^{\mathcal{L}_b}_{/-1:1}$ gives a classic Boolean algebra.

It is remarkable however, that such a priori obvious properties as impotency of conjunction and disjunction, are only satisfied in this limit Boolean case. Indeed in general, the natural logical consequence of combining more and more fuzzy propositions will sooner or later necessarily end up with a completely undetermined proposition. The same is true when combining conjunctively or disjunctively a number of times the same fuzzy proposition. Indeed, $\forall (x, r_x), (y, r_y) \in E^{\mathcal{L}_b}$ such that $r_x \neq 0$ we have:

$$\begin{aligned} |\, r_x \,| &> |\, r_x \curlywedge r_y \,|, \\ |\, r_x \,| &> |\, r_x \curlyvee r_y \,|. \end{aligned}$$

We recover here a similar situation as in classic error propagation. The more we operate with imprecise numbers, we more we increase the imprecision of the out-coming result, and this imprecision is essentially related to the imprecision of the initial inputs.

Finally, to validate now the naturality property of the $\mathcal{L}_b$ calculus, we must show that the curly operators $\curlyvee$ and $\curlywedge$ verify Equation 7. In order to do so, it is again sufficient to show that for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_b}$ and both the curly operators we have:

$$\begin{aligned} \mu(\pi(x \vee y, r_x \curlyvee r_y)) &= \pi(\mu(x \vee y, r_x \curlyvee r_y)), \\ \mu(\pi(x \wedge y, r_x \curlywedge r_y)) &= \pi(\mu(x \wedge y, r_x \curlywedge r_y)). \end{aligned}$$

Indeed, for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ or $r_y > 0$, $\mu(\pi(x \vee y, r_x \curlyvee r_y))) = \mu(x \vee y, 1) = (x \vee y, 1) = \pi(x \vee y, r_x \curlyvee r_y) = \pi(\mu(x \vee y, r_x \curlyvee r_y)$; if $r_x < 0$ and $r_y < 0$, $\mu(\pi(x \vee y, r_x \curlyvee r_y)) = \mu(x \vee y, -1) = (\neg(x \vee y), 1) = \pi(\neg(x \vee y), r_x \curlywedge r_y) = \pi(\mu(x \vee y, r_x \curlyvee r_y))$.

And for any $(x, r_x), (y, r_y) \in E^{\mathcal{L}_o}$, if $r_x > 0$ and $r_y > 0$, $\mu(\pi(x \wedge y, r_x \curlywedge r_y)) = \mu(x \wedge y, 1) = (x \wedge y, 1) = \pi(x \wedge y, r_x \curlywedge r_y) = \pi(\mu(x \wedge y, r_x \curlywedge r_y)$; if $r_x < 0$ or $r_y < 0$, $\mu(\pi(x \wedge y, r_x \curlywedge r_y)) = \mu(x \wedge y, -1) = (\neg(x \wedge y), 1) = \pi(\neg(x \wedge y), r_x \curlyvee r_y) = \pi(\mu(x \wedge y, r_x \curlywedge r_y))$.

This concludes the demonstration that $\mathcal{L}_b$ does indeed implements a natural credibility calculus.

4 Moving on

In order to situate now the whole family of natural credibility calculus one may define on propositional expressions, let us explore two directions for further investigations.

Following the general properties of the $\mathcal{L}_o$ calculus, we may want to consider the t-norm concept as potential generalization. Unfortunately, the split truth/falseness semantics is not quite compatible with the formal properties of a t-norm. Indeed, let us recall that a t-norm T defined on the interval $[-1;1]$ should verify the following four axioms:

$$T(1, r_x) = r_x, \forall r_x \in [-1;1] \quad (8)$$

$$T(r_x, r_y) = T(r_y, r_x), \forall r_x, r_y \in [-1;1] \quad (9)$$

$$T(r_x, r_y) \leq T(r_u, r_v) \quad \text{if} \quad -1 \leq r_x \leq r_u \leq 1, -1 \leq r_y \leq r_v \leq 1 \quad (10)$$

$$T(r_x, T(r_y, r_z)) = T(T(r_x, r_y), r_z), \forall r_x, r_y, r_z \in [-1;1]. \quad (11)$$

It is easily verified that the multiplicative conjunctive operator $\curlywedge$ verifies three of these axioms, i.e. all except the third one. This is not astonishing, as this axiom is not so "naturally" a logical axiom but rather a geometrical axiom underlying the "triangularity" heritage of the t-norm concept.

What axiom could advantageously replace the "triangular" t-norm condition in order to make fit conceptually the t-norm to a natural credibility calculus on the rational interval $[-1,1]$?

A possibility might be the following:

$$| T(r_x, r_y) | \leq | T(r_u, r_v) | \ \text{ if } 0 \leq | r_x | \leq | r_u | \leq 1, 0 \leq | r_y | \leq | r_v | \leq 1.$$

In some sense we would recover the triangular axiom in some absolute terms. But this idea has still to be further explored.

Finally, more following the semiotical intuitions of C.S. Peirce, we may interpret the classic ordinal $\mathcal{L}_o$ credibility calculus and the above introduced Bochvar-like $\mathcal{L}_b$ credibility calculus as some limit constructions of a same semiotical foundation of logical fuzziness. Indeed, the $\mathcal{L}_o$ calculus to be applicable in a practical setting supposes a same closed universal semiotical reference for all ground propositions $p \in P$ as is usual in a mathematical logic context for instance, where as the multiplicative model apparently supposes shared semiotical references for all determined parts and disjoint semiotical references for the logically undetermined parts of each proposition $p \in P$ as is usual for instance in repetitive physical measures with error propagation.

These general considerations leave open the case where each ground expression $p \in P$ is completely supported by a different semiotical reference. In this last case we would get as third limit case some kind of aggregational logic (see Bisdorff [3]) as implemented by the concordance principle in the multicriteria approach to preference aggregation for instance.

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A review of construction and representation results for fuzzy weak orders

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Weak orders, i.e. reflexive, transitive, and complete binary relations, are among the most fundamental concepts in preference modeling. It is well-known that weak orders are nothing else but linear orders of equivalence classes, where the corresponding equivalence relation is the symmetric kernel of the weak order. If the underlying set of alternatives X is finite, a weak order can be represented by a single score function [2].

In analogy to the crisp case, fuzzy weak orders are fundamental concepts in fuzzy preference modeling [3, 4, 5]. Given a non-empty set of alternatives X, a fuzzy relation $R : X^2 \to [0,1]$ is a *fuzzy weak order* if it fulfills the following three axioms for all $x,y,z \in X$ (where T is a left-continuous t-norm):

$R(x,x) = 1$	(reflexivity)
$T\big(R(x,y),R(y,z)\big) \leq R(x,z)$	(T-transitivity)
$R(x,y) = 1$ or $R(y,x) = 1$	(strong completeness)

In this contribution, we give an overview of construction and representation results for fuzzy weak orders. This includes both known results and new insights:

(i) Every fuzzy weak order can be represented as a union of a crisp linear order and a fuzzy equivalence relation—which is a full analogue to the crisp case [1]. Based on this discovery, we are able to construct fuzzy weak orders from pseudo-metrics if the t-norm T is continuous Archimedean [1].

(ii) For the case that X is finite, we give a necessary and sufficient condition that a fuzzy weak order is determined only by the degrees to which two consecutive equivalence classes are related to each other.

(iii) Every fuzzy weak order can be represented by score functions [6], but not necessarily by a single one, not even if X is finite [3]. A necessary and sufficient condition for the representability by a single score function is given.

(iv) Fuzzy weak orders can be represented by an embedding to the fuzzy power set $\mathcal{F}(X)$ equipped with the fuzzy inclusion induced by the t-norm T [1].

All these reviews and new results are demonstrated by means of detailed examples.

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A bridge between fuzzy set theory and coherent conditional probabilities (II)

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In this talk (strictly linked with that by Romano Scozzafava with the same title) we start from our approach to fuzzy set theory in terms of conditional events and *coherent* conditional probabilities, showing also how the concept of *possibility function* naturally arises in this context. Coherent conditional probability is looked on as a general non-additive "uncertainty" measure $\psi(\cdot) = P(E|\cdot)$ of the conditioning events. In particular, we show that ψ can be interpreted as a *possibility* measure, giving a relevant characterization: ψ is a *possibility* if and only if it is a *capacity*. Moreover, we give also a characterization of the measure ψ as an (antimonotone) *information measure*. Any *coherent* extension of a membership function is between these two extreme cases, but the converse is not true. So we discuss also a characterization of *coherence* of such extensions in terms of a suitable weighted mean of conditional probabilities.

We recall from the previous talk (with the same title) the following basic notions.

Let φ be any *property* related to a random quantity X : notice that a *property*, even if expressed by a statement, does not single–out an *event*, since the latter needs to be expressed by a *nonambiguous* proposition that can be either *true* or *false*.

Consider now the **event** E_φ = "You claim φ" and a coherent conditional probability $P(E_\varphi|A_x)$, looked on as a real function $\mu_{E_\varphi}(x) = P(E_\varphi|A_x)$ defined on C_X, the range of X. Then a *fuzzy subset* E^*_φ of C_X is the pair

$$E^*_\varphi = \{E_\varphi \,,\, \mu_{E_\varphi}\},$$

with $\mu_{E_\varphi}(x) = P(E_\varphi|A_x)$ for every $x \in C_X$.

So a coherent conditional probability $P(E_\varphi|A_x)$ is a measure of how much You, given the event $A_x = \{X = x\}$, are willing to *claim* the property φ, and it plays the role of the membership function of the fuzzy subset E^*_φ. We recall that we have been able not only to define fuzzy subsets, but also to introduce in a very natural way the basic continuous T-norms and the relevant dual T-conorms, bound to the former by *coherence*.

On the other hand, if we consider the weakest T-norm

$$T_o(x,y) = \begin{cases} \min(x,y) & \text{if} \max(x,y) = 1\,, \\ 0 & \text{otherwise} \end{cases}$$

we can prove that the choice of $p = P(E_\varphi \wedge E_\psi|A_x \wedge A_y)$ agreeing with T_o is *not* coherent.

The three coherent choices discussed in the previous talk correspond to the particular values $\lambda = 0$, $\lambda = 1$, $\lambda = \infty$, respectively, of the fundamental (archimedean) Frank t-norms T_λ and t-conorms S_λ (see [6]), with $\lambda \in [0, \infty]$, that is (for λ different from the above three values)

$$T_\lambda(x,y) = log_\lambda\left(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1}\right), \; S_\lambda(x,y) = 1 - log_\lambda\left(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1}\right).$$

In our framework (where, given a t-norm singling-out the value $P(E_\varphi \wedge E_\psi | A_x \wedge A_y)$ of the conjunction, then the corresponding choice of the t-conorm, which determines the value of the disjunction $P(E_\varphi \vee E_\psi | A_x \wedge A_y)$, is *uniquely* driven by the coherence of the relevant conditional probability) we are able to capture also Frank t-norms and t-conorms (for any $\lambda \in [0, \infty]$), archimedean or not.

In our setting it is completely natural to consider fuzzy measures, taking as starting point a membership function, regarded as a pointwise distribution. This requires in fact only to extend a coherent conditional probability assessment on the family $\{E_\varphi | A_x\}$ to the larger family of events $\{E_\varphi | A\}$, with A element of the algebra $\mathcal{A}$ spanned by events $\{A_x\}$.

Intuitively, $P(E_\varphi | A)$ is the probability that "You claim φ" in the hypothesis that the value of the variable X belongs to A.

The results that follow are mainly taken from [3] and [4]. Let us introduce the following ("natural") definitions:

(D1) Let E be an arbitrary event and P any coherent conditional probability on the family $\mathcal{G} = \{E\} \times \{A_x\}_{x \in C_X}$, admitting $P(E|\Omega) = 1$ as (coherent) extension. A *distribution of possibility* on C_X is the real function π defined by $\pi(x) = P(E|A_x)$.

Actually, along the same lines we can as well introduce any general distribution ψ, to be called just *uncertainty measure*.

(D2) Under the same conditions of **(D1)**, a *distribution of uncertainty measure* on C_X is the real function ψ defined by $\psi(x) = P(E|A_x)$.

When C_X is finite, since every extension of $P(E|\cdot)$ must satisfy the axioms of a conditional probability, condition $P(E|\Omega) = 1$ gives

$$P(E|\Omega) = \sum_{x \in C_x} P(A_x|\Omega) P(E|A_x) \qquad \text{and} \qquad \sum_{x \in C_x} P(A_x|\Omega) = 1\,.$$

Then $1 = P(E|\Omega) \leq \max_{x \in C_x} P(E|A_x)$; therefore $P(E|A_x) = 1$ for at least one event A_x.

On the other hand, we notice that in our framework (where *null probabilities for possible conditioning events are allowed*) it does not necessarily follow that $P(E|A_x) = 1$ for every x; in fact we may well have $P(E|A_y) = 0$ (or else equal to any other number between 0 and 1) for some $y \in C_X$. Obviously, the constraint $P(E|A_x) = 1$ for some x is not necessary when the cardinality of C_X is infinite.

From now on, given an arbitrary event E, let $\mathcal{C}$ be a family of conditional events $\{E|H_i\}_{i \in I}$, where $card(I)$ is arbitrary and events H_i's are a *partition* of Ω, $P(E|\cdot)$ an arbitrary (coherent) conditional probability on $\mathcal{C}$, $\mathcal{H}$ the algebra spanned by the H_i's, and $\mathcal{H}^o = \mathcal{H} \setminus \{\emptyset\}$.

Here we list some of the main results:

(A) Any coherent extension of P to $\mathcal{C}' = \{E|H : H \in \mathcal{H}^o\}$ is such that, for every $H, K \in \mathcal{H}$, with $H \wedge K = \emptyset$,

$$(1) \qquad \min\{P(E|H), P(E|K)\} \leq P(E|H \vee K) \leq \max\{P(E|H), P(E|K)\}.$$

It follows that any coherent extension of P to $\mathcal{C}' = \{E|H : H \in \mathcal{H}^o\}$ is such that, for every $H, K \in \mathcal{H}$, with $H \wedge K = \emptyset$,

$$P(E|H \vee K) \leq P(E|H) + P(E|K).$$

On the other hand

(B) Any real function f defined on $\mathcal{H}$ such that, if $H \wedge K = \emptyset$,

$$\min\{f(H), f(K)\} \leq f(H \vee K) \leq \max\{f(H), f(K)\},$$

is a *capacity* if and only if, for every $H, K \in \mathcal{H}$,

$$f(H \vee K) = \max\{f(H), f(K)\}.$$

So the function $f(H) = P(E|H)$, with P a coherent conditional probability, in general *is not a capacity*.

The question now is: are there coherent conditional probabilities $P(E|\cdot)$ monotone with respect to $\subseteq$? We reached a positive answer by means of the following result (given in [5]), which represents the main tool to introduce *possibility measures* in our context referring to coherent conditional probabilities.

(C) Let $f : \mathcal{C} \to [0,1]$ be *any* function such that

$$(2) \qquad f(E|H_i) = 0 \text{ if } E \wedge H_i = \emptyset \text{ and } f(E|H_i) = 1 \text{ if } H_i \subseteq E$$

holds. Then any P extending f on $\mathcal{K} = \{E\} \times \mathcal{H}^o$ and such that

$$(3) \qquad P(E|H \vee K) = \max\{P(E|H), P(E|K)\}, \quad \text{for every } H, K \in \mathcal{H}^o$$

is a coherent conditional probability.

(D3) Let $\mathcal{H}$ be an algebra of subsets of C_X and E an arbitrary event. If P is any coherent conditional probability on $\mathcal{K} = \{E\} \times \mathcal{H}^o$, with $P(E|\Omega) = 1$ and such that

$$P(E|H \vee K) = \max\{P(E|H), P(E|K)\}, \quad \text{for every } H, K \in \mathcal{H}^o,$$

then a *possibility measure* on $\mathcal{H}$ is the real function Π defined by $\Pi(H) = P(E|H)$ for $H \in \mathcal{H}^o$ and $\Pi(\emptyset) = 0$.

In our context, **(C)** assures that any possibility *measure* can be obtained as coherent extension (unique, in the finite case) of a possibility *distribution*. Vice versa, given any possibility measure Π on an algebra $\mathcal{H}$, there exists an event E and a coherent conditional probability P on $\mathcal{K} = \{E\} \times \mathcal{H}^o$ agreeing with Π, *i.e.* whose extension to $\{E\} \times \mathcal{H}$ (putting $P(E|\emptyset) = 0$) coincides with Π.

So an immediate consequence of **(B)** and **(C)** is that **any coherent P extending f on $\mathcal{K} = \{E\} \times \mathcal{H}^o$ is a capacity if and only if it is a possibility**.

Going back to our interpretation of a membership function $\mu(x)$ through a suitable coherent conditional probability (a measure of how much You, given the event A_x, are willing to *claim* the relevant property φ), and putting

$$H_o = \{x \in \mathcal{C}_X : \mu(x) = 0\}\,,\; H_1 = \{x \in \mathcal{C}_X : \mu(x) = 1\}\,,$$

the conditional probability $P(E|H^c)$, with $H = H_o \vee H_1$, is *a measure of how much You are willing to claim property* φ if the only fact you know is that $x \in H^c$. On the other hand, *every membership function can be regarded as a possibility distribution*. If $\mathcal{A}$ is an algebra of subsets of $\mathcal{C}_X$, the ensuing possibility *measure* can be interpreted in the following way: it is a sort of "global" membership (relative to each finite $A \in \mathcal{A}$) which takes, among all the possible choices for its value on A, *i.e.* among all possible extensions satisfying (2), the *maximum* of the membership in A.

Moreover, we can regard every possibility measure Π as a decreasing function of the elements of the *zero-layer set* $\{0,1,2,\ldots,k\}$ associated to the *class* $\{P_\alpha\}$ of unconditional probabilities that are used to represent a coherent conditional probability in our main characterization theorem (see [2], p.81).

In conclusion, *the coherent extensions of a conditional probability $P(E|A_x)$ that satisfy (3) give rise to different zero-layers* for the *atoms* A_x corresponding to different $P(E|A_x)$, so that such a coherent conditional probability $P(E|\cdot)$ can be suitably associated to a measure of your "disbelief" in the events $A \in \mathcal{A}$.

Then some of the usual arguments may appear counterintuitive: in fact, the "global" membership should possibly decrease when the information is not concentrated on a given x, but is "spread" over a larger set (for example, considering the statement "Mary is young", you may be willing, if you know that Mary's age is $x = 39$, to put $\mu(x) = .2$, while if you know that her age is $y = 26$, you may be willing to put $\mu(y) = .9$; on the other hand, knowing that her age is between 26 and 39, the corresponding possibility is still .9).

So our results may suggest to take as such global measure a function which is **not** a capacity, yet satisfying the weaker conditions under **(A)**.

With the aim of studying *information* measures in the framework of coherent conditional probabilities, we gave also the following definition, which parallels, in a sense, those **(D1)** and **(D2)** for uncertainty (including *possibility*) measures.

(D4) Let F be an arbitrary event and P any coherent conditional probability on the family $\mathcal{G} = \{F\} \times \{A_x\}_{x \in C_X}$, admitting $P(F|\Omega) = 0$ as (coherent) extension. We define *pointwise information measure* on C_X the real function ψ defined by $\psi(x) = P(F|A_x)$.

When C_X is finite, since every extension of $P(F|\cdot)$ must satisfy the axioms of a conditional probability, considering the condition $P(F|\Omega) = 0$, we necessarily have

$$P(F|\Omega) = \sum_{x \in C_x} P(A_x|\Omega)P(F|A_x) \qquad \text{and} \qquad \sum_{x \in C_x} P(A_x|\Omega) = 1\,.$$

Then $0 = P(F|\Omega) \geq \min\limits_{x \in C_x} P(F|A_x)$, so $P(F|A_x) = 0$ for at least one event A_x.

On the other hand, we notice that in our framework it does not necessarily follow that $P(F|A_x) = 0$ for every x; in fact we may well have $P(F|A_y) = 1$ (or to any other number between 0 and 1) for some

$y \in C_X$. Obviously, the constraint $P(F|A_x) = 0$ for some x is not necessary when the cardinality of C_X is infinite.

Under the same conditions mentioned before **(A)**, we get an immediate consequence of **(A)** itself:

(B1) Any real function f defined on $\mathcal{H}$ such that, if $H \wedge K = \emptyset$,

$$\min\{f(H), f(K)\} \leq f(H \vee K) \leq \max\{f(H), f(K)\},$$

is *antimonotone* with respect to $\subseteq$ if and only if, for every $H, K \in \mathcal{H}$,

$$f(H \vee K) = \min\{f(H), f(K)\}.$$

The following result proves the existence of coherent conditional probabilities $P(F|\cdot)$ antimonotone with respect to $\subseteq$. It represents also the main tool to introduce *information measures* in our context referring to coherent conditional probabilities.

(C1) Let $f : \mathcal{C} \to [0,1]$ be *any* function such that

$$f(F|H_i) = 0 \text{ if } F \wedge H_i = \emptyset \text{ and } f(F|H_i) = 1 \text{ if } H_i \subseteq F$$

holds . Then any P extending f on $\mathcal{K} = \{F\} \times \mathcal{H}^o$ and such that

$$P(F|H \vee K) = \min\{P(F|H), P(F|K)\}, \quad \text{for every } H, K \in \mathcal{H}^o,$$

is a coherent conditional probability.

In the case that the assessment $P(F|H_i)$ admits $P(F|\Omega) = 0$ as coherent extension, we obtain as well a coherent extension by requiring both $P(F|\Omega) = 0$ and choosing "min" as combination rule to make the extension of P.

Are the two extreme cases

– $P(E|A_x)$ extended to the disjunction of conditioning events by taking the *maximum* (possibility measure, **monotone**)

– $P(E|A_x)$ extended to the disjunction of conditioning events taking the *minimum* (**antimonotone** measure)

the most natural ways to extend membership functions?

We recall that coherence implies

$$\min\{P(E|H), P(E|K)\} \leq P(E|H \vee K) \leq \max\{P(E|H), P(E|K)\}$$

but the converse is NOT true. So, in general, a value between the two extremes is not necessarily a coherent choice for the conditional probability $P(E|H \vee K)$ (which can be looked on as a sort of "global" membership ...).

Coherent choices have been characterized in [1]: they are **weighted means** of $P(E|H)$ and $P(E|K)$ (weights equal to zero or one are allowed). More generally, this result can be stated with reference to the disjunction of any finite number of *conditioning* events.

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Stable commutative copulas in pairwise comparison models

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The purpose of this lecture is twofold. Firstly, we revise the class of *stable copulas* [7], i.e. the class of copulas that coincide with their *survival copula* [11]. Secondly, we describe two comparison models, a *deterministic* one and a *stochastic* one, in which stable commutative copulas play a simplifying role.

We propose a method for constructing copulas which largely generalizes the ordinal sum construction method. The method is based on a grid structure and the use of what we have called *foreground* and *background* copulas [2]. It can be applied in particular to construct *commutative copulas* and *stable commutative copulas*. Requiring associativity as well leads to the usual ordinal sum construction of t-norms, which for the purpose of constructing stable copulas reduces to the well-known 'symmetric' ordinal sums of Frank t-norms [7].

In the deterministic model, objects are represented by feature vectors that indicate presence or absence of certain properties. A typical way of comparing objects is by means of *cardinality-based similarity measures* operating on the corresponding feature vectors [4]. The generalization to fuzzy feature vectors requires the choice of an appropriate model of fuzzy intersection along with *fuzzification rules* for other set-theoretic operations [1]. For more than two decades now, t-norms have become the standard model for that purpose, and their use is hardly questioned. However, here we show the power of stable commutative copulas. Indeed, $T_{\mathbf{L}}$-transitivity and $T_{\mathbf{P}}$-transitivity of the cardinality-based similarity measures are preserved in the fuzzification process when using a stable commutative copula as model for fuzzy set intersection [9]. Links with *Bell-type inequalities* for copulas and t-norms will be discussed as well [8, 10].

The second comparison model deals with random variables. For a random vector $(X_1, X_2, \ldots, X_n)$, its components are compared pairwisely by considering the *'winning probabilities'* of one over the other. More specifically, a probabilistic relation Q is defined: $Q(X_i, X_j) = \mathcal{P}(X_i > X_j) + 1/2\,\mathcal{P}(X_i = X_j)$. This relation indeed satisfies $Q(X_i, X_j) + Q(X_j, X_i) = 1$. Moreover, its computation requires only the knowledge of the bivariate marginal distributions, which are in turn uniquely determined from the univariate marginal distributions and the copula that binds them. We consider the case where all pairs of variables are coupled by a same commutative copula C. One of the key issues in comparison models is the transitivity exhibited by the model. For probabilistic relations, we have previously developed the rich framework of *cycle-transitivity* [6]. Remarkably, the transitivity of the probabilistic relation expressing the winning probabilities can be classified within this framework, and the corresponding upper bound function only depends on the commutative copula C considered [5]. In case C is stable, this upper bound function is given by $U(\alpha,\beta,\gamma) = \beta + C(1-\beta,\gamma) = \gamma + C(\beta,1-\gamma)$. In particular, when $C = T^{\mathbf{F}}_{\lambda}$ is a Frank t-norm, then $U(\alpha,\beta,\gamma) = S^{\mathbf{F}}_{1/\lambda}(\beta,\gamma)$. In the specific case of independent random variables, i.e. $C = T_{\mathbf{P}}$, we recover the previously studied *dice model* [6] characterized by $U(\alpha,\beta,\gamma) = \beta+\gamma-\beta\gamma$, i.e. dice-transitivity.

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Vague ordered fields: towards an axiomatic theory of vague real line

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The notion of fuzzy function based on many-valued equivalence relations (many-valued similarity relations (equalities) [17, 18, 19], fuzzy equivalence relations [4, 6, 7, 21, 22, 26], similarity relations [1, 2, 3, 15, 28], indistinguishability operators [27], etc.) has been introduced by several authors, and applied to category theory [5], approximate reasoning and fuzzy control theory [8, 10, 15, 22]. The author of this talk [8, 9, 10] later proposed other versions of this kind of fuzzy function, known as strong fuzzy function and perfect fuzzy function, which have more desirable and powerful representation properties than the others. Many-valued equivalence relation-based fuzzy orderings have been studied by Höhle-Blanchard [16] and Bodenhofer [1, 2, 3] w.r.t. different special integral, commutative cqm-lattices. Later on, these fuzzy orderings are generalized on the basis of a fixed and a general integral, commutative cqm-lattice $M = (L, \leq, *)$ under the name M-vague orderings [13, 14]. For a given nonempty set X and an M-equivalence relation E on it, an M-vague ordering on X is a special L-fuzzy relation on X satisfying some further properties by means of E.

Strong (perfect) fuzzy functions [8, 9, 10] form the elementary tools of vague algebra [9, 11, 12] and vague lattices [13, 14]. In contrast to fuzzy algebra [23] and fuzzy lattices [25], vague algebra and vague lattices basically involve vaguely defined binary operations (M-vague binary operations [9, 11, 12]) and vaguely defined ordering relations (M-vague orderings), where the integral, commutative cqm-lattice $M = (L, \leq, *)$ [10, 20] denotes the many-valued logical basis of these studies. A vague binary operation $\tilde{\circ}$ on X can be roughly described as a special L-fuzzy relation (more precisely, a special strong fuzzy function) from $X \times X$ to X with some reasonable properties formulated in terms of E [9, 11, 12]. Strong (perfect) fuzzy functions propose a new approach to the fuzzy setting of numerous different branches of mathematics. Vague algebra and vague lattices are only two important cases of such an approach. The development of a sound theory of real line equipped with M-vague orderings, M-vague addition operations and M-vague multiplication operations [9, 12], which will be called vague real line, lies at the heart of future studies in the theory of many-valued equivalence relation-based fuzzy functions. It is well-known that basic axioms of the real line in the classical sense have been derived starting from an abstract ordered field in the classical sense. For this reason, in an analogue manner to the real line in the classical case, it is natural to start from a vague ordered field for the establishment of an axiomatic theory of vague real line. Vague ordered fields and the transition from vague ordered fields to the vague real line will be the main subjects of this presentation. The outline of this talk can be expressed as follows. After a brief introduction of strong (perfect) fuzzy functions and vague algebraic notions, we will define many-valued equivalence relation-based strict fuzzy orderings, which will be an essential tool of the vague ordered fields, and establish the connection between these kinds of strict fuzzy orderings and M-vague orderings. Then we will introduce vague ordered fields, and touch on the problem of the derivation of the basic axioms of vague real line starting from vague ordered fields. All necessary fundamental axioms of vague real line, which have not yet been revealed in their entirity, are crucial problems in developing a sound theory for vague real line. The aim of this talk can be summarized as the introduction of vague real line and the invitation of the researchers to this new and bachelor field.

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MV-algebras and semirings

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Semirings are algebraic structures with two associative binary operations, where one distributes over the other, introduced by Vandiver [10] in 1934. In more recent times semirings have been deeply studied, especially in relation with applications ([5]). For example semirings have been used to model formal languages and automata theory (see [4]), to deal with scheduling problems ([3]) and semirings over real numbers ($(\max, +)$-semirings) are the basis for the idempotent analysis [7].

In this work, we establish a relationship between semirings and many-valued logics.

Many-valued logic has been proposed to model phenomena in which uncertainty and vagueness are involved. One of the more general classes of many-valued logics is the Basic logic defined in [6] as the logic of continuous t-norms. Special cases of Basic logics are Łukasiewicz, Godel and Product logic. In particular Łukasiewicz logic has been deeply investigated, together with its algebraic counterpart, MV-algebras, introduced by Chang in [1] to prove completeness theorem of Łukasiewicz logic.

MV-algebras have nice algebraic properties and can be considered as intervals of lattice-ordered groups (see [2]). Łukasiewicz disjunction and conjunction are interpreted by the operations $\oplus$ and $\odot$ of the MV-algebra $[0,1]$ given by

$$x \oplus y = \min\{1, x+y\}, \quad x \odot y = \max\{0, x+y-1\}.$$

In spite of satisfying theoretical results regarding Łukasiewicz logic, all the attempts to use it as an instrument to deal with uncertainty phenomena, for example in the fuzzy context, had to deal with one of its main characteristic: conjunction and disjunction do not distribute one with respect to the other.

In this paper we stress that operations $\odot$ and $\oplus$ in any MV-algebra A both come from the same operation in the lattice ordered group associated with A. In order to model the notion of conjunction and disjunction one have instead to consider a lattice operation $\wedge$ (or dually, $\vee$) together with the MV-algebraic operation $\oplus$ (or dually $\odot$).

An example of how this representation can be useful to model fuzzy phenomena will be given in the field of automata. Indeed in [4], semirings have been proposed to give a generalization of automata, the so called K-Σ- automata. More recently, automata with values in semirings over the natural numbers or the real numbers sets have been deeply investigated both to finding results on nondeterminism or infinite behavior of finite automata, and in the context of formal power series (see [8], [9]). We shall give a description of automata having values in BL-algebras and MV-algebras.

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On different ways of ordering conjoint evaluations

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Introduction

Operations for combining [0,1]–valued fuzzy set membership functions pointwisely, such as triangular norms or co-norms, uni-norms, null-norms, etc, have been extensively studied. Such operations indeed provide two services by returning a real number as a result of the combination of the membership degrees: i) a numerical degree of conjoint membership is assessed ; ii) one can take advantage of the linear order of the real numbers for comparing the degrees.

However, in many practical problems (such as multiple criteria analysis, flexible constraints satisfaction problems), the scale [0,1] is too rich for being used, and more qualitative scales having a finite number of levels have to be preferred. But, the internal operations that can be defined on the latter scales (e.g., Godo and Sierra, 1988; Mas *et al*., 1999; Fodor, 2000) have a limited discriminating power since they take values on a finite range.

In order to escape the dilemma of using either too expressive a scale which would enable an accurate discrimination between the degrees, or a more appropriate scale leading to too many ties, we investigate another route in this preliminary note. We are no longer looking for global evaluations which then can be compared, but we are rather handling the comparison of vectors of the membership degrees directly (following ideas already outlined in (Dubois and Prade, 2001)) by introducing refinements of Pareto ordering.

Let $\mathbf{L} = \{\alpha_0 = \mathbf{0} < \alpha_1 < \ldots < \alpha_L = \mathbf{1}\}$ be a finite scale. Vectors of a given size N $(\alpha^1, \ldots, \alpha^k, \ldots, \alpha^N)$ made of values in $\mathbf{L}$, can be partially ordered by Pareto ordering, denoted by $<_P$. Let us, for instance, consider the case N = 2. We have $(\alpha_0, \alpha_0) <_P (\alpha_0, \alpha_1) <_P (\alpha_1, \alpha_1) <_P \ldots <_P (\alpha_{L-1}, \alpha_L) <_P (\alpha_L, \alpha_L)$. For notational simplicity we shall write (i, j) < (i', j'), in place of $(\alpha_i, \alpha_j) <_P (\alpha_{i'}, \alpha_{j'})$. We assume *symmetry*, thus pairs (i, j) and (j, i) are equivalent, and by convention when we write (i, j) it is assumed that $i \le j$. More generally, we have $(i, j) < (k, l)$ as soon as $i \le k$ and $j < l$, or $i < k$ and $j \le l$. The only undetermined cases are such that $i < k$ and $j > l$.

Motivating example

Once Pareto ordering is applied, what remains to specify is the ordering between pairs (i, j) and (k, l) such that $i > k$ and $j < l$ (Moura-Pires and Prade, 2000). The situation for the case N = 2, with $\mathbf{L} = \{\alpha_0 = \mathbf{0} < \alpha_1 < \alpha_2 < \alpha_3 = \mathbf{1}\}$, is pictured in Fig. 1.

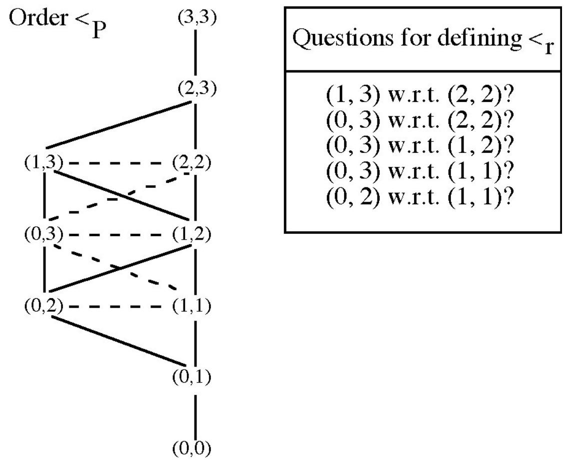

Fig. 1. Refinements of Pareto ordering

The pending decisions are indicated in dotted lines. Such a refinement of $<_P$ will be denoted by $<_r$ (refined ordering). For instance, as pictured in Fig. 1, (1, 3) $<_P$ (2, 3) and (2, 2) $<_P$ (2, 3), while (2, 2) is $<_P$-incomparable to (1, 3) and to (0, 3). Moreover, when specifying a refinement $<_r$, one should obey the transitivity requirement. For instance, it is impossible to enforce (1, 3) $<_r$ (2, 2) and (2, 2) $<_r$ (0, 3) in the same time. One may also choose to complete the Pareto ordering by enforcing equalities, e.g. (1, 1) $=_r$ (0, 2).

It can be checked that there are 12 different "linearizations" of $<_P$ without ties for N = 2 and L = 3. Here are four examples (the added decisions are indicated in bold):

(0, 0) $<_r$ (0, 1) $<_r$ (0, 2) $<_r$ (0, 3) $\mathbf{<_r}$ (1, 1) $<_r$ (1, 2) $<_r$ (1, 3) $\mathbf{<_r}$ (2, 2) $<_r$ (2, 3) $<_r$ (3, 3)

(0, 0) $<_r$ (0, 1) $<_r$ (1, 1) $\mathbf{<_r}$ (0, 2) $<_r$ (1, 2) $<_r$ (2, 2) $\mathbf{<_r}$ (0, 3) $<_r$ (1, 3) $<_r$ (2, 3) $<_r$ (3, 3)

(0, 0) $<_r$ (0, 1) $<_r$ (0, 2) $\mathbf{<_r}$ (1, 1) $\mathbf{<_r}$ (0, 3) $\mathbf{<_r}$ (1, 2) $<_r$ (1, 3) $\mathbf{<_r}$ (2, 2) $<_r$ (2, 3) $<_r$ (3, 3)

(0, 0) $<_r$ (0, 1) $<_r$ (1, 1) $\mathbf{<_r}$ (0, 2) $<_r$ (1, 2) $\mathbf{<_r}$ (0, 3) $<_r$ (1, 3) $\mathbf{<_r}$ (2, 2) $<_r$ (2, 3) $<_r$ (3, 3).

The two first orderings are just the lexi-min and the lexi-max ordering respectively. In case **L** is an interval scale, the third ordering above would correspond to the one that would be given by an arithmetic mean refined by minimum, while the last ordering seems to be less simple to interpret. Note that the number of levels that are thus obtained amounts to 10 elements (from (0, 0) to (3, 3)), while **L** has 4 levels only.

Conversely, one may also consider the possibility of a coarsening ($<_c$) of the Pareto ordering, if some pairs, ordered by $<_P$, are found as equally good, e. g., $\forall$i,j (0, i) $=_c$ (0, j), or $\forall$i,j (i, k) $=_c$ (j, k) for some k. We can thus express a form of absorption-like property for some levels. Thus, other combination schemes can be recovered, by both completing and coarsening the Pareto ordering, including the minimum:

(0, 0) $=_c$ (0, 1) $=_c$ (0, 2) $=_c$ (0, 3) $\mathbf{<_r}$ (1, 1) $=_c$ (1, 2) $=_c$ (1, 3) $\mathbf{<_r}$ (2, 2) $=_c$ (2, 3) $<_P$ (3, 3).

General framework

The above example has shown that it is possible to specify a variety of ranking modes that are sufficiently discriminating, but still remaining in a qualitative setting. Generally speaking, the problem is how to efficiently describe any Pareto-compatible ranking using a small number of conditions on the relative positioning of a few tuples and to study the characteristic properties of such rankings.

Lexi-f and other refinements

A natural idea is to start with an operation f from $\mathbf{L}\times\ldots\times\mathbf{L}$ to $\mathbf{L}$, defined by an aggregation structure $(f^1, f^2, f^3, \ldots, f^k, \ldots)$ where f^1 is the identity and f^k is defined on $\mathbf{L}^k$ has k arguments, and to use a refinement principle. One may for instance apply the lexi-f, a generalization of the lexi-min or the lexi-max, defined for any globally increasing f (f strictlty increases when all its arguments strictly increase), defined in the following way (Dubois, Prade, 2001). Let us consider two N-vectors of evaluations $I = (i_1, \ldots, i_N)$ and $J = (j_1, \ldots, j_N)$. Then $I >_{\text{lexi-f}} J \Leftrightarrow f(M(I) - M(J)) > f(M(J) - M(I))$ where M(I) is the multi-set of evaluations α_{i_k} associated with I, and thus identical evaluations are discarded before applying f. This means that in the above example with a 4-level scale, $f(\mathbf{L}\times\mathbf{L}) = \{f^2(0, 0), f^2(1, 1), f^2(2, 2), f^2(3, 3)\}$. Thus, we cannot represent in this way an ordering such that $(1, 1) < (0, 3) < (2, 2)$ for instance (as it is the case for the two last examples of linear orderings of the previous section), since $f^2(0, 3) \in f(\mathbf{L}\times\mathbf{L})$ and the lexi-f cannot provide any refinement for the considered pairs. It shows that any complete pre-order cannot be generated has a lexi-f ordering for some qualitative aggregation structure f. An open question is how to characterize the descriptive power of the lexi-f? Are there other meaningful refinement principles based on a N-ary operation closed on $\mathbf{L}$? We might think of using a transposition of Lorenz dominance defined for real-valued vectors by $\mathbf{u} <_{\text{Lorenz}} \mathbf{v} \Leftrightarrow L(\mathbf{u}) <_P L(\mathbf{v})$ where $L(\mathbf{u}) = (u_1, u_1 + u_2, \ldots, u_1 + \ldots + u_N)$, assuming $u_1 \geq u_2 \geq \ldots \geq u_N$. Taking $L(\mathbf{u}) = (u_1, f(u_1, u_2), \ldots, f(u_1, \ldots, u_N))$ enables the non-trivial refinement of Pareto ordering for suitable choices of f.

Possible requirements

Indeed the above example indicates that there exists a large set of worth investigating refinements which can be specified without using an aggregation structure f. Obviously, it raises the question of how requirements on the ordering between the vectors can be expressed in the general case, i.e., for N larger than 2, or when $\mathbf{L}$ has more than 4 levels. What would be natural in order to moderate the combinatorics for defining a complete relation $<_r$ in the general case is to introduce various requirements on the ordering. We already mentioned the symmetry condition (the comparison of two vectors should not depend on the way the components of the vectors are displayed). Other possible natural requirements that may be thought of are the following ones:

- A weak form of *preferential independence* holds, namely:

$$(\alpha_i, \alpha_j) <_r (\alpha_{i'}, \alpha_{j'}) \Rightarrow (\alpha^{\mathbf{I}}, \alpha_i, \alpha^{\mathbf{J}}, \alpha_j, \alpha^{\mathbf{K}}) <_r (\alpha^{\mathbf{I}}, \alpha_{i'}, \alpha^{\mathbf{J}}, \alpha_{j'}, \alpha^{\mathbf{K}}) \qquad \text{(PI)}$$

where $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ are subsets of exponents and $\alpha^{\mathbf{I}}$ stands for the vector of α^k's where $k \in \mathbf{I}$. Note that this requirement is still in the spirit of the lexi-f. This leads using Pareto ordering to

$$(\alpha_i, \alpha_j) <_r (\alpha_{i'}, \alpha_{j'}),\ \alpha^{\mathbf{I}} \leq_P \alpha^{\mathbf{I'}},\ \alpha^{\mathbf{J}} \leq_P \alpha^{\mathbf{J'}},\ \alpha^{\mathbf{K}} \leq_P \alpha^{\mathbf{K'}} \Rightarrow (\alpha^{\mathbf{I}}, \alpha_i, \alpha^{\mathbf{J}}, \alpha_j, \alpha^{\mathbf{K}}) <_r (\alpha^{\mathbf{I'}}, \alpha_{i'}, \alpha^{\mathbf{J'}}, \alpha_{j'}, \alpha^{\mathbf{K'}})$$

Thus, we will have $(1, 1) <_r (0, 2) \Rightarrow (1, 1, 1) <_r (0, 1, 2) <_P (0, 2, 2)$.

- However, the use of this principle can be seriously questioned as suggested by the following example. Assume $(0, 3) <_r (1, 2)$. Then $(0, 3, 4, 5) < (1, 2, 4, 5)$ applying PI. But it may be the case that $(3, 4) >_r (2, 5)$ and $(0, 5) >_r (1, 4)$, which would rather lead to state $(0, 3, 4, 5) >_r (1, 2, 4, 5)$, at least if we assume the other natural cumulative principle and we use symmetry:

$$\alpha^{\mathbf{I}} <_r \alpha^{\mathbf{I'}},\ \alpha^{\mathbf{J}} <_r \alpha^{\mathbf{J'}} \Rightarrow (\alpha^{\mathbf{I}}, \alpha^{\mathbf{J}}) <_r (\alpha^{\mathbf{I'}}, \alpha^{\mathbf{J'}}) \qquad \text{(C)}$$

But this new principle itself cannot lead to a safe extension of $<_r$, as shown by the following example. Assume $(0, 3) <_r (1, 2)$, $(3, 6) <_r (4, 5)$, $(2, 4) <_r (3, 3)$, $(1, 5) <_r (0, 6)$. (C) applied to the two first relations entail (0, 3, 3, 6) $<_r$ (1, 2, 4, 5), while the last two lead to (0, 3, 3, 6) $>_r$ (1, 2, 4, 5)!

- Other examples of generic principles are "*translation*" rules. Namely,

$(\alpha^{I}, \alpha_i, \alpha^{J}, \alpha_j, \alpha^{K}) <_r (\alpha^{I'}, \alpha_{i'}, \alpha^{J'}, \alpha_{j'}, \alpha^{K'}) \Rightarrow (\alpha^{I}, \alpha_{i+x}, \alpha^{J}, \alpha_{j+x}, \alpha^{K}) <_r (\alpha^{I'}, \alpha_{i'+x}, \alpha^{J'}, \alpha_{j'+x}, \alpha^{K'})$

for x> 0, and i+x, j+x, i'+x, j'+x less than L. A similar condition can be used changing + x into – x.

- Lastly, one may also use a "*transference*" principle of the form (i, i) < (i – 1, i + 1) for any i ≥ 1, or more generally that (i,j) < (i – 1, j + 1) (we may also think of the converse principle). This latter condition applied to the case of a 5-level scale, for instance, considerably reduces the number of remaining questions to answer in order to define a complete ordering. Namely we have, applying Pareto ordering together with the latter condition, (0, 0) < (0, 1) < (1, 1) < (0, 2) < (1, 2) < (0, 3)? (2, 2) < (1, 3) < (0, 4)? (2, 3) < (1, 4)? (3, 3) < (2, 4) < (3, 4) < (4, 4), where the question marks stand for undetermined relations. We would even have only one indetermination, (0, 3)? (2, 2), if we add the previous translation constraints (i, j) < (i', j') ⇒ (i, j + 1) < (i', j' +1) and (i + 1, j) < (i' + 1, j') with i + 1 < j and i' + 1 < j'.

Note that the transference principle is in the spirit of Pigou-Dalton transferring in social choice, which enables Pareto ordering on vectors of real numbers $\mathbf{u} = (u_1, \ldots, u_N)$ to be extended by stating $(\ldots, u_i, \ldots, u_j, \ldots) < (\ldots, u_i - \varepsilon, \ldots, u_j + \varepsilon, \ldots)$ where $0 \le \varepsilon \le u_i - u_j$. It is known that this refinement is equivalent to Lorenz dominance. See (Spanjaard, 2003) for details and references. Note that in our framework, the counterpart of this idea is written

$(\alpha_i, \alpha_j) < (\alpha_{i-1}, \alpha_{j+1})$ where $\forall k, \alpha_k \in \mathbf{L}$, since neither $\alpha_j + \varepsilon$ nor $\alpha_j + \alpha_k$ make sense.

Using conditions on the rank of the elements of the scale

As it can be seen, the refinements of Pareto ordering raise problems, and anyway do not lead to a complete ordering generally. A more efficient way for getting complete orderings is to define them through conditions and operations on the indices numbering the elements of the scale. Namely

$f(i, j) < f(i', j') \Rightarrow (\alpha_i, \alpha_j) <_r (\alpha_{i'}, \alpha_{j'})$

If f is associative, it is simple to extend the definition to N-vectors. This is compatible with Pareto ordering if f is non-decreasing, i.e.

$(\alpha_i, \alpha_j) <_P (\alpha_{i'}, \alpha_{j'}) \Rightarrow f(i, j) < f(i', j')$

In case f(i, j) = f(i', j') it might be further refined by another condition. For example,

$(0, 0) <_r (0, 1) <_r (0, 2) <_{\mathbf{r}} (1, 1) <_{\mathbf{r}} (0, 3) <_{\mathbf{r}} (1, 2) <_r (1, 3) <_{\mathbf{r}} (2, 2) <_r (2, 3) <_r (3, 3)$

is generated by f(i, j) = i +j refined by min(i, j) < min(i', j') if i + j = i' + j'. Note that this ordering violates the transference property.

However, generally speaking, it is not clear that any complete pre-ordering refining Pareto ordering can be specified in such a way using integer-valued arithmetic operations.

Conclusion

This informal discussion is not intended to bring any new substantial result. Still it is a preliminary attempt at understanding how to characterize complete pre-order structures capable of modeling different behaviors for comparing qualitative evaluation profiles.

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A definition of subjective possibility

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Quantitative possibility theory (QPT) was proposed as an approach to the representation of quantified uncertainty (Zadeh, 1978; Dubois and Prade 1988, 2000). In order to sustain this claim, operational semantics could be instrumental. In the subjectivist context, quantitative possibility theory somehow competes with probability theory in its personalistic or Bayesian views and with the Transferable Belief Model (TBM) (Smets and Kennes 1994; Smets 1998), both of which also intend to represent degrees of belief. We use the term 'subjectivist' to mean that we consider the concepts of beliefs (how much we believe) and betting behaviors (how much would we pay to enter into a game) without regard to the possible random nature and repeatability of the events. An operational definition, and the assessment methods that can be derived from it, provides a meaning to the value .7 encountered in statements like 'my degree of belief is .7'. Bayesians claim that any state of incomplete knowledge of an agent can be modeled by a single probability distribution on the appropriate referential, and that degrees of belief coincide with probabilities that can be revealed by a betting experiment in which the agent provides betting odds under an exchangeable bet assumption. A similar setting exists for imprecise probabilities (Walley, 1991), relaxing the assumption of exchangeable bets, and more recently for the TBM as well (Smets, 1997), introducing several betting frames corresponding to various partitions of the referential. In that sense, numerical values encountered in these three theories are well defined.

QPT seems to be a theory worth exploring as well, and rejecting it because of the current lack of convincing semantics would be unfortunate. The recent revival, by De Cooman and colleagues (1999), of a form of subjectivist QPT due to Giles (1982), and the development of possibilistic networks based on incomplete statistical data (Borgelt and Kruse, 2003) suggests on the contrary that it is fruitful to investigate various operational semantics for possibility theory. This is due to several reasons: first possibility theory is a special case of most existing non-additive uncertainty theories, be they numerical or not. Hence progress in one of these theories usually has impact in possibility theory. Another major reason is that possibility theory is very simple, certainly the simplest competitor for probability theory, for instance when using fuzzy numbers in fuzzy optimization problems. The aim of this paper is to propose subjectivist semantics for numerical possibility theory.

Such subjectivist semantics differs from the upper and lower probabilistic setting proposed by Giles and followers, without questioning its merit. Instead of making the bets non-exchangeable, we assume that the exchangeable betting rates only imperfectly reflect an agent's beliefs.

For long, it had been realized that possibility functions are mathematically identical to consonant plausibility functions (Shafer, 1976) so using the semantics of the TBM for producing a semantics for quantitative epistemic possibility theory is an obvious approach, even if not explored in depth so far.

Consider what beliefs held by an agent on what is the actual value of a variable ranging on a set Ω, called the frame of discernment. It is assumed that such beliefs can be represented by a belief function. A belief function can be mathematically defined from a finite random set that has a very specific interpretation. The so-called basic belief mass assigned to each set is understood as the weight given to the fact that all the agent may know is that the value of the variable of interest lies somewhere in that set. A plausibility function evaluates to what extent events are consistent with the available evidence. When the sets with positive mass are nested, the plausibility function is called a possibility measure, and can be characterized, just like probability, by an assignment of weights to singletons, called a possibility distribution.

The agent's beliefs cannot be directly assessed. All that can be known is the value of the `pignistic' probabilities the agent would use to bet on the frame Ω (Smets, 1991). The pignistic probability induced by a mass function is built by defining a uniform probability on each set of positive mass, and performing the convex mixture of these probabilities according to the mass function. In terms of game theory it corresponds to the Shapley value of a game; in terms of upper and lower probabilities it is the centre of gravity of the set of probabilities dominating the belief function. The pignistic probability is what is obtained by means of the random simulation of a fuzzy number, picking a cut at random followed by a random choice of an element in the cut, as studied by Chanas and Nowakowski (1988), among others.

The knowledge of the values of the probability p allocated to the elements of Ω is not sufficient to construct a unique underlying belief function whose pignistic transform is p. Many belief functions induce the same probability distribution. For instance, uniform betting rates on Ω either correspond to complete ignorance on the values of the variable, or to the knowledge that the variable is random and uniformly distributed. So all that is known about the mass function that represents the agent's beliefs is that it belongs to the ones that induce the supplied probability. Under this scheme, we do not question the exchangeability of bets, as done by Walley, Giles and others. What we question is the assumption of a one-to-one correspondence between betting rates produced by the agent, and the actual beliefs entertained by the agent. Betting rates do not tell if the uncertainty of the agent results from the perceived randomness of the phenomenon under study or from a simple lack of information about it.

The belief functions whose pignistic transform is *p* are called *isopignistic* belief functions and form the set *IP(p)*. Since several mass functions lead to the same betting rates, one has to select one that most plausibly reflects the actual state of belief of the agent. A cautious approach is to obey a `least commitment principle' that states that one should never presuppose more beliefs than justified. Then, one should select the 'least committed' element in the family of mass functions compatible with the pignistic probability function prescribed by the obtained betting rates. The first main result of this paper is that the least committed belief function, among the ones which share the same pignistic transform, is consonant, that is, the corresponding plausibility function is a possibility function. This possibility function is the unique one in the set of plausibility functions having this prescribed pignistic probability, because the pignistic transformation is a bijection between possibilities and probabilities. So this possibility function corresponds to the least committed mass function whose transform is equal to the probability supplied by the agent.

This result is formalized on the basis of a measure of non-commitment of a belief function, namely the average of the cardinalities of its focal elements weighted by the mass function. Let m be a mass function from 2^{Ω} to [0, 1], and let $I(m) = \sum_{A \subseteq \Omega} m(A)card(A)$ be its imprecision measure estimating the extent to which it is non-committal. Let p be the probability distribution obtained by eliciting an agent's betting rates on the frame Ω. It is assumed that the actual belief of the agent is modeled by a mass function on Ω such that p = Pig(m) is the pignistic transform of m, that is :

$$p(w) = \sum_{A:\, w \in A} m(A)/\, card(A) \qquad (1)$$

This is an extension of Laplace indifference principle, according to which equally possible outcomes have equal probability. It is a weighted form thereof. It is suggested that the least debatable representation of an agent's belief is the mass function m* which maximizes I(m) under the constraint (1) induced by betting rates.

Theorem 1: The mass function m* which maximizes I(m) under the constraint Pig(m) = p is consonant. It defines a unique possibility distribution π defined by

$$\pi(w) = \sum_{u \in \Omega} \min(p(w), p(u)),\ \ w \in \Omega. \qquad (2)$$

It is the converse of the pignistic transform of a possibility distribution, the converse of the transformation used by Chanas and Nowakovski. This probability/possibility transform was already proposed without formal justification by Dubois and Prade (1983).

This result was already announced by the authors in (Dubois et al. 2001), but its proof is still unpublished. It contrasts with a similar result by Smets (2000) that uses a notion of information index based on the commonality function.

Moreover, Smets (2000) suggested that the least specific isopignistic belief function according to the commonality ordering (based on $Q(A) = \sum_{A \subseteq E} m(E)$) is also $Pig^{-1}(Pig(m))$. This ordering is less intuitive than the specialization ordering and the inclusion of *Bel-Pl* intervals. However, there is indeed a unique minimally *Q*-informative belief function in *IP(p)*, and it is precisely the one found by maximizing *I(m)*. But the commonality ordering turns to be more natural than one could think at first glance, since, in order to show the above result expressed by Theorem 2 below, we first prove that, for ensuring comparability in the sense of the Q-informativeness ordering between a consonant belief function and a belief function, it is enough to rely on singletons:

Lemma *:* Consider a belief function with mass function m and a possibility distribution π with respective commonality functions Q and Q_{π}. Then $Q_{\pi}(A) \geq Q(A),\ \forall A \subseteq \Omega$ if and only if $\pi(\omega) \geq Pl(\{\omega\}),\ \forall \omega \in \Omega$.

Theorem 2: The unique consonant mass function in $IP(p)$ (induced by the possibility distribution defined by (2)), is minimally Q-informative.

These results provide a first reply to objections raised by Bayesian subjectivists against the use of fuzzy numbers and numerical possibility theory in decision-making and uncertainty modeling tasks. Interestingly, this approach does not refute the Bayesian operational setting; it only questions the interpretation of betting rates as full-fledged degrees of belief.

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Partially ordered monads and powerset Kleene algebras

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Composing various powerset functors with the term monad gives rise to the concept of generalised terms. The goal is to extend traditional term unification with unification involving powersets of terms. This enables a study of substitutions and unifiers within Kleisli categories related to particular monads.

As constructions of monads involve complicated calculations with natural transformations, proofs are supported by a graphical approach that provides a useful tool for handling various conditions, such as those for distributive laws.

Monads equipped with order structures extends suitably to so called partially ordered monads. We will show how these partially ordered monads, together with their subconstructions, contribute to providing a generalised notion of powerset Kleene algebras. This generalisation builds upon more general powerset functor setting far beyond just strings (Kleene, 1956) and relations (Tarski, 1941)

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Structured lattices and ground categories of L-sets

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It is quite well known since [4] in the contest of fuzzy mathematics that in many disciplines and especially in fuzzy topology it is very useful to set up the classes of objects and of morphisms to deal with (e.g. the working category, dubbed "ground category") as well as to associate to each morphism between two objects suitable operators, in both directions,(namely powerset operators) between the lattices of "canonical subobjects"(namely powersets) of the considered objects.

Among papers mainly devoted to this topic we quote [2, 3, 5, 6] : the ground categories constructed in [5, 6], either in the fixed-basis or in the variable-basis context, contain only objects associated to (crisp) sets; the objects of the ground categories considered in [2, 3] are arbitrary L-sets (L a suitable, fixed complete lattice).

Though not explicitly listed among the elements of the ground categories, powersets associated to objects and powerset operators associated to morphisms (i.e. powerset functors, as they are defined in [2]) are fundamental in most applications of this sort of set theory based on ground categories; for instance, in fuzzy topology, which in any case lies between classical topology and pointless topology, topologies are (M)-subsets of some ground object and (special) morphisms are maps satisfying properties expressed in terms of the powerset operators. In [5, 6] one can find a detailed and motivated justification for extending powersets and powerset operators from the traditional case of classical set theory to a more general context, including, as a first step, the Zadeh powerset operators. These operators are also the fundamental tool for the construction of powerset operators in [2, 3] and so they will be in this new approach.
Here an original idea of [1] is extended and developed so as to allow the construction of powerset operators to be applied in more general situation, including those considered in [2, 3] and a special case of variable-basis fuzzy set theory extended to arbitrary L-sets.

The fundamental aspect of the construction presented here is a sort of localization of the process leading to the definition of forward and backward powerset operators both of which can be obtained in the same way, by using the corresponding Zadeh operators.

This process could be further extended by considering fuzzy sets as lattice-bundles so as to extend and include the general case of Rodabaugh's variable-basis fuzzy set theories.

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Fuzzy relation equations and fuzzy control — some old and some new ideas

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In fuzzy control, it is a well known approach to transfer, with reference to the compositional rule of inference, a list of linguistic control rules of the form

$$\text{IF } \alpha \text{ is } A_i, \text{ THEN } \beta \text{ is } B_i, \quad i = 1, \ldots, n$$

into a system of fuzzy relation equations

$$A_i \circ R = B_i, \qquad 1 \leq i \leq n,$$

for a fuzzy relation R which has to be determined as a solution of this system of relation equations.

The presentation shall have its focus on methodological considerations, will remind some approaches toward solvability considerations for such systems as well as toward approximate solutions like [4, 3], and extend them slightly with reference to some recent results explained e.g. in the papers [1, 2, 5].

But we will also give an embedding of this methodology to treat fuzzy control problems into a wider perspective of handling an interpolation problem in an approximative way.

And we shall go on to look at some open problems from a rather general point of view.

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Capacities on lattices

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1 Introduction

Capacities [3] have been introduced by Choquet, and rediscovered by Sugeno [13] under the name of *fuzzy measures*. On a mathematical point of view, these are monotonic set functions $\mu : \mathcal{P}(N) \longrightarrow [0,1]$ over some set N (assumed to be finite in this paper), or otherwise said, isotone mappings from the Boolean lattice $(2^N, \subseteq)$ to the linear lattice $([0,1], \leq)$, preserving top and bottom. Usual tools used in capacity theory are the Möbius transform [11], the Choquet integral, and interaction index [5].

Recently, Grabisch and Labreuche have proposed the concept of *bi-capacities* [7, 6], which generalizes capacities for bipolar scales in a context of decision making. Mathematically speaking, these are functions $v : \mathcal{Q}(N) \longrightarrow [-1,1]$, where $\mathcal{Q}(N) := \{(A,B) \in 2^N \times 2^N \mid A \cap B = \emptyset\}$, being increasing in first coordinate and decreasing in second one. More abstractly, a bi-capacity is an isotone mapping from the lattice $(3^N, \sqsubseteq)$ to the linear lattice $([-1,1], \leq)$ preserving top and bottom, where $(A,B) \sqsubseteq (C,D)$ iff $A \subseteq C$ and $B \supseteq D$. Usual tools of capacity theory mentionned above have all been generalized to bi-capacities.

Taking this as a starting point, one may define capacities as isotone mappings from some lattice L to $([-1,1], \leq)$, preserving top and bottom. This can be interpreted in decision making and even larger domains such as knowledge discovery [10]. The aim of the paper is to show how to generalize usual tools of capacity theory to this general setting, using the less possible restrictions on the lattice L. For the Choquet integral, we refer the reader to [9].

We will make a particular mention of belief functions (see a pioneering work by Barthélemy defining belief functions on lattices [1]), and refer the reader to [8] for the case of possibility measures.

2 Capacities on lattices

(for a reference on lattices, see [2]) Let $(L, \leq)$ be a finite lower locally distributive lattice, we denote as usual $\vee, \wedge, \top, \perp$ supremum, infimum, top and bottom. Any such lattice can be represented uniquely by its $\vee$-irreducible elements in an irredundant decomposition [4]. An element $i \in L$ is a *$\vee$-irreducible element* if $i \neq \perp$ and it has only one predecessor. Let us call $\mathcal{J}(L)$ the set of all $\vee$-irreducible elements of L. For any $x \in L$, we denote by $\eta^*(x)$ its unique irredundant decomposition in join-irreducible elements.

For $x, y \in L$, we say that x *covers* y (or y is a *predecessor* of x), denoted $x \succ y$, if there is no $z \in L, z \neq x, y$ such that $x \leq z \leq y$.

Let $v : L \longrightarrow \mathbb{R}$ be a real-valued function on L. v is a *capacity* if v is *isotone*. Bottom and top have to be preserved if one replaces $\mathbb{R}$ by any closed interval.

3 Möbius transform

The first fundamental concept in capacity theory is the Möbius transform. Following the general definition of Rota [11] (see also [2, p. 102]), we have already a definition for the general case. For any function f on $(L, \leq)$, the *Möbius transform* of f is the function $m : L \longrightarrow \mathbb{R}$ solution of the equation:

$$f(x) = \sum_{y \leq x} m(y).$$

The expression of m is obtained through the Möbius function μ by:

$$m(x) = \sum_{y \leq x} \mu(y,x) f(y)$$

where μ is defined inductively by

$$\mu(x,y) = \begin{cases} 1, & \text{if } x = y \\ -\sum_{x \leq t < y} \mu(x,t), & \text{if } x < y \\ 0, & \text{otherwise.} \end{cases}$$

4 Derivative of functions on lattices

Let $(L, \leq)$ be a finite lower locally distributive lattice, and $f : L \longrightarrow \mathbb{R}$ a real-valued function on it.

Definition 1. Let $i \in \mathcal{J}(L)$. The *derivative* of f w.r.t. i at point $x \in L$ is given by:

$$\Delta_i f(x) := f(x \vee i) - f(x).$$

Note that $\Delta_i f(x) = 0$ if $i \leq x$. We say that the derivative $\Delta_i f(x)$ is *Boolean* if $[x, x \vee i]$ is the Boolean lattice 2^1, otherwise said $x \vee i \succ x$.

Using the irredundant decomposition, the derivative w.r.t any element y can be defined.

Definition 2. Let $x, y \in L$, and $y = \vee_{k=1}^{n} i_k$ be the irredundant decomposition of y into join-irreducible elements. Then the derivative of f w.r.t y at point x is given by:

$$\Delta_y f(x) = \Delta_{i_1}(\Delta_{i_2}(\cdots \Delta_{i_n} f(x) \cdots)).$$

The derivative is *Boolean* if $[x, x \vee y]$ is the Boolean lattice 2^n. The derivative is 0 if for some k, $i_k \leq x$.

We express the derivative in terms of the Möbius transform of f.

Theorem 3. *Let $x, y \in L$, such that $\Delta_y f(x)$ is Boolean. Then*

$$\Delta_y f(x) = \sum_{z \in [y, x \vee y]} m(z).$$

5 Shapley value and interaction index

We need some additional structure on L at this point. We consider finite lower locally distributive lattices $L_1, \dots, L_n$, with top and bottom of L_i denoted $\top_i, \bot_i$, $i = 1, \dots, n$, and L is the product lattice $L := L_1 \times \cdots \times L_n$ with the product order. A *vertex* of L is an element $x = (x_1, \dots, x_n)$ of L where x_i is either $\top_i$ or $\bot_i$, for $i = 1, \dots, n$. We denote $\Gamma(L)$ the set of vertices of L. Note that if L is a Boolean lattice, then $L = \Gamma(L)$.

We begin by defining the importance index as the interaction index w.r.t. a single join-irreducible element.

Definition 4. Let $i = (\bot_1, \dots, \bot_{j-1}, i_0, \bot_{j+1}, \dots, \bot_n)$ be a join-irreducible element of L. The *interaction w.r.t. i* of v is any function of the form

$$I(i) := \sum_{x \in \Gamma(\prod_{k=1}^{j-1} L_k) \times \{\underline{i_0}\} \times \Gamma(\prod_{k=j+1}^{n} L_k)} \alpha^1_{h(x)} \Delta_i v(x), \tag{1}$$

where $\underline{i_0}$ is the (unique) predecessor of i_0 in L_j, $h(x)$ is the number of components of x equal to $\top_l$, $l = 1, \dots, n$, and $\alpha^1_k \in \mathbb{R}$ for any integer k.

Observe that the constants $\alpha^1_{h(x)}$ do not depend on i. Also, the derivative is Boolean.

Let us generalize Definition 4 to a class of elements of L denoted $\tilde{L}$ and defined as follows: $\tilde{L} := \bigcup_{J \subseteq N} \tilde{L}_J$, with

$$\tilde{L}_J := \{x \in L \mid \forall k \in J, \exists \underline{i_k} \in L_k \text{ such that } \forall i \in \eta^*(x_k), i \succ \underline{i_k}, \\ \text{and } x_k = \bot_k \text{ if } k \in N \setminus J\}$$

In words, it is the set of elements whose coordinates are either bottom or such that the irredundant decomposition covers a unique element. Observe that for the case where L_k is a linear lattice or a Boolean one (i.e. practical cases fo interest), $\tilde{L} = L$.

Definition 5. Let $K \subseteq N$, and $x \in \tilde{L}_K$, and denote as above for all $k \in K$, $\underline{i_k}$ the element covered by all $i \in \eta^*(x_k)$. The *interaction w.r.t. x* of v is any function of the form

$$I(x) := \sum_{y \mid y_k = \top_k \text{ or } \bot_k \text{ if } k \notin K, y_k = \underline{i_k} \text{ else}} \alpha^{|J|}_{h(y)} \Delta_x v(y) \tag{2}$$

where J is the set of join-irreducible elements in the decomposition of x.

The derivative is Boolean if in addition the L_k's are modular (and hence distributive).

We have the following general result.

Theorem 6. *Let $K \subseteq N$, and assume distributivity holds for every L_k, $k \in K$. The expression of the interaction index for $x \in \tilde{L}_K$ in terms of the Möbius transform is given by:*

$$I(x) = \sum_{z \in [x, \check{x}]} \beta^{|J|,|K|}_{k(z)} m(z),$$

with $\check{x}_k := (\top_k)$ for $k \notin K$, and $\check{x}_k = x_k$ else, J is the set of join-irreducible elements in the decomposition of x, and $k(z)$ is the number of coordinates of z not equal to $\bot_l$, $l = 1, \dots, n$. Moreover, the real constants $\beta^{|J|,|K|}_{k(z)}$ are related to the $\alpha^{|J|}_{h(x)}$'s by:

$$\beta^{|J|,|K|}_{k(z)} = \sum_{l=0}^{n-k(z)} \binom{n-k(z)}{l} \alpha^{|J|}_{(k(z)-|K|+l)} \tag{3}$$

6 Belief functions on lattices

Let L be a lattice. Following the classical definition, we say that a capacity $v : L \longrightarrow [0,1]$ on L is a *belief function* iff its Möbius transform is non negative, and v preserves top and bottom. Barthélemy has shown in [1] that this is equivalent to say that v is k-monotone for all $k > 2$, the definition of k-monotonicity being adapted in the obvious way for our general setting.

In fact, most of properties of belief functions are still true when defined on a lattice. We show in the sequel the decomposition of belief functions into simple support functions, which generalizes the classical result of Shafer [12].

For any belief function b on L, we define the corresponding *commonality function* q by $q(x) := \sum_{y \geq x} m(y)$, where m is the Möbius transform of b.

Let b_1, b_2 be two belief functions on L, m_1, m_2 their Möbius transform, and q_1, q_2 their commonality functions. The *Dempster rule of combination* of b_1, b_2, denoted $b_1 \oplus b_2$ is defined in terms of its Möbius transform by

$$m_1 \oplus m_2(x) = \sum_{y_1 \wedge y_2 = x} m_1(y_1) m_2(y_2)$$

It is easy to show that the commonality function $q_1 \oplus q_2$ associated to $b_1 \oplus b_2$ is

$$q_1 \oplus q_2(x) = q_1(x) q_2(x).$$

Definition 7. We call *simple support function focussed on y*, denoted y^{ω}, the function of which the Möbius transform satisfies

$$m(x) = \begin{cases} 1 - \omega, & \text{if } x = y \\ \omega, & \text{if } x = \top \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition of some belief function b in terms of simple support functions is thus to write b under the form:

$$b(x) = \bigoplus_{y \in L} y^{\omega_y}(x).$$

It can be shown that the coefficients ω_y of this decomposition write

$$\omega_y = \prod_{x \geq y} q(x)^{-\mu(x,y)}$$

where $\mu(x,y)$ is the Möbius function. Note that as in the classical case, these coefficients may be strictly greater than 1, hence corresponding simple support functions have negative Möbius transform.

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Order-reversing involutions and residuated lattices

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Abstract

Given a complete lattice $(L, \leq)$ with an order–reversing involution, we find conditions to exist a residuated binary operation $*$ such that the order–reversing involution is determined by the residuation associated to the binary operation $*$. Particularly, if $(L, \leq,)$ is a completely distributive lattice with an order–reversing involution $'$ we prove that there exists an operation $*$ such that $(L, \leq, *)$ is an integral, commutative Frobenius lattice in which $\alpha \longrightarrow \perp = \alpha'$ for each $\alpha \in L$ if and only if $\alpha \leq \beta'$ whenever $\alpha \wedge \beta$.

Keywords: Order–reversing involution, *po*–semigroup, residuation, Heyting algebra.
AMS Classification: 18A40, 54A40.

1 Preliminaries

Let $(L, \leq)$ be a complete lattice with universal bounds $\perp$ and $\top$. In particular $\bigvee \varnothing = \perp$ and $\bigwedge \varnothing = \top$. A unary operation $'$ is an *order–reversing involution* (or a *quasi-complementation*) if it is an involution (i.e. $\alpha'' = \alpha$ for all $\alpha \in L$) that inverts the ordering (i.e. $\alpha \leq \beta$ implies $\beta' \leq \alpha'$).

A *po–groupoid* (short for partially ordered groupoid) is a poset $(L, \leq)$ with a binary operation $*$ on L which satisfies the *isotonicity* condition: $\alpha \leq \beta$ implies $\alpha * \gamma \leq \beta * \gamma$ and $\gamma * \alpha \leq \gamma * \beta$ for all $\alpha, \beta, \gamma \in L$. When $*$ is commutative or associative, $(L, \leq, *)$ is called a *commutative po*–groupoid or *po–semigroup*, respectively.

In a *po*–groupoid $(L, \leq, *)$ an element α is called *ideal* element if $\alpha * \beta \leq \alpha \wedge \beta$ for all $\alpha, \beta \in L$. An *po*–groupoid $(L, \leq, *)$ is called *integral* if and only if the universal upper bound $\top$ acts as unit element w.r.t. $*$. In an integral *po*–groupoid $(L, \leq, *)$ all elements are ideal.

Let $(L, \leq, *)$ be a *po*–groupoid and $\alpha, \beta \in L$. The *right–residual* $\alpha \overset{*}{\longrightarrow}_r \beta$ of β by α is the largest $\gamma \in L$ (if it exists) such that $\alpha * \gamma \leq \beta$; the *left–residual* $\alpha \overset{*}{\longrightarrow}_l \beta$ of β by α is the largest $\gamma \in L$ (if it exists) such that $\gamma * \alpha \leq \beta$. A *residuated lattice* is an *m*–lattice $(L, \leq, *)$ in which $\alpha \overset{*}{\longrightarrow}_r \beta$ and $\alpha \overset{*}{\longrightarrow}_l \beta$ always exists for any $\alpha, \beta \in L$. Obviously, in case $(L, \leq, *)$ is commutative, both $\alpha \overset{*}{\longrightarrow}_r \beta$ and $\alpha \overset{*}{\longrightarrow}_l \beta$ coincide. We shall denote them by $\alpha \overset{*}{\longrightarrow} \beta$ and call it the *implication* associated to $*$. The existence of residuals implies that the operation $*$ preserves all existing supremss in each argument.

A *po*–groupoid $(L, \leq, *)$ in which $\left(\alpha \overset{*}{\longrightarrow}_l \perp\right) \overset{*}{\longrightarrow}_r \perp = \alpha$ for every right-ideal element α and $\left(\beta \overset{*}{\longrightarrow}_r \perp\right) \overset{*}{\longrightarrow}_l \perp = \beta$ for every left-ideal element β is a *Frobenius po–groupoid* (cf. [1, page

341]). In particular, if $(L,\leq,*)$ is an integral commutative residuated lattice, then it is Frobenius if and only if $\left(\alpha \xrightarrow{*} \bot\right) \xrightarrow{*} \bot = \alpha$ for every $\alpha \in L$.

A lattice $(L,\leq)$ is said to be a *Heyting algebra* if $(L,\leq,\wedge)$ is a residuated lattice. Obviously, $(L,\leq,\wedge)$ is an integral commutative residuated lattice.

An element p in a lattice L is called *prime* if and only if the relation $p \geq \alpha \wedge \beta$ always implies $p \geq \alpha$ or $p \geq \beta$. The set of all prime elements is denoted $\mathrm{PRIME}\, L$. Dually, an element q in a lattice L is called *coprime* if and only if the relation $q \leq \alpha \vee \beta$ always implies $q \leq \alpha$ or $q \leq \beta$. The set of all coprime elements is denoted $\mathrm{COPRIME}\, L$.

2 Order-reversing involutions and residuated lattices

We shall try to answer the following question:

> *Given a lattice with an order–reversing involution $(L,\leq,')$, does there exist a binary operation $*$ such that the order–reversing involution $'$ is determined by the implication $\xrightarrow{*}$ associated to $*$, i.e. $\alpha \xrightarrow{*} \bot = \alpha'$ for each $\alpha \in L$?*

In view of the structures considered in the preliminaries, we can reformulate the previous question in a more precise way:

> *Given a lattice with an order–reversing involution $(L,\leq,')$, does there exist an integral commutative Frobenius lattice $(L,\leq,*)$ such that the order–reversing involution $'$ is determined by the implication $\xrightarrow{*}$ associated to $*$, i.e. $\alpha \xrightarrow{*} \bot = \alpha'$ for each $\alpha \in L$?*

This question has been studied by Esteva and Godo in [2] in the case of bounded chains.

The answer to the previous question is obviously NOT in general. In fact, we have the following example:

Example *Let $L = \{\bot,\alpha,\beta,\top\}$ where $\alpha \wedge \beta = \bot$, $\alpha \vee \beta = \top$, $\alpha' = \alpha$ and $\beta' = \beta$. Let assume that there exists an integral, commutative m–lattice $(L,\leq,*)$ such that the order–reversing involution $'$ is determined by the implication $\xrightarrow{*}$. Then $\alpha * \beta \leq \alpha \wedge \beta = \bot$ and so $\beta \leq \alpha \xrightarrow{*} \bot = \alpha' = \alpha$.*

Consequently we see that in order to have a positive answer the lattice must satisfy some additional condition. Particularly, the existence of such a binary operation requires that the order–reversing involution satisfies the following condition:

$$\forall \alpha,\beta \in L \qquad \alpha \wedge \beta = \bot \quad \Longrightarrow \quad \alpha \leq \beta' \tag{$\bigstar$}$$

Now we can reformulate the previous question:

> *Given a lattice with an order–reversing involution $(L,\leq,')$ satisfying condition $\bigstar$, does there exist an integral, commutative, Frobenius lattice $(L,\leq,*)$ such that the order–reversing involution $'$ is given by the implication $\xrightarrow{*}$, that is, $\alpha' = \alpha \xrightarrow{*} \bot$ for all $\alpha \in L$?*

In order to have an answer to this question we shall use the residuation associated to the infimum and consequently from now on we shall assume that $(L,\leq)$ is a Heyting algebra. We have then the following lemmata:

Lemma 1. *Let $(L,\leq,')$ be a Heyting algebra with an order–reversing involution $'$. Then condition $\bigstar$ is equivalent to the following condition:*

$$\forall\alpha\in L \qquad \alpha \xrightarrow{\wedge} \bot \leq \alpha' \tag{$\bigstar\bigstar$}$$

Lemma 2. *Let $(L,\leq,')$ be a Heyting algebra with an order–reversing involution $'$. Then for each $\alpha,\beta\in L$ such that β is coprime we have*

$$\alpha \xrightarrow{\wedge} \beta' = \begin{cases} \top, & \text{if } \alpha \leq \beta'; \\ \beta', & \text{if } \alpha \not\leq \beta'. \end{cases}$$

Consequently

$$\alpha \wedge (\alpha \xrightarrow{\wedge} \beta')' = \begin{cases} \bot, & \text{if } \alpha \leq \beta'; \\ \alpha\wedge\beta, & \text{if } \alpha \not\leq \beta'. \end{cases} \leq \gamma \quad \Longleftrightarrow \quad \beta \leq \alpha' \vee (\alpha \xrightarrow{\wedge} \gamma)$$

Corollary 3. *Let $(L,\leq,')$ be a Heyting algebra with an order–reversing involution $'$. Then for each $\alpha,\beta\in L$ such that α and β are coprime we have*

$$(\alpha \xrightarrow{\wedge} \beta')' \wedge (\beta \xrightarrow{\wedge} \alpha')' \;=\; \beta \wedge (\beta \xrightarrow{\wedge} \alpha')' \;=\; (\alpha \xrightarrow{\wedge} \beta')' \wedge \alpha.$$

Theorem 4. *Let $(L,\leq,')$ be a complete lattice with an order–reversing involution $'$ such that:*
(i) *$(L,\leq)$ is a Heyting algebra and*
(ii) *any element of L is the supremum of all coprime elements below it.*
Then the following binary operation $*$ defined for each $\alpha,\beta\in L$ by

$$\alpha*\beta \;=\; \vee\{q_1\wedge q_2 : q_1,q_2 \text{ coprime}, q_1\leq\alpha, q_2\leq\beta \text{ and } q_1\not\leq q_2'\}$$

determines a commutative, residuated lattice structure $(L,\leq,*)$. The corresponding residuation $\xrightarrow{*}$ is defined for each $\beta,\gamma\in L$ by

$$\beta \xrightarrow{*} \gamma \;=\; \wedge\{q'\vee p : q \text{ coprime}, p \text{ prime}, q\leq\beta, \gamma\leq p\}.$$

Moreover, if $(L,\leq,')$ satisfies condition $\bigstar$, then $(L,\leq,*)$ is an integral, commutative Frobenius lattice satisfying $\alpha' = \alpha \xrightarrow{*} \bot$ for all $\alpha\in L$.

As a consequence on the previous theorem we have the following corollaries which are the announced answers to the stated question:

Corollary 5. *Let $(L,\leq,')$ be a Heyting algebra with an order–reversing involution $'$ such that any element is the supremum of all coprime elements below it. Then there exists an operation $*$ such that $(L,\leq,*)$ is an integral, commutative Frobenius lattice in which $\alpha \xrightarrow{*} \bot = \alpha'$ for each $\alpha\in L$ if and only if condition $\bigstar$ is satisfied.*

If the lattice is continuous, then condition (i) in the theorem is equivalent to distributivity of the lattice (see [4]). Moreover, a lattice is completely distributive if and only if it is continuous and satisfies condition (ii) in the theorem. Consequently, we have the following:

Corollary 6. *Let* $(L, \leq, ')$ *be a completely distributive lattice with an order–reversing involution* $'$*. Then there exists an operation* $*$ *such that* $(L, \leq, *)$ *is an integral, commutative Frobenius lattice in which* $\alpha \xrightarrow{*} \perp = \alpha'$ *for each* $\alpha \in L$ *if and only if condition* ★ *is satisfied.*

Particularly in the case of a bounded chain, condition ★ is always satisfied and we have the following:

Corollary 7. (*Proposition A.4 in* [2]) *Let* $(L, \leq, ')$ *be a bounded chain with an order–reversing involution* $'$*. Then there exists an operation* $*$ *such that* $(L, \leq, *)$ *is an integral, commutative Frobenius lattice in which* $\alpha \xrightarrow{*} \perp = \alpha'$ *for each* $\alpha \in L$.

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Fuzzy predicate logic – a survey

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Mathematical fuzzy logic (or fuzzy logic in the narrow sense) is understood as a kind of many-valued logic with comparative notion of truth and with the real unit interval $[0,1]$ as the standard set of truth values. We further postulate truth-functionality (existence of truth-functions of connectives) and base the theory of truth functions on the notion of a t-norm as a truth function of conjunction. The basic fuzzy logic BL works with continuous t-norms as truth functions of conjunction and their residua as corresponding truth functions of implication; more generally, the monoidal t-norm based logic MTL works with left-continuous t-norms and their residua. Other generalizations will be mentioned.

Part 1 of the talk will be devoted to a very quick survey of propositional logics BL, MTL and stronger logics related to particular t-norms (Łukasiewicz, Gödel, product logic and some others). (For a detailed survey see [8].)

Part 2 will describe in some details the predicate logics BL∀ and MTL∀ and other predicate logics built over the propositional logics of Part 1 ([9, 6, 7, 10]). Here again we shall distinguish standard semantics (of [0,1]-fuzzy relational structures) and general semantics (fuzzy relational structures over linearly ordered BL-algebras, MTL-algebras and similar algebras). Tarski style truth definition will be given and completeness of very natural axiom systems with respect to the general semantics will be presented. But several important predicate fuzzy logic are not recursively axiomatizable with respect to their standard semantics; some others are. This will be surveyed and degree of undecidability of most of these logics will be explicitly stated ([12, 13, 1, 2, 3, 4, 19, 17, 18]). Main examples: the set of standard tautologies of the fuzzy predicate logic BL∀ is not arithmetical, whereas the set of standard tautologies of the logic MTL∀ coincides with the set of general tautologies of this logic and therefore is recursively enumerable. When defining the general semantics of BL∀ we cannot restrict ourselves to interpretations over BL-chains that are completely ordered; this would lead again to a non-arithmetical set of tautologies [20]. On the other hand, we can give up linear order of the algebras; one gets a complete axiomatization of this semantics just by deleting one axiom from the corresponding axiomatization based on linearly ordered algebras. Then we refer on corresponding falsity-free (positive) logics and their semantics based on algebras called hoops [7]. Completeness and conservativity results will be presented.

Part 3 will deal with mathematics based on fuzzy predicate logic. We go into some details concerning set theory. We describe a Zermelo-Fraenkel-like fuzzy set theory over Basic predicate logic and Cantor-like set theory with full comprehension over Łukasiewicz predicate logic. The latter theory can be shown to contain full Peano arithmetic with its classical logic ([23, 24, 25, 15, 16, 14]).

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Fuzzy sets and sheaves

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It is a remarkable fact that the historic development of fuzzy set theory (cf. [2]) proceeds completely isolated from sheaf theory[3]. Also the long lasting debate on categorical foundations of fuzzy set theory (cf. [6]) does not open the horizon for sheaf-theoretic arguments in the formulation of such fundamental notions as *membership function*, *measurement of membership*, *similarity*, *fuzzy ordering*, *fuzzy relational equation*, etc.

The aim of this paper is to explain that large parts of fuzzy set theory are actually subfields of sheaf theory. We show that fuzzy sets are *subsheaves* of simple sheaves — so-called sheaves of level cuts, similarity relations are *sheaves of ordinary equivalence relations*, fuzzy subgroups are *subsheaves of subgroups* of simple sheaves of groups, and stratified Ω-valued topological spaces are *topological space objects* in the category of sheaves. Further, intersections, unions, images and inverse images of fuzzy sets, the max – min-composition of fuzzy relations are special categorical constructions in the category of sheaves. Fuzzy power sets are nothing but power sheaves of simple sheaves. Fuzzy relational equations are equations in the Kleisli category associated with the power object monad in the category of sheaves. Moreover, fuzzy theorists are not able to give a proper solution of the quotient problem w.r.t. similarity relations and a proper construction of fuzzy factor groups w.r.t. invariant fuzzy subgroups.

In order to overcome these shortcomings some fundamental knowledge from sheaf theory is inevitable. Therefore we begin with some basic facts from sheaf theory including the role of the so-called *espace étalé*, the concept of Ω-*valued sets* and the *tilde-construction*. We recall the construction of the subobject classifier and the identification of subobjects with characteristic morphisms in the category of sheaves, resp. complete Ω-valued sets. The importance of these constructions will appear immediately for every fuzzy set theoretist, when their relationships to standard techniques in fuzzy set theory are explained — e.g. level cut techniques or the interpretation of fuzzy sets by their prototypes. Further, we discuss the set-theoretical operations on fuzzy sets in the light of sheaf theory and quote the important *categorical axioms* for fuzzy preorderings, similarity relations and fuzzy partial orderings. We solve the quotient problem w.r.t. similarity relations in terms of an *exact diagram* and show by using only categorical arguments that the *symmetrization* of (fuzzy) preorders leads always to a (fuzzy) partial ordering on the respective quotient.

Further, we describe group objects in the category of sheaves, resp. complete Ω-valued sets and characterize fuzzy subgroups as subgroup objects of simple sheaves of groups. Since we have already solved the quotient problem w.r.t. similarity relations, we are in the position to give a proper construction of fuzzy factor groups which are again of course a part of an exact diagram. Finally, we study higher order constructions and give a detailed description of the *formation of union* of fuzzy systems

[3] A historic account on sheaf theory can be found in [3].

of fuzzy sets. After having understood the power object monad in the category of sheaves, resp. complete Ω-valued sets, we recall the axioms of topological space objects, and show that topological space objects on simple sheaves and *stratified* Ω*-valued topological spaces* are the same things. We close this talk with two important examples of topological space objects: One is generated by fibrewise topological spaces, while the other one is construced from separated presheaves of ordinary topological spaces. In this context it is interesting to see that there exists an adjoint situation between topological space objects on complete Ω-valued sets and separated presheaves of ordinary topological spaces on Ω.

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Joint distributions on MV-algebras as interactions of fuzzy events

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In recent years many papers have been written generalizing some theorems, known from the Kolmogorovian probability theory, to MV-algebras. To achieve such results, so-called product MV-algebras were introduced and, using the product, the joint probability distribution was defined. In this paper we present an approach how to define the joint distributions on MV-algebras which are not necessarily closed under product. First we construct conditional measures on a given MV-algebra. And, using these conditional measures, we define the joint probability distributions.

We will work with a semi-simple MV-algebra, $\mathcal{M}$, which is represented by a system of integrable functions defined on a probability space $(\Omega, \mathcal{S}, \mu)$ with their range in $[0;1]$ and such that $\emptyset \in \mathcal{M}$ and the system $\mathcal{M}$ is closed under the operations $*$ and $\oplus$ defined pointwise by

$$f^*(x) = 1 - f(x), \quad (f \oplus g)(x) = \min\{1, f(x) + g(x)\}$$

The *conditional probability distribution*, γ, on the MV-algebra $\mathcal{M}$ is an additive normed measure on $\mathcal{M}$, defined as follows

$$\nu(f) = \nu(g)\gamma(f|g) + \nu(g^*)\gamma(f|g^*)$$

with the following conditions holding for γ

$$\int fg\, d\mu = 0 \qquad \Rightarrow \qquad \gamma(f|g) = 0$$
$$\int fg^*\, d\mu = 0 \qquad \Rightarrow \qquad \gamma(f|g^*) = 0$$

where $\nu(f) = \int f\, d\mu$.

Now, we will show that such conditional distibutions on $\mathcal{M}$ are not given uniquely.

Denote $\mathcal{T}$ the system of all transformations $\tau : \mathcal{M} \longrightarrow [0;1]^\Omega$ such that for each $f \in \mathcal{M}$

1. $\tau(f)$ is $\mathcal{S}$-measurable
2. $\int f\, d\mu = \int \tau(f)\, d\mu$
3. for any $x \in \Omega$ there holds $f(x) = 0 \quad \Rightarrow \quad (\tau(f))(x) = 0$.

Theorem 1. *Let $\tau \in \mathcal{T}$ be such that for any $g \in \mathcal{M}$*

$$\tau(g^*) = 1 - \tau(g) \tag{1}$$

Define for any $f, g \in \mathcal{M}$

$$\gamma(f|g) = \begin{cases} \frac{\int f \cdot \tau(g)\, d\mu}{\int \tau(g)\, d\mu} & \text{if } 0 < \nu(g) < 1 \\ \nu(f) & \text{if } \nu(g) = 1 \\ 0 & \text{if } \nu(g) = 0. \end{cases} \tag{2}$$

Then for any $g \in \mathcal{M}$ such that $\nu(g) > 0$, $\gamma(.|g)$ is a conditional measure.

We will say that event f is *independent* of g with respect to a conditional measure γ iff $\nu(f) = \gamma(f|g)$. γ will always denote the conditional measure defined by Formula 2 from Theorem 1.

Remark 2. As we will see in the next example, the independence of event f of g does not imply the independence of the event g of f. This nonsymmetric relation of independence allows us to distinguish between a cause and its effects. Similar results concerning the ortho-modular lattices have been achieved also by O. Nánásiová in [4].

Example 3. Let $\Omega = [0;1]$ and μ be Lebesgue measure. Let τ be the transformation given by

$$(\tau(f))(x) = \begin{cases} \frac{1}{\mu(A(f))} \int_{A(f)} f\, d\mu & \text{iff } f(x) \in]0.5;1[\text{ and } A(f) = \{x \in \Omega; f(x) \in]0.5;1[\} \\ \frac{1}{\mu(B(f))} \int_{B(f)} f\, d\mu & \text{iff } f(x) \in]0;0.5[\text{ and } B(f) = \{x \in \Omega; f(x) \in]0;0.5[\} \\ f(x) & \text{otherwise} \end{cases}$$

provided $\mu(A(f)) \neq 0$, $\mu(B(f)) \neq 0$. If e.g. $\mu(A(f)) = 0$, we can put any value to $(\tau(g))(x)$ for $x \in A(f)$.
Take $f(x) = x$ and $g(x) = \frac{1}{2}x$.Then we get

$$(\tau(f))(x) = \begin{cases} 0.25 & \text{iff } x \in]0;0.5[\\ 0.75 & \text{iff } x \in]0.5;1[\\ x & \text{otherwise} \end{cases} \qquad (\tau(g))(x) = \begin{cases} 0.25 & \text{iff } x \in]0;1[\\ x & \text{otherwise} \end{cases}$$

Now, compute the conditional measure

$$\gamma(f|g) = \frac{\int_0^1 f \cdot \tau(g)\, d\mu}{\int_0^1 g\, d\mu} = \frac{0.25 \int_0^1 x\, d\mu}{0.25} = 0.5 = \nu(f)$$

$$\gamma(g|g) = \frac{\int_0^1 g \cdot \tau(g)\, d\mu}{\int_0^1 g\, d\mu} = \frac{0.25 \int_0^1 0.5x\, d\mu}{0.25} = 0.25 = \nu(g)$$

$$\gamma(g|f) = \frac{\int_0^1 g \cdot \tau(f)\, d\mu}{\int_0^1 f\, d\mu} = \frac{0.25 \int_0^{0.5} 0.5x\, d\mu + 0.75 \int_{0.5}^1 0.5x\, d\mu}{0.5} = \frac{5}{16} \neq \nu(g) = 0.25$$

$$\gamma(f|f) = \frac{\int_0^1 f \cdot \tau(f)\, d\mu}{\int_0^1 f\, d\mu} = \frac{0.25 \int_0^{0.5} x\, d\mu + 0.75 \int_{0.5}^1 x\, d\mu}{0.5} = \frac{5}{8} \neq \nu(f) = 0.5$$

Hence we get that g is dependent on f and f is also dependent on f. On the other hand, f is independent of g and also g is independent of itself. In the Kolmogorovian probability theory we are not used to the fact that an event is independent of itself. But even this can happen when dealing with MV-algebtras instead of Boolean algebras.

Remark 4. Once having defined for any pair f, g of elements of the MV-algebra $\mathcal{M}$ the measure $\gamma(f|g)$, the conditional measure if $\nu(g) > 0$, we can define also the two-dimensional distribution on $\mathcal{M} \times \mathcal{M}$ – the measure (probability) of occurence of this pair f, g. This, in fact represents the interaction of f and g. And the interaction can be different if we change the order.

The *measure of interaction* of a pair $f,\ g \in \mathcal{M}$ will be denoted by $p(f,g)$ and defined as

$$p(f,g) = \gamma(f|g)\gamma(g|1) \tag{3}$$

Theorem 5 (Basic properties of p). *Let p be a measure of interaction on the MV-algebra $\mathcal{M}$ and f,g be any elements of $\mathcal{M}$. Then*

1. $p(f,1) = p(1,f) = \nu(f)$
2. $p(f,g) = p(g,f) = 0$, *if* $\int fg\,d\mu = 0$
3. $p(f,g) \leq \min\{\nu(f);\nu(g)\}$, *particularly* $p(f,f) \leq \nu(f)$
4. *the variables of p do not commute, i.e. in general* $p(f,g) \neq p(g,f)$

Example 6. Assume that $\Omega = [0;1]$ and μ is the Lebesgue measure. The transformation τ will be defined by the following

$$(\tau(f))(x) = \begin{cases} 0, & \text{if } f(x) = 0 \\ 1, & \text{if } f(x) = 1 \\ \frac{1}{\mu(\mathcal{A}(f))}\int_{\mathcal{A}(f)} f(x)\,d\mu(x) & \text{otherwise, where } \mathcal{A}(f) = \{x; 0 < f(x) < 1\} \end{cases}$$

Let $f(x) = x$ and $g(x) = \min\{0, x-0.5\}$. Then

$$(\tau(f))(x) = \begin{cases} 0, & \text{if } x = 0 \\ 1, & \text{if } x = 1 \\ 0.5 & \text{otherwise} \end{cases}$$

$$(\tau(g))(x) = \begin{cases} 0, & \text{if } x \leq 0.5 \\ 0.25 & \text{if } x > 0.5 \end{cases}$$

Then

$$p(g,f) = \int_0^1 g0.5\,d\mu = 0.5\int_{0.5}^1 (0.5-x)\,d\mu = \frac{1}{16}$$

$$p(f,g) = \int_{0.5}^1 x0.25\,d\mu = \frac{1}{4}\frac{3}{8} = \frac{3}{32}$$

$$p(f,f) = \int_0^1 x0.5d\mu = \frac{1}{4}$$

$$p(g,g) = \int_{0.5}^1 (x-0.5)0.25d\mu = \frac{1}{32}$$

We add some references where you can find papers with related topics.

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The concept of independence in the context of similarity relations

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The success of fuzzy systems in real world applications is based on their capability to model human expert knowledge in an easily understandable way using simple rules incorporating vague concepts represented by fuzzy sets. However, the use of rules in conjunction with vague concepts alone does not guarantee the interpretability of a fuzzy system.

There is a number of other aspects that have to be considered.

- The shape of the fuzzy sets should be chosen in such a way that they really correspond to real world vague concepts.
- The number of rules should be strictly limited, especially the number of rules firing at the same time.
- The number of attributes or variables occurring in a single rule should be kept very small.
- Finally, the way in which the fuzzy sets are aggregated to determine the firing degree of a rule, implies a certain independence assumption of the underlying vague concepts.

Here we will mainly concentrate on the last of these aspects.

Understanding fuzzy sets as induced concepts in the context of similarity or equality relations [9, 6, 8, 2, 5, 7, 1] leads to a rigorous and consistent interpretation the vague concepts. Fuzzy sets can no longer be chosen arbitrarily, but have to be in accordance with the underlying similarity relations. A very simple way to define suitable equality relations is based on the concept of scaling [4].

The similarity relations specify how exact values have to be distinguished in a certain range of a domain in order to solve the task for which the fuzzy system is designed. Taking a look at standard fuzzy systems, the underlying similarity relations for the single domains are assumed to be independent, i.e. the similarity of two tuples of values depends only on the similarities of the single values. However, this assumption is only partly satisfied in most real world applications.

Here we take a closer look at the notion of independence in the context of similarity relations. It turns out [3] that independence in the context of similarity relations is a non-symmetric concept in contrast to the well known probabilistic independence notion, where for instance $P(A|B) = P(A) \Rightarrow P(B|A) = P(B)$ holds.

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Triangular norms as special semigroups

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1 Introduction

Clearly, each triangular norm [15, 26] is a special semigroup operation on the unit interval $[0,1]$. To be precise, $([0,1],T,\leq)$ is a fully ordered abelian semigroup with neutral element 1. Several results and constructions from the theory of general semigroups [3, 6, 9] have been carried over to t-norms. Well-known examples are [24, 25] and the full characterization of continuous t-norms based on I-semigroups [5, 19, 20]. In this contribution we give a survey on recent advances in this context (for an extensive survey see [17]).

2 Archimedean components

To simplify terminology, we shall identify, if T is a triangular norm, the fully ordered semigroup $([0,1],T,\leq)$ with the t-norm T since the underlying set and the order are clear in this context. In particular, we shall also speak about subsemigroups of t-norms (which are necessarily fully ordered) without mentioning the order $\leq$ explicitly.

In semigroups $(X,*)$ with $X \subseteq \mathbb{R}$, in particular for $([0,1],T)$ where T is a t-norm, we shall write $x_*^{(n)}$ and $x_T^{(n)}$, respectively, or simply $x^{(n)}$ if the semigroup operation is clear, in order to distinguish it from the usual power x^n (with respect to the multiplication of real numbers).

Let T be a t-norm and let $(X,*)$ be a subsemigroup of T. Then it is evident that $(X,*)$ is a fully ordered commutative semigroup where the operation $*$ is bounded from above by the minimum, i.e., $x*y \leq \min(x,y)$ for all $x,y \in X$. If 0 and 1 are contained in X then they are annihilator and neutral element of $(X,*)$, respectively.

In general, it is not clear whether for each semigroup $(X,*,\leq)$, where $X \subseteq [0,1]$, where the operation $*$ is bounded from above by the minimum and where 1, whenever it is contained in X, acts as neutral element, the operation $*$ can be extended to a triangular norm.

However, in the special case when X is a convex subset of $[0,1]$, i.e., a subinterval of $[0,1]$, we shall see that such an extension is always possible. In order to show this, we use the following notions going back to [16] and [10]. Note that the name tosab is an acronym for *totally ordered semigroup, abelian, bounded by the minimum*.

Definition 1. Let I be a non-empty subinterval of the closed unit interval $[0,1]$.

(i) A fully ordered commutative semigroup $(I,*)$ where $*$ is bounded from above by the minimum will be called a *tosab*.

(ii) If $([0,1],*)$ is a tosab then the operation $*$ is called a *t-subnorm*.

When investigating the structure of t-norms, their Archimedean subsemigroups play an important role (compare [6, 14]).

Definition 2. Let T be a t-norm. Two elements $x,y \in [0,1]$ are called *Archimedean equivalent* if there is an $n \in \mathbb{N}$ such that $x^{(n)} \leq y \leq x$ or $y^{(n)} \leq x \leq y$. For each $x \in [0,1]$ the equivalence class I_x containing x is called a *T-Archimedean class* of T or *Archimedean class* if T is either irrelevant or clear from context.

Clearly, as noted in [7], each Archimedean class is a convex subset of $[0,1]$. Obviously, by complete analogy we may define the Archimedean classes of tosabs and, in particular, of t-subnorms. The following result can be found in [14, Proposition 3.2].

Proposition 3. *Let T be a t-norm.*

(i) For all $(x,y) \in [0,1]^2$ we have $I_{T(x,y)} = I_{\min(x,y)}$.

(ii) For each $x \in [0,1]$ the pair $(I_x, T|_{I_x^2})$ is a subsemigroup of $([0,1],T)$ (and, hence, a tosab), and it is called an Archimedean component *of T.*

As a consequence, for two t-norms T_1 and T_2 with the same Archimedean components we have $x_{T_1}^{(n)} = x_{T_2}^{(n)}$ for each $x \in [0,1]$ and $n \in \mathbb{N}$.

A necessary and sufficient condition for a singleton $\{x\}$ to be a (trivial) Archimedean class for a t-norm T is that $T(y,z) = x$ holds if and only if $\min(y,z) = x$. As a consequence, $\{1\}$ is an Archimedean class of each t-norm T.

It is easy to see that a triangular norm is Archimedean if and only if its only non-trivial Archimedean class is either $[0,1[$ or $]0,1[$. Similarly, a non-trivial tosab is Archimedean if and only if it has only one non-trivial Archimedean class.

From [15, Proposition 1.6 and Theorem 2.12] the following characterization of Archimedean components follows immediately.

Lemma 4. *A fully ordered commutative semigroup* $(I,*)$ *is an Archimedean component of some t-norm* T *if and only if either* $I=\{1\}$ *or* I *is a convex subset of* $[0,1[$ *such that for all* $x \in I$ *we have* $\lim_{n \longrightarrow \infty} x_*^{(n)} = \inf I$.

The following result, whose proof is straightforward, will be helpful for determining the uniqueness of t-norms with given Archimedean components.

Lemma 5. *Let* T *be a t-norm and* $\{I_x \mid x \in [0,1]\}$ *the set of Archimedean components. Then the following are equivalent:*

(i) *For each t-norm* $\tilde{T}$ *with* $\tilde{T} \neq T$ *there is an element* $x \in [0,1]$ *such that the Archimedean component* $(\tilde{I}_x, \tilde{T}|_{(\tilde{I}_x)^2})$ *of* $\tilde{T}$ *and the Archimedean component* $(I_x, T|_{I_x^2})$ *of* T *are different.*

(ii) *For all* $(x,y) \in [0,1]^2$ *with* $x \leq y$ *there is a unique fully ordered commutative semigroup* $(I_{\{x,y\}}, *)$*, where the operation* $*$ *is bounded from above by the minimum, such that both* $(I_x, T|_{I_x^2})$ *and* $(I_y, T|_{I_y^2})$ *are subsemigroups of* $(I_{\{x,y\}}, *)$*.*

Lemma 6. *Assume that* I_u *equals* $[a,b[$ *or* $]a,b[$ *and let* $(I_u, *_u)$ *be an Archimedean component of some t-norm* T *such that for each* $x \in]a,b[$ *there is a* $y \in]a,b[$ *with* $x *_u y > a$ *and such that the conditional cancellation law holds. Then, putting* $I = I_u \cup I_b$*, the semigroup* $(I, T|_{I^2})$ *is the ordinal sum of* $(I_u, *_u)$ *and* $(I_b, T|_{I_b^2})$*.*

Theorem 7. *Let* T *be a t-norm and suppose that each of its non-trivial Archimedean components satisfies the hypotheses of Lemma* 6*. Then there is no other t-norm* $\tilde{T}$ *having the same Archimedean components as* T*.*

Corollary 8. *Let* T *be a t-norm, suppose that each of its non-trivial Archimedean components is continuous and satisfies the hypotheses of Lemma* 6 *and, additionally,* $\lim_{z \nearrow b_x} T(y,z) = y$ *if* $x \in [0,1]$*,* $y \in I_x$ *and* $b_x = \sup I_x$*. Then* T *is a continuous t-norm, and it is uniquely determined by its Archimedean components.*

Example 9. Assume that T is a t-norm whose Archimedean components are $\left([0,\frac{1}{2}[, *_1\right)$ with $x *_1 y = x \cdot y$, $\left([\frac{1}{2},1[, *_2\right)$ with $x *_2 y = \frac{1}{2}$, and the trivial component $(\{1\}, *)$. Then we get

$$T(x,y) = \begin{cases} x \cdot y & \text{if } (x,y) \in [0,\frac{1}{2}[^2, \\ \frac{1}{2} & \text{if } (x,y) \in [\frac{1}{2},1[^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

i.e., T necessarily is the ordinal sum of its Archimedean components (see Proposition 13).

Note also that Archimedean components play a key role in the characterization of several specific semigroups. For example, in the *torsion semigroups* introduced in [22] for each x the set $\{x^1, x^2, \ldots, x^n, \ldots\}$ is finite. Therefore, for a *torsion t-norm* T and for each $x \in [0,1]$ there is an $n \in \mathbb{N}$ such that $x_T^{(n)}$ is an idempotent element of T. However, this is equivalent to the fact that each Archimedean component $(I,*)$ of T is a torsion semigroup which, in addition, satisfies $\inf I \in I$. Observe that, for a continuous t-norm T, $([0,1],T)$ is a torsion semigroup if and only if each Archimedean summand of T is nilpotent. A special subclass of torsion t-norms are the so-called *n-contractive t-norms* studied in [1], in which case $x_T^{(n)}$ is an idempotent element for each $x \in [0,1]$ (so n-contractive t-norms can be viewed as uniform torsion semigroups). A characterization of n-contractive t-norms by means of their Archimedean components, together with a construction method for n-contractive t-norms, can be found in [18].

Another interesting algebraic property closely linked to Archimedean components is the weak cancellativity investigated in [23]. A semigroup $(X,*)$ is said to be *weakly cancellative* if $x*x = x*y = y*y$ implies $x = y$, which, in the case of a t-norm T, is equivalent with saying that $T(x,x) = T(y,y)$ implies $x = y$, because of the monotonicity of T. Observe that a continuous t-norm T is weakly cancellative if and only if each Archimedean summand of T is strict. In general, a t-norm T is weakly cancellative if and only each Archimedean component of T is weakly cancellative. Note that a weakly cancellative Archimedean t-norm never has zero divisors, but it is not necessarily cancellative (an example for that is the Krause t-norm [15, Appendix B]).

3 Ordinal sums

Ordinal sums of abstract semigroups were introduced by A. H. Clifford in [2] (see also [8, 21]), foreshadowed in [4, 12], yielding a semigroup structure on the union of pairwise disjoint semigroups. We recall this fundamental result for convenience.

Theorem 10. *Let $(A,\preceq)$ be a linearly ordered set with $A \neq \emptyset$ and $((X_\alpha,*_\alpha))_{\alpha\in A}$ a family of semigroups such that $X_\alpha \cap X_\beta = \emptyset$ whenever $\alpha \neq \beta$. Put $X = \bigcup_{\alpha\in A} X_\alpha$ and define the operation $*\colon X^2 \longrightarrow X$ by*

$$x*y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha^2, \\ x & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \alpha \prec \beta, \\ y & \text{if } (x,y) \in X_\alpha \times X_\beta \text{ and } \beta \prec \alpha. \end{cases} \quad (1)$$

Then $(X,*)$ is a semigroup, and it will be called the ordinal sum of the semigroups $((X_\alpha,*_\alpha))_{\alpha\in A}$.

This result can be directly applied (see [25, 26] and Theorem 7.1 in Chapter 1) to construct new triangular norms from a given family of t-norms. The t-norm obtained via this construction will be referred to as an *ordinal sum of t-norms*:

Theorem 11. *Let $(T_\alpha)_{\alpha\in A}$ be a family of t-norms and $(]a_\alpha,b_\alpha[)_{\alpha\in A}$ be a family of non-empty, pairwise disjoint open subintervals of $[0,1]$. Then the following function $T\colon [0,1]^2 \longrightarrow [0,1]$ is a t-norm:*

$$T(x,y) = \begin{cases} a_\alpha + (b_\alpha - a_\alpha)\cdot T_\alpha\left(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}\right) & \text{if } (x,y) \in [a_\alpha,b_\alpha[^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \quad (2)$$

Proposition 12. *Let $(A,\preceq)$ be a linearly ordered set with $A \neq \emptyset$ and $((X_\alpha,*_\alpha))_{\alpha\in A}$ a family of semigroups such that $(X_\alpha)_{\alpha\in A}$ is a partition of the closed unit interval $[0,1]$. If the operation $*: [0,1]^2 \longrightarrow [0,1]$ given by (1) is a triangular norm, then we have:*

- *(i) Each X_α is a subinterval of $[0,1]$.*
- *(ii) Each semigroup $(X_\alpha,*_\alpha)$ is a fully ordered commutative semigroup where the operation $*_\alpha$ is bounded from above by the minimum, i.e., we have $x *_\alpha y \leq \min(x,y)$ for all $x,y \in X_\alpha$.*
- *(iii) The order $\preceq$ on A is compatible with the usual order $\leq$ on $[0,1]$, i.e., for $\alpha,\beta \in A$ we have $\alpha \prec \beta$ if and only if $x < y$ for all $x \in X_\alpha$ and $y \in X_\beta$.*

(iv) For all $(x,y) \in [0,1]^2$ *we have*

$$x * y = \begin{cases} x *_\alpha y & \text{if } (x,y) \in X_\alpha^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \quad (3)$$

Proposition 13. *Let* $([0,1],*)$ *be the ordinal sum of a family* $((X_\alpha, *_\alpha))_{\alpha\in A}$ *of semigroups. Then the operation* $*$ *is a t-norm if and only if each* $(X_\alpha, *_\alpha)$ *is a tosab, if the order* $\preceq$ *on* A *is compatible with the usual order* $\leq$ *on* $[0,1]$*, and if there is an* $\alpha_0 \in A$ *such that* 1 *is the neutral element of* $*_{\alpha_0}$.

Theorem 14. *Let* I *be a non-empty subinterval of* $[0,1]$*. A semigroup* $(I,*)$ *is a continuous tosab if and only if it is an ordinal sum of idempotent tosabs and continuous Archimedean tosabs with neutral element with possibly one exception if for some summand* $(I_{\alpha_0}, *_{\alpha_0})$ *we have* $\sup I_{\alpha_0} = \sup I \in I_{\alpha_0} \cup ([0,1] \setminus I)$*, in which case* $(I_{\alpha_0}, *_{\alpha_0})$ *need not have a neutral element.*

Definition 15. A tosab is called *ordinally irreducible* if it cannot be expressed as an ordinal sum of two or more non-singleton tosabs.

Proposition 16. *Let* T *be a t-norm. Then the following are equivalent:*

(i) T *is ordinally irreducible.*
(ii) For each $x \in]0,1[$ *there exist* $y,z \in [0,1]$ *with* $y < x < z$ *and* $T(y,z) < y$.

The following modification of Theorem 11, where the resulting t-norm T will be referred to as an *ordinal sum of t-subnorms*, was proved in [11].

Theorem 17. *Let* $(V_\alpha)_{\alpha\in A}$ *be a family of t-subnorms and* $(]a_\alpha, b_\alpha[)_{\alpha\in A}$ *be a family of non-empty, pairwise disjoint open subintervals of* $[0,1]$*. Further, if* $b_{\alpha_0} = 1$ *for some* $\alpha_0 \in A$ *then assume that* V_{α_0} *is a t-norm, and if* $b_{\alpha_0} = a_{\beta_0}$ *for some* $\alpha_0, \beta_0 \in A$ *then assume either that* V_{α_0} *is a t-norm or that* V_{β_0} *has no zero divisors. Then the following function* $T\colon [0,1]^2 \longrightarrow [0,1]$ *is a t-norm:*

$$T(x,y) = \begin{cases} a_\alpha + (b_\alpha - a_\alpha) \cdot V_\alpha\left(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}\right) & \text{if } (x,y) \in]a_\alpha, b_\alpha]^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \quad (4)$$

The construction in Theorem 17 is not identical to the one in Theorem 10 (for instance, $T|_{]a_\alpha,b_\alpha]^2}$ is not necessarily a semigroup operation on $]a_\alpha, b_\alpha]$). However, in Theorem 18 below we shall show that each t-norm T where $([0,1],T)$ is an ordinal sum of semigroups as in Theorem 10 can be rewritten as an ordinal sum of t-subnorms as in Theorem 17.

In [16, Theorem 3.1] it was shown that the construction in Theorem 17 is the most general way to obtain a t-norm as an ordinal sum of semigroups.

Theorem 18. *Let* T *be a t-norm. Then the following are equivalent:*

(i) $([0,1],T)$ *is an ordinal sum of semigroups.*
(ii) T *is an ordinal sum of t-subnorms.*

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A characterization and composition of quasi-copulas

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1 Introduction

In this contribution the main attention will be paid to two–dimensional quasi–copulas, which are a special type of binary 1–Lipschitz aggregation operators. Quasi–copulas will be characterized as solutions to a certain functional equation. We also show that quasi–copulas and dual quasi–copulas are important for describing the structure of 1–Lipschitz aggregation operators with any neutral element or annihilator in the unit interval. Finally, we will study under which conditions the composition of any two quasi–copulas is again a quasi–copula. The study of these problems was motivated by several papers on fuzzy preference modeling [5, 6], and by papers concerning some problems in fuzzy probability calculus, e.g., [10] and others. Therefore we expect applications of obtained results in these areas.

Recall first the definitions and properties of basic notions which are used throughout the paper.

Definition 1. Let $n \in \mathbb{N}, n \geq 2$. *An n–ary aggregation operator A* is a non–decreasing function $A : [0,1]^n \longrightarrow [0,1]$ satisfying the boundary conditions $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$.

In this paper we will deal with binary aggregation operators only. Therefore if no confusion can arise, we will use for them the name aggregation operators only.

Aggregation operators satisfying the standard Lipschitz condition with constant 1, i.e., satisfying the property

$$|A(x_1,y_1) - A(x_2,y_2)| \leq |x_1 - x_2| + |y_1 - y_2|,$$

for all $x_1, x_2, y_1, y_2 \in [0,1]$, will be called *1–Lipschitz aggregation operators*.

From well–known types of binary aggregation operators, for example, the arithmetic mean M, the product operator Π, *Min* and *Max* operators, as well as weighted means, OWA operators, copulas, quasi–copulas, Choquet integral-based aggregation operators, Sugeno intergal–based aggregation operators are 1–Lipschitz aggregation operators. More details on these classes of aggregation operators can be found, e.g., in [2].

Distinguished classes of 1–Lipschitz aggregation operators are the classes of copulas and quasi–copulas.

Definition 2. A (two–dimensional) *copula* C is a function $C : [0,1]^2 \longrightarrow [0,1]$ with the properties:

- $C(0,x) = C(x,0) = 0$ and $C(x,1) = C(1,x) = x$ for all $x \in [0,1]$;
- $C(x_1,y_1) + C(x_2,y_2) \geq C(x_2,y_1) + C(x_1,y_2)$ for all $x_1, x_2, y_1, y_2 \in [0,1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

Definition 3. [9] A (two–dimensional) *quasi–copula* Q is a function $Q : [0,1]^2 \longrightarrow [0,1]$ with the properties:

- $Q(0,x) = Q(x,0) = 0$ and $Q(x,1) = Q(1,x) = x$ for all $x \in [0,1]$;
- Q is non–decreasing in each of its arguments;
- Q satisfies Lipschitz's condition with constant 1.

Copulas are also non–decreasing functions in each variable and 1–Lipschitz. Each copula is evidently a quasi–copula. Due to the 1–Lipschitz property, copulas as well as quasi–copulas are continuous functions on the unit square.

Note that the conditions in the first two items of the definition of a quasi–copula mean that quasi–copulas are aggregation operators with zero annihilator and neutral element equal to 1. One of the last two properties is superfluous because for 1–Lipschitz aggregation operators they are equivalent. Therefore quasi–copulas can be equivalently characterized as

- 1–Lipschitz aggregation operators with neutral element 1,

  or as

- 1–Lipschitz aggregation operators with zero annihilator.

The set of all quasi–copulas will be denoted by $\mathcal{Q}$.

The following claim is only a slight modification of a given definition of a quasi–copula.

Lemma 4. A function $Q : [0,1]^2 \longrightarrow [0,1]$ is a quasi–copula if and only if it satisfies the following conditions:

(i) Q is non–decreasing;

(ii) Q is 1–Lipschitz;

(iii) $Q(0,1) = Q(1,0) = 0$ and $Q(1,1) = 1$.

Since an aggregation operator A is always monotone and satisfies the property $A(1,1) = 1$, we obtain the following result.

Corollary 5. An aggregation operator A is a quasi–copula if and only if it is 1–Lipschitz and $A(0,1) = A(1,0) = 0$.

For any $Q \in \mathcal{Q}$, the function Q^*, so–called dual of a quasi–copula Q, is defined by

$$Q^* : [0,1]^2 \longrightarrow [0,1], \qquad Q^*(x,y) = x + y - Q(x,y).$$

The dual of any quasi–copula is also a non–decreasing and 1–Lipschitz function, but with zero neutral element and annihilator equal to 1.
Denote by $\mathcal{D}$ the set of all functions $f : [0,1]^2 \longrightarrow [0,1]$ which are non–decreasing, 1–Lipschitz and with zero neutral element (and 1 as neutral element). The set $\mathcal{D}$ will be called the set of all dual quasi–copulas.

2 Characterization of quasi–copulas

In [12], cf. [16], 1–Lipschitz aggregation operators have been characterized as solutions to a simple functional equation, similar to the Frank functional equation [8], in the following way.

Theorem 6. A binary aggregation operator A is 1–Lipschitz if and only if there is a binary aggregation operator B, such that for all $x, y \in [0,1]$ it holds

$$A(x,y) + B(x,y) = x + y. \tag{1}$$

Commutative quasi–copulas can also be characterized as solutions to the following type of a functional equation.

Theorem 7. A commutative aggregation operator A is a commutative quasi–copula if and only if there exists an aggregation operator B such that for all $x, y \in [0,1]$ we have

$$A(x,y) + B(1-x,y) = y. \tag{2}$$

Remark 8. The previous claim without the commutativity condition must be reformulated in the following way: An aggregation operator A is a quasi–copula if and only if there exist aggregation operators B and C such that for each $x, y \in [0,1]$ we have

$$A(x,y) + B(1-x,y) = y \quad \text{and} \quad A(x,y) + C(x,1-y) = x.$$

3 The structure of binary 1–Lipschitz aggregation operators with annihilator or neutral element

Quasi–copulas also play an important role in the characterization of 1–Lipschitz aggregation operators with annihilator or neutral element from the unit interval. We first show that each 1–Lipschitz aggregation operator with annihilator $a \in]0,1[$ can be built up from a quasi–copula, dual quasi–copula and the value a. Then we also clarify the structure of 1–Lipschitz aggregation operators with neutral element $e \in]0,1[$.

For a given aggregation operator A denote $A^*(x,y) = x + y - A(x,y)$, $(x,y) \in [0,1]^2$. Then $(A^*)^* = A$ and due to Theorem 6 it holds that the operator A is 1–Lipschitz if and only if A^* is a 1–Lipschitz aggregation operator.

If a 1–Lipschitz aggregation operator A has neutral element e_A, then for $\forall x \in [0,1]$, $A^*(x,e_A) = A^*(e_A,x) = e_A$, which means that the element e_A is the annihilator of the operator A^*, i.e., $e_A = a_{A^*}$. Analogously, for the annihilator of A, if it exists, we have $a_A = e_{A^*}$.

The structure of 1–Lipschitz aggregation operators with annihilator

Let A be a 1–Lipschitz aggregation operator with annihilator $a_A \in [0,1]$. According to the previous discussions:

- if $a_A = 0$ then A is a quasi–copula;
- if $a_A = 1$ then A is a dual quasi–copula.

• In the case that $a_A = a \in]0,1[$, define, similarly as in the case of nullnorms [3], the mappings φ_a, ψ_a by

$$\varphi_a(x) = \frac{x}{a}, \quad , \quad \psi_a(x) = \frac{x-a}{1-a}. \tag{3}$$

Then the function $Q_A : [0,1]^2 \longrightarrow [0,1]$,

$$Q_A(x,y) = \psi_a\left(A\left(\psi_a^{-1}(x), \psi_a^{-1}(y)\right)\right) = \frac{A(a+(1-a)x, a+(1-a)y) - a}{1-a} \tag{4}$$

is a quasi–copula, and the function $D_A : [0,1]^2 \longrightarrow [0,1]$

$$D_A(x,y) = \varphi_a\left(A\left(\varphi_a^{-1}(x), \varphi_a^{-1}(y)\right)\right) = \frac{A(ax,ay)}{a} \tag{5}$$

is a dual quasi–copula.
Therefore the operator A can be expressed on the squares $[0,a]^2$ and $[a,1]^2$, as a transformation of some dual quasi–copula and some quasi–copula, respectively, i.e.,

$$A(x,y) = \begin{cases} \varphi_a^{-1}\left(D_A\left(\varphi_a(x), \varphi_a(y)\right)\right) & \text{if } (x,y) \in [0,a]^2 \\ \psi_a^{-1}\left(Q_A\left(\psi_a(x), \psi_a(y)\right)\right) & \text{if } (x,y) \in [a,1]^2. \end{cases}$$

If $(x,y) \in [0,a[\times]a,1]$, then

$$a = A(x,a) \le A(x,y) \le A(a,y) = a,$$

which means that $A(x,y) = a$, and the same is true for the rest of the unit square $]a,1] \times [0,a[$.

The structure of 1–Lipschitz aggregation operators with neutral element

A similar situation to the previous one is for 1–Lipschitz aggregation operators with neutral element.

Let A be a 1–Lipschitz aggregation operator with neutral element $e_A \in [0,1]$. Trivially,

- if $e_A = 1$ then A is a quasi–copula;
- if $e_A = 0$ then A is a dual quasi–copula.

• If $e_A = e \in]0,1[$, then the function $Q_A : [0,1]^2 \longrightarrow [0,1]$,

$$Q_A(x,y) = \varphi_e\left(A\left(\varphi_e^{-1}(x), \varphi_e^{-1}(y)\right)\right) \tag{6}$$

is a quasi–copula, and the function $D_A : [0,1]^2 \longrightarrow [0,1]$,

$$D_A(x,y) = \psi_e\left(A\left(\psi_e^{-1}(x), \psi_e^{-1}(y)\right)\right) \tag{7}$$

is a dual quasi–copula. Therefore

$$A(x,y) = \begin{cases} \varphi_e^{-1}\left(Q_A\left(\varphi_e(x), \varphi_e(y)\right)\right) & \text{if } (x,y) \in [0,e]^2 \\ \psi_e^{-1}\left(D_A\left(\psi_e(x), \psi_e(y)\right)\right) & \text{if } (x,y) \in [e,1]^2. \end{cases}$$

In the case of uninorms [7] which is similar to this one, the values on the rest parts of the unit square are not determined uniquely, they are between the values of *Min* and *Max* operators, in general. In the case of 1–Lipschitz aggregation operators the values at the points $(x,y) \in [0,e[\times]e,1] \cup]e,1] \times [0,e[$ are determined uniquely. Indeed, if the operator A is 1–Lipschitz aggregation operator, the same is true for A^*, and moreover, $a_{A^*} = e$. Using the results of the previous part, the values of A^* at these points are $A^*(x,y) = e$, that is, $A(x,y) = x+y-e$ at all points $(x,y) \in [0,e[\times]e,1] \cup]e,1] \times [0,e[$.

4 On composition of quasi–copulas

For arbitrary binary aggregation operators A, B and F, the function $F(A,B) : [0,1]^2 \longrightarrow [0,1]$ defined by

$$F(A,B)(x,y) = F(A(x,y), B(x,y)),$$

is also a binary aggregation operator and is called a *composed aggregation operator*. It is easy to verify that $F(A,B)$ really possesses the properties of an aggregation operator.

In this section we give a necessary and sufficient condition under which composition of any two quasi–copulas is again a quasi–copula.

Preserving the 1–Lipschitz property: It is known, that although all three aggregation operators A, B, F are 1–Lipschitz, the composed aggregation operator $F(A,B)$ need not be of this property. For example, despite the Łukasiewicz t–conorm S_L is a 1–Lipschitz aggregation operator, the composed operator $S_L(S_L, S_L)$ does not possess this property [12]. However, if the outer operator F is a kernel aggregation operator, and A, B are 1–Lipschitz, then $F(A,B)$ is always 1–Lipschitz aggregation operator [4, 12].
Recall that a binary aggregation operator F has the *kernel property* if for all $u_1, u_2, v_1, v_2 \in [0,1]^2$ we have

$$|F(u_1,v_1) - F(u_2,v_2)| \leq \max\left(|u_1 - u_2|, |v_1 - v_2|\right).$$

It is clear that each kernel aggregation operator is also 1–Lipschitz. More details on kernel aggregation operators can be found in [13, 14, 15]. It can be shown that the kernel property of an outer operator is also a necessary condition for the 1–Lipschitz property of a composed aggregation operator [16].

Theorem 9. Let F be a binary aggregation operator. Then for any binary 1-Lipschitz aggregation operators A and B the composed aggregation operator $F(A,B)$ is 1-Lipschitz if and only if F is a kernel aggregation operator.

As a consequence of this theorem we obtain the sufficient condition for quasi–copulas.

Corollary 10. If the outer operator F is kernel, then composition of any two quasi–copulas is a quasi–copula.

The 1–Lipschitz property of the composed operator $F(Q_1, Q_2)$ is preserved by Theorem 9. Observe that due to the property $F(0,0) = 0$ the operator $F(Q_1, Q_2)$ possesses zero as annihilator.

However, for quasi–copulas, as a special type of 1–Lipschitz aggregation operators, the kernel property of F on $[0,1]^2$ can be relaxed, because the points with coordinates $(Q_1(x,y), Q_2(x,y))$ for any two quasi–copulas Q_1, Q_2 and all points $(x,y) \in [0,1]^2$, never fill in the whole unite square.

Lemma 11. Denote $K = \{(Q_1(x,y), Q_2(x,y))\,;\,(x,y) \in [0,1]^2, Q_1,\, Q_2 \in \mathcal{Q}\}$. Then

$$K = \left\{(u,v)\,;\, u \in [0,1],\, v \in \left[\max(2u-1,0), \frac{u+1}{2}\right]\right\}.$$

Because of this property of quasi–copulas we obtain the following claim.

Theorem 12. Let F be an aggregation operator. For any quasi–copulas Q_1, Q_2, a composed aggregation operator $F(Q_1, Q_2)$ is a quasi–copula if and only if the operator F has the kernel property on the set K defined in Lemma 11.

Note that for composition of copulas the claim analogous to that one in Corollary 2, is not true. Despite the outer operator is kernel, the composition of two copulas need not be a copula, as we can see in the following example.

Example 13. Let $F = \text{med}_k$, $k \in [0,1]$, i.e., $F(x,y) = \text{med}(x,y,k)$. Set $C_1 = T_L$ and $C_2 = T_P$, where T_P is the product t–norm. Then the composed operator is $A_k = \text{med}_k(T_L, T_P)$.
The operators C_1 and C_2 are copulas and each operator $F = \text{med}_k$ is a kernel aggregation operator on $[0,1]^2$. According to Theorem 9, the composed operator A_k is always 1–Lipschitz. For example, for $k = 0.5$ we obtain the operator

$$A_{0.5}(x,y) = \begin{cases} T_L(x,y) & \text{if } T_L(x,y) \geq 0.5 \\ T_P(x,y) & \text{if } T_P(x,y) \leq 0.5 \\ 0.5 & \text{if } T_L \leq 0.5 \leq T_P(x,y). \end{cases}$$

The operator $A_{0.5}$ is not a copula because it is not 2–monotone. To show this, consider the points $x = \frac{2}{3}$, $x' = \frac{3}{4}$, $y = \frac{2}{3}$ and $y' = \frac{3}{4}$. Then we have

$$A_{0.5}(\frac{3}{4}, \frac{3}{4}) + A_{0.5}(\frac{2}{3}, \frac{2}{3}) - A_{0.5}(\frac{2}{3}, \frac{3}{4}) - A_{0.5}(\frac{3}{4}, \frac{2}{3}) = 0.5 + \frac{4}{9} - 0.5 - 0.5 = -\frac{1}{18} < 0,$$

which contradicts the 2–monotonicity of $A_{0.5}$.
Note that by the previous theorem, all operators A_k, $k \in [0,1]$, are quasi–copulas. The claim follows from the facts that T_L and T_P are quasi–copulas (each copula is also a quasi–copula) and the outer operator $\text{med}(x,y,k)$ is kernel on $[0,1]^2$ and thus also on the set K.

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Modifying L-sets: two views based on level-sets

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In this paper we present two views of modifiers defined by means of level-sets. The author has studied these modifiers earlier and the current paper serves mainly as a survey of this work.

The author has studied *compositional modifier operators* (see e.g. [4, 5, 6, 11]), and especially modifiers which are also interior operators in Alexandroff topologies. In an Alexandroff topology the intersection of every family of open sets is open (see e.g. [1]). *L-sets on U* ([2]) are generalizations of *fuzzy sets* ([13]), defined as mappings $\mathcal{A} : U \longrightarrow L$, and they are modified by operating its level-sets by means of interior operators in Alexandroff topologies. These generalized operators are called *level-set generated modifiers* and denoted by $\mathcal{F}_L$ ([10]). In this case Representation Theorems presented by C. V. Negoita and D. A. Ralescu (see [12] and also [3]) are applicable when representing L-sets by means of level-sets. In this paper we demand that $L = (L, \leq, \wedge, \vee, \otimes)$ is a *cl-quasi-monoid* ([3]), and axioms for *L-interior operators* and *L-topologies* are given in [3]. Under certain conditions the level-set generated modifiers are also L-interior operators ([10]).

The author has also studied *coarsening operators* in [7, 8, 9]. Certain coarsening operators, namely *natural coarsening operators* denoted by $\mathcal{C}_L$, can be defined by means of open sets of Alexandroff topologies, and L-sets are modified by omitting those level-sets which are not open. In this case also Representation Theorems are applicable when representing L-sets by means of level-sets.

Because under certain conditions the level-set generated modifiers are L-interior operators, the image of this operator is a L-topology, say $\mathfrak{T}_1$. In this case we will show that the image of a natural coarsening operator is also a L-topology, say $\mathfrak{T}_2$, while the natural coarsening operators do not generally need to be L-interior operators (see [9]). We will show that $\forall \mathcal{A} \in L^U$, $\mathcal{C}_L(\mathcal{A}) \subset \mathcal{F}_L(\mathcal{A})$ and $\mathfrak{T}_1 = \mathfrak{T}_2$. Still, the category of produced L-topological spaces is isomorphic to the category of crisp Alexandroff spaces.

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Integrals of random fuzzy sets

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1 Introduction

Concepts of random fuzzy sets, often also called fuzzy random variables, have been introduced to extend the classical notion of random variables to random experiments with outcomes in form of fuzzy subsets of $\mathbb{R}^k$. The idea behind is to represent outcomes of random experiments in a more adequate way by integrating those inherent aspects of vagueness which are of non-random nature.

From a technical point of view it had been turned out that the notion of random fuzzy sets by Puri and Ralescu (cf. [15]) is the most general suggestion which admits also a probobability theory with extensions of the classical limit theorems (cf. [10], [13]). The talk deals with the notion of integrals of random fuzzy sets in the sense of Puri and Ralescu. The aim of the talk is to develop different ways to define integrals and then to investigate their mutual relationships.

The seminal paper by Puri and Ralescu has offered the mostly accepted approach. As they defined random fuzzy sets as extended random compact sets they could transfer Aumann's concept to define integrals for random compact sets, the so called Aumann-integral. A new direction has been initialized by Diamond and Kloeden who introduced the class of L_p-metrics on sample spaces consisting of fuzzy subsets of $\mathbb{R}^k$ ([4], for extensions see [12]). These metrics yield other concepts of random fuzzy sets as random elements in Banach spaces which makes possible to embed the probability theory with fuzzy observations into the general probability theory in Banach spaces ([9], [14], [11], [13]). The L_p-metrics work on subspaces of the sample spaces considered by Puri and Ralescu. Moreover, it has been shown that in most cases the different notions of random fuzzy sets coincide ([10], [13]). Therefore, reasonable alternatives to define integrals might be obtained by adaption of Bochner- and Pettis-integration.

Then two problems will be tackled within the talk. If the range of a random fuzzy set is restricted to a sample space where one of the L_p-metrics works, when does the Aumann-integral belong to this sample space? Secondly what are the mutual relationships between Aumann- and the adaptions of Bochner- as well as Pettis-integration? Both problems are not investigated systematically in literature. A first attempt concerning the first problem had been offered by the talk "Probability theory in sample spaces of fuzzy subsets" held at the 23rd Linz-Seminar of Fuzzy Sets 2003 (cf.[13]). Answers to the second problem w.r.t. the L_2-metric are given in [9] and [14], a comprehensive account was presented at the 23rd Linz-Seminar (cf. [13]). However these results suffer from quite unsatisfactory conditions of integrability that the random fuzzy sets should fulfil. Moreover, only sufficient conditions are available which ensure that the Aumann-integral of a random fuzzy set belongs to the sample space under consideration. The conditions of integrability will be improved and the sufficient conditions will be completed by necessary ones.

As applications of the investigations dominated convergence theorems and strong laws of large numbers as well as central limit theorems will be derived. The obtained versions generalize and improve already known results from literature, especially those that were presented at the 23rd Linz-Seminar.

2 Random fuzzy sets

Let $\mathcal{K}_{co}^{+}(\mathbb{R}^k)$ gather all nonvoid convex compact subsets of $\mathbb{R}^k$. We will restrict ourselves to the sample space $F_{coc}^{no}(\mathbb{R}^k)$ which consists of all fuzzy subsets of $\mathbb{R}^k$ with $\alpha-$cuts belonging to $\mathcal{K}_{co}^{+}(\mathbb{R}^k)$. Applying Zadeh's extension principle we can define on $F_{coc}^{no}(\mathbb{R}^k)$ a semilinear structure $\{\oplus_F, \lambda\odot_F \mid \lambda \in \mathbb{R}\}$. It turns out that it is inherited from the Minkowski operations on $\mathcal{K}_{co}^{+}(\mathbb{R}^k)$ on the $\alpha-$cuts, that is

$$[\widetilde{A} \oplus_F \widetilde{B}]^\alpha = [\widetilde{A}]^\alpha \oplus [\widetilde{B}]^\alpha,\ [\lambda \odot_F \widetilde{A}]^\alpha = \lambda \odot [\widetilde{A}]^\alpha$$

for all $\widetilde{A}, \widetilde{B} \in F_{coc}^{no}(\mathbb{R}^k),\ \lambda \in \mathbb{R},\ \alpha \in]0,1]$ (c.f. e.g. [4]). The fuzzy subset of $\mathbb{R}^k$ with $1_{\{0\}}$ as membership function will be denoted by $\widetilde{0}$. It is the neutral element w.r.t. $\oplus_F$.

Due to a widely used suggestion by Puri and Ralescu ([15]) we can extend the notion of random compact sets to $F_{coc}^{no}(\mathbb{R}^k)$ in the following way:

Each mapping $\widetilde{Y} : \Omega \to F_{coc}^{no}(\mathbb{R}^k)$ is associated with its $\alpha-$cut-mappings

$$[\widetilde{Y}]^\alpha : \Omega \to \mathcal{K}_{co}^{+}(\mathbb{R}^k),\ \omega \mapsto [\widetilde{Y}(\omega)]^\alpha\ (\alpha \in]0,1])$$

Puri and Ralescu called a mapping $\widetilde{Y} : \Omega \to F_{coc}^{no}(\mathbb{R}^k)$ a fuzzy random variable over some probability space $(\Omega, \mathcal{F}, \mathrm{P})$ if all the $\alpha-$cut-mappings are convex-valued random compact sets over $(\Omega, \mathcal{F}, \mathrm{P})$. However from the point of view of general probability theory this definition is not convenient since there is not any natural notion of distribution emerging from it. Therefore it is more reasonable to conceptualize random fuzzy sets as $F_{coc}^{no}(\mathbb{R}^k)-$valued measurable mappings. For this purpose we need a suitable $\sigma-$algebra on $F_{coc}^{no}(\mathbb{R}^k)$. The suggestion below was introduced the first time in [10].

Since every fuzzy subset of $\mathbb{R}^k$ is uniquely determined by it positive rational $\alpha-$cuts we may deduce a topology $\tau_{F_{coc}^{no}}$ on $F_{coc}^{no}(\mathbb{R}^k)$ from the product topology $\tau_{p\delta_\infty}$ on $\mathcal{K}_{co}^{+}(\mathbb{R}^k)^{]0,1]\cap\mathbb{Q}}$ w.r.t. the Hausdorff metric δ_∞, which is separably metrizable.

Now a mapping $\widetilde{Y} : \Omega \to F_{coc}^{no}(\mathbb{R}^k)$ is defined to be a **random fuzzy set** if it is Borel-measurable w.r.t. $\tau_{F_{coc}^{no}}$. The image measure under $\widetilde{Y}$ is called the **distribution of** $\widetilde{Y}$. Since $\tau_{F_{coc}^{no}}$ is metrizable, every random fuzzy set is a random element in $F_{coc}^{no}(\mathbb{R}^k)$ w.r.t. to every metric which induces $\tau_{F_{coc}^{no}}$. Therefore it is natural to speak of a **simple random fuzzy set** in the case that a random fuzzy set has only a finite range. In fact the introduced notion of random fuzzy sets is equivalent with the concept of fuzzy random variables by Puri and Ralescu (cf. [10]).

Other concepts to define random fuzzy sets are based on the identification of each fuzzy subset $\widetilde{A}$ from $F_{coc}^{no}(\mathbb{R}^k)$ with its support function $s_{\widetilde{A}} : [0,1] \times S^{k-1} \to \mathbb{R}$, where S^{k-1} denotes the euclidean unit sphere in $\mathbb{R}^k$. Every support function is measurable w.r.t the product $\sigma-$algebra consisting of the Borel subsets of $[0,1] \times S^{k-1}$ ([12]). This property is the basis to build the subspaces of fuzzy subsets with integrable support functions. Integrability will be defined w.r.t. $\lambda^1 \otimes \lambda^{S^{k-1}}$, the product measure of the Lebesgue-Borel measure λ^1 on $[0,1]$ and $\lambda^{S^{k-1}}$, the unit Lebesgue-Borel measure on S^{k-1}.

Let for $p \in [1,\infty[$ define the space $F^{no}_{cocp}(\mathbb{R}^k)$ to consist of fuzzy subsets from $F^{no}_{coc}(\mathbb{R}^k)$ with support functions being $\lambda^1 \otimes \lambda^{S^{k-1}}-$integrable of order p. Additionally, let $F^{no}_{coc\infty}(\mathbb{R}^k)$ be the space of all fuzzy subsets from $F^{no}_{coc}(\mathbb{R}^k)$ with support functions being essentially bounded w.r.t. $\lambda^1 \otimes \lambda^{S^{k-1}}$. Indeed this space gathers all the fuzzy subsets from $F^{no}_{coc}(\mathbb{R}^k)$ with bounded supports. The restriction of the semilinear structure $\{\oplus_F, \lambda \odot_F \mid \lambda \in \mathbb{R}\}$ to $F^{no}_{cocp}(\mathbb{R}^k)$ is well defined for every $p \in [1,\infty]$ (cf. [13]).

By the mapping $j_{F^{no}_{cocp}(\mathbb{R}^k)} : F^{no}_{cocp}(\mathbb{R}^k) \to L_p([0,1] \times S^{k-1})$ $(p \in [1,\infty])$, which identifies fuzzy subsets with the respective equivalence classes of their support functions every space $F^{no}_{cocp}(\mathbb{R}^k)$ can be embedded into the L_p-space $L_p([0,1] \times S^{k-1})$ w.r.t. $\lambda^1 \otimes \lambda^{S^{k-1}}$ as a positive cone (cf. [13]).

Using the L_p-norm on $L_p([0,1] \times S^{k-1})$ one can define a metric ρ_p on $F^{no}_{cocp}(\mathbb{R}^k)$, called the L_p-**metric**. Another custom concept is the so called $L_{p,\infty}-$**metric**, which is a completion of the metric on $F^{no}_{coc\infty}(\mathbb{R}^k)$ introduced by Klement, Puri and Ralescu in [8] (cf. [12]). Each pair ρ_p, d_p induces the same topology (cf. [12]), and in the case of $p = \infty$ both metrics are even identical (cf. [13]).

The L_p- and $L_{p,\infty}-$metrics give the opportunity to consider random elements in the subspaces $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. the respective L_p- or $L_{p,\infty}-$metric. Those random elements can be identified, via the embeddings $j_{F^{no}_{cocp}(\mathbb{R}^k)}$, with random elements in the L_p-spaces $L_p([0,1] \times S^{k-1})$. It turns out that in fact all these random elements are random fuzzy sets ([10], [13]).

Proposition 1. *Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space and let $\widetilde{Y} : \Omega \to F^{no}_{coc}(\mathbb{R}^k)$. Then we can state:*

.1 If $\widetilde{Y}$ is $F^{no}_{cocp}(\mathbb{R}^k)-$valued for $p \in [1,\infty[$, then the following statements are equivalent:

- *(i) $\widetilde{Y}$ is a random fuzzy set over $(\Omega, \mathcal{F}, \mathrm{P})$.*
- *(ii) $\widetilde{Y}$ is a random element in $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. ρ_p over $(\Omega, \mathcal{F}, \mathrm{P})$.*
- *(iii) $\widetilde{Y}$ is a random element in $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. d_p over $(\Omega, \mathcal{F}, \mathrm{P})$.*

.2 If $\widetilde{Y}$ is $F^{no}_{coc\infty}(\mathbb{R}^k)-$valued, then the following statements are equivalent:

- *(i) $\widetilde{Y}$ is a random fuzzy set over $(\Omega, \mathcal{F}, \mathrm{P})$.*
- *(ii) Every mapping $[\widetilde{Y}]^\alpha : \Omega \to \mathcal{K}^+_{co}(\mathbb{R}^k)$, $\omega \mapsto [\widetilde{Y}(\omega)]^\alpha$ $(\alpha \in [0,1])$ is a convex-valued random compact set over $(\Omega, \mathcal{F}, \mathrm{P})$, where $[\widetilde{Y}(\omega)]^0$ denotes the topological closure of the support of $\widetilde{Y}(\omega)$.*

.3 If $\widetilde{Y}$ is a random element in $F^{no}_{coc\infty}(\mathbb{R}^k)$ w.r.t. $\rho_\infty = d_\infty$ over $(\Omega, \mathcal{F}, \mathrm{P})$, then it is a random fuzzy set over $(\Omega, \mathcal{F}, \mathrm{P})$. The converse is not necessarily true.

♦

3 Integrably bounded random fuzzy sets

Since every random fuzzy set can be regarded as an extended convex-valued random compact set, it suggests itself to define integrals for random fuzzy sets by carrying over Aumann's well accepted concept (cf. [1]). Related to a random fuzzy set $\widetilde{Y}$ the task is to find a fuzzy subset $\widetilde{E}^A\widetilde{Y} \in F^{no}_{coc}(\mathbb{R}^k)$ which satisfies $[\widetilde{E}^A\widetilde{Y}]^\alpha = E^A[\widetilde{Y}]^\alpha$ for all $\alpha \in]0,1]$. Since the sets $E^A[\widetilde{Y}]^\alpha$ $(\alpha \in]0,1])$ should belong to $\mathcal{K}^+_{co}(\mathbb{R}^k)$, the $\alpha-$cut mappings $[\widetilde{Y}]^\alpha$ $(\alpha \in]0,1])$ have to be integrably bounded convex-valued random compact sets.

A random fuzzy set with integrably bounded $\alpha-$cut mappings is known as an **integrably bounded random fuzzy set** (cf. [15]). Indeed the concept of integrably bounded random fuzzy sets is sufficient to find the desired extended **Aumann-integral** (cf. [15], Theorem 3.1).

The Aumann-integral for a simple random fuzzy set $\widetilde{Y}$ with distribution $Q_{\widetilde{Y}}$ and different outcomes $\widetilde{A}_1,...,\widetilde{A}_m \in F^{no}_{coc}(\mathbb{R}^k)$ may be easily calculated as

$$\widetilde{E}^A\widetilde{Y} = (Q_{\widetilde{Y}}\widetilde{Y}^{-1}(\widetilde{A}_1) \odot_F \widetilde{A}_1) \oplus_F ... \oplus_F (Q_{\widetilde{Y}}\widetilde{Y}^{-1}(\widetilde{A}_m) \odot_F \widetilde{A}_m)$$

Since $\tau_{F^{no}_{coc}}$ is separably metrizable, every integrably bounded random fuzzy set can be approximated pointwise by a sequence of simple random fuzzy sets. So we can raise the question whether the Aumann-integral of integrably bounded random fuzzy sets can be described as a kind of Bochner-integral? Is it possible to attain the Aumann-integral of integrably bounded random fuzzy sets as limit points of sequences of Aumann-integrals of simple random fuzzy sets? The answer is affirmative as the following theorem shows.

Theorem 2. *Let d be a metric on $F^{no}_{coc}(\mathbb{R}^k)$ that induces $\tau_{F^{no}_{coc}}$, let δ_∞ be the Hausdorff metric on $\mathcal{K}^+_{co}(\mathbb{R}^k)$, and let $\widetilde{Y} : \Omega \to F^{no}_{coc}(\mathbb{R}^k)$ be a random fuzzy set over some probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then the following statements are equivalent:*

.1 $\widetilde{Y}$ is integrably bounded.

.2 There exists some $A \in \mathcal{F}, \mathrm{P}A = 1$, and a sequence $(\widetilde{Y}_n)_n$ of simple random fuzzy sets over $(\Omega, \mathcal{F}, \mathrm{P})$ such that

(i) $\lim\limits_{n\longrightarrow\infty} d(\widetilde{Y}_n(\omega), \widetilde{Y}(\omega)) = 0$ *for all* $\omega \in A$.

(ii) $\sup\limits_n \delta_\infty([\widetilde{Y}_n]^\alpha, \{0\})$ *is* $\mathrm{P}-$*integrable for every* $\alpha \in]0,1] \cap \mathbb{Q}$.

If one of the statements .1, .2 is satisfied, then $\lim\limits_{n\longrightarrow\infty} d(\widetilde{E}^A\widetilde{Y}_n, \widetilde{E}^A\widetilde{Y}) = 0$ *holds for any sequence* $(\widetilde{Y}_n)_n$ *of simple random fuzzy sets as in statement .2.* ♦

Remark:

Theorem 2 is an extension of a classical result from the theory of random compact sets: Debreu suggested a kind of Bochner-integral for convex-valued random compact sets. He has shown that it coincides with the Aumann-integral in the case of integrably bounded convex-valued random compact sets (cf. [3]; see also [7]). The characterization of the integral by statement .2 of Theorem 2 may be regarded as an generalization of Debreu's concept. Moreover, the extensions of the Aumann- as well as Debreu-integrals coincide. ♦

Considering random fuzzy sets with outcomes in the spaces $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. the respective L_p- or $L_{p,\infty}-$metrics it is interesting to find necessary and sufficient conditions which characterize them as integrably bounded with Aumann-integrals belonging to the respective subspace. The following theorem gives a complete answer to this problem.

Theorem 3. *Let $p \in [1,\infty]$ and let $\widetilde{Y} : \Omega \to F^{no}_{cocp}(\mathbb{R}^k)$ denote a random fuzzy set over some probability space $(\Omega, \mathcal{F}, \mathrm{P})$.*

.1 For $p = 1$ the random fuzzy set $\widetilde{Y}$ is integrably bounded with $\widetilde{E}^A\widetilde{Y} \in F^{no}_{coc1}(\mathbb{R}^k)$ if and only if either $\rho_1(\widetilde{Y}, \widetilde{0})$ or $d_1(\widetilde{Y}, \widetilde{0})$ is $\mathrm{P}-$integrable. In this case $j_{F^{no}_{coc1}(\mathbb{R}^k)} \circ \widetilde{Y}$ is $\mathrm{P}-$Pettis-integrable and $j_{F^{no}_{coc1}(\mathbb{R}^k)}(\widetilde{E}^A\widetilde{Y})$ coincides with the Pettis integral of $j_{F^{no}_{coc1}(\mathbb{R}^k)} \circ \widetilde{Y}$.

.2 For $p \in]1,\infty[$ the random fuzzy set $\widetilde{Y}$ is integrably bounded with $\widetilde{E}^A\widetilde{Y} \in F^{no}_{cocp}(\mathbb{R}^k)$ if and only if $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$ is $\mathrm{P}-$Pettis-integrable as well as either $\rho_1(\widetilde{Y}, \widetilde{0})$ or $d_1(\widetilde{Y}, \widetilde{0})$ is $\mathrm{P}-$integrable. In this case $j_{F^{no}_{cocp}(\mathbb{R}^k)}(\widetilde{E}^A\widetilde{Y})$ coincides with the Pettis integral of $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$.

.3 For $p = \infty$ the random fuzzy set $\widetilde{Y}$ is integrably bounded with $\widetilde{E}^A\widetilde{Y} \in F^{no}_{coc\infty}(\mathbb{R}^k)$ if and only if $\rho_\infty(\widetilde{Y}, \widetilde{0}) = d_\infty(\widetilde{Y}, \widetilde{0})$ is $\mathrm{P}-$integrable.

♦

Remark:

In [13] it has been shown that some $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy set $\widetilde{Y}$ over a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ is integrably bounded with $\widetilde{E}^A\widetilde{Y} \in F^{no}_{cocp}(\mathbb{R}^k)$ if $\rho_p(\widetilde{Y}, \widetilde{0})$ is

$$\begin{cases} \mathrm{P}-\text{integrable of order } p & : \; p \in [1,\infty[\\ \mathrm{P}-\text{integrable} & : \; p = \infty \end{cases}.$$

This result is now improved by Theorem 3. Moreover, the converse direction has been found. ♦

4 Pettis-integrable random fuzzy sets

If $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy sets are considered as random elements w.r.t. the L_p-metric ρ_p or the $L_{p,\infty}-$metric d_p, they can be identified, via the standard embedding, with random elements in a real Banach space. So the concepts of Pettis- or Bochner-integrals may be used as alternative ways to define integrals for random fuzzy sets. This section deals with the approach inspired by the Pettis-integration.

Definition 4. Let $p \in [1,\infty]$ and let $\widetilde{Y} : \Omega \to F^{no}_{cocp}(\mathbb{R}^k)$ denote a random fuzzy set over some probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then $\widetilde{Y}$ will be defined as $\mathrm{P}-$**Pettis-integrable w.r.t.** ρ_p if it satisfies the following properties

(i) $\widetilde{Y}$ is a random element in $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. ρ_p over $(\Omega, \mathcal{F}, \mathrm{P})$.

(ii) $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$ is $\mathrm{P}-$Pettis-integrable.

(iii) There exists some $\widetilde{E}^P\widetilde{Y} \in F^{no}_{cocp}(\mathbb{R}^k)$ with $j_{F^{no}_{cocp}(\mathbb{R}^k)}(\widetilde{E}^P\widetilde{Y})$ being identical with the Pettis-integral of $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$.

If $\widetilde{Y}$ is $\mathrm{P}-$Pettis-integrable w.r.t. ρ_p, then $\widetilde{E}^P\widetilde{Y}$ will be called the **Pettis-integral of** $\widetilde{Y}$. ♦

Remark:
Since the embedding $j_{F^{no}_{cocp}(\mathbb{R}^k)}$ is injective, the fuzzy subset $\widetilde{E}^P\widetilde{Y}$ is unique if it exists. ♦

One can derive the following relationship between Aumann- and Pettis-integration of $F^{no}_{cocp}(\mathbb{R}^k)$-valued random fuzzy sets for $p \in [1,\infty]$.

Theorem 5. *Let $p \in [1,\infty]$ and let $\widetilde{Y} : \Omega \to F^{no}_{cocp}(\mathbb{R}^k)$ denote a random fuzzy set over some probability space $(\Omega, \mathcal{F}, \mathrm{P})$.*

.1 If $p \in [1,\infty[$, and if $\widetilde{Y}$ is integrably bounded with $\widetilde{E}^A\widetilde{Y} \in F^{no}_{cocp}(\mathbb{R}^k)$, then $\widetilde{Y}$ is $\mathrm{P}-$Pettis-integrable and $\widetilde{E}^A\widetilde{Y} = \widetilde{E}^P\widetilde{Y}$.

.2 If $d_\infty(\widetilde{Y},\widetilde{0}) = \rho_\infty(\widetilde{Y},\widetilde{0})$ is $\mathrm{P}-$integrable, then $\widetilde{Y}$ is $\mathrm{P}-$Pettis-integrable with $\widetilde{E}^A\widetilde{Y} = \widetilde{E}^P\widetilde{Y}$.

♦

Remark:
As remarked above after Theorem 3, Theorem 5 improves a former result for $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy sets $\widetilde{Y}$ with $\rho_p(\widetilde{Y},\widetilde{0})$ being integrable of order p for $p \in [1,\infty[$. In particular, it also improves a result by Näther who has shown the result of Theorem 5 for $F^{no}_{coc2}(\mathbb{R}^k)-$valued random fuzzy sets $\widetilde{Y}$ with $\rho_2(\widetilde{Y},\widetilde{0})$ being integrable of order 2 (cf. [14]). ♦

5 Bochner-integrable random fuzzy sets

Analogously to Pettis-integrability one can develop Bochner-integration of random fuzzy sets.

Definition 6. Let $p \in [1,\infty]$ and let $\widetilde{Y} : \Omega \to F^{no}_{cocp}(\mathbb{R}^k)$ denote a random fuzzy set over some probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then $\widetilde{Y}$ will be defined as $\mathrm{P}-$**Bochner-integrable w.r.t.** ρ_p if it satisfies the following properties

(i) $\widetilde{Y}$ is a separably-valued random element in $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. ρ_p over $(\Omega, \mathcal{F}, \mathrm{P})$.

(ii) $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$ is $\mathrm{P}-$Bochner-integrable.

(iii) There exists some $\widetilde{E}^B\widetilde{Y} \in F^{no}_{cocp}(\mathbb{R}^k)$ with $j_{F^{no}_{cocp}(\mathbb{R}^k)}(\widetilde{E}^B\widetilde{Y})$ being identical with the Bochner-integral of $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$.

If $\widetilde{Y}$ is $\mathrm{P}-$Bochner-integrable w.r.t. ρ_p, then $\widetilde{E}^B\widetilde{Y}$ will be called the **Bochner-integral of** $\widetilde{Y}$. ♦

Remarks:

1) Since each embedding $j_{F^{no}_{cocp}(\mathbb{R}^k)}$ is injective, the fuzzy subset $\widetilde{E}^B\widetilde{Y}$ is unique if it exists.

2) Observe that for every $p \in [1,\infty[$ the $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy sets are exactly the separably-valued random elements in $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. ρ_p since ρ_p is separable.

3) For every $p \in [1,\infty]$ each simple random element in $F^{no}_{cocp}(\mathbb{R}^k)$ w.r.t. ρ_p is integrably bounded as well as Bochner-integrable, and $\widetilde{E}^A\widetilde{Y} = \widetilde{E}^B\widetilde{Y}$ holds.

♦

It turns out that for $p \in [1,\infty[$ Aumann- and Bochner-integration of $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy sets are closely related.

Theorem 7. *Let $p \in [1,\infty[$, and let $\widetilde{Y} : \Omega \to F^{no}_{cocp}(\mathbb{R}^k)$ denote a random fuzzy set over some probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then the following statements are equivalent:*

.1 $\widetilde{Y}$ *is* $\mathrm{P}-$*Bochner-integrable.*

.2 $\widetilde{Y}$ *is integrably bounded with* $\widetilde{E}^A\widetilde{Y} \in F^{no}_{cocp}(\mathbb{R}^k)$, *and* $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$ *is* $\mathrm{P}-$*Bochner-integrable with* $j_{F^{no}_{cocp}(\mathbb{R}^k)}(\widetilde{E}^A\widetilde{Y})$ *being identical with the Bochner-integral of* $j_{F^{no}_{cocp}(\mathbb{R}^k)} \circ \widetilde{Y}$.

.3 Either $\rho_p(\widetilde{Y},\widetilde{0})$ *or* $d_p(\widetilde{Y},\widetilde{0})$ *is* $\mathrm{P}-$*integrable.*

.4 There exists some sequence $(\widetilde{Y}_n)_n$ *of simple* $F^{no}_{cocp}(\mathbb{R}^k)-$*valued random fuzzy sets over* $(\Omega, \mathcal{F}, \mathrm{P})$ *which satisfy*

(i) $\lim_{n\longrightarrow\infty} \rho_p(\widetilde{Y}_n(\omega), \widetilde{Y}(\omega)) = 0$ *for all* $\omega \in \Omega$

(ii) $\lim_{n\longrightarrow\infty} E\rho_p(\widetilde{Y}_n, \widetilde{Y}) = 0$

If any of the statements .1 - .4 is fulfilled, then $\widetilde{Y}$ *is integrably bounded with* $\widetilde{E}^A\widetilde{Y} \in F^{no}_{cocp}(\mathbb{R}^k)$, *and*

$$\widetilde{E}^A\widetilde{Y} = \widetilde{E}^B\widetilde{Y} \text{ as well as } \lim_{n\longrightarrow\infty} \rho_p(\widetilde{E}^A\widetilde{Y}_n, \widetilde{E}^A\widetilde{Y}) = 0$$

whenever $(\widetilde{Y}_n)_n$ *is a sequence of simple* $F^{no}_{cocp}(\mathbb{R}^k)-$*valued random fuzzy sets as in statement .4.* ♦

Remark:

Theorem 7 improves a former result in [13] where it has been shown that a $F^{no}_{cocp}(\mathbb{R}^k)-$ valued random fuzzy set $\widetilde{Y}$ satisfies statement .2 of Theorem 7 if $\rho_p(\widetilde{Y},\widetilde{0})$ is integrable of order p. In particular also a result by Körner is improved who has proved that statement .2 of Theorem 7 holds for a $F^{no}_{coc2}(\mathbb{R}^k)-$valued random fuzzy set $\widetilde{Y}$ with $\rho_2(\widetilde{Y},\widetilde{0})$ being integrable of order 2 (cf. [9]). ♦

In the case of $F^{no}_{coc\infty}(\mathbb{R}^k)-$valued random fuzzy sets the concept of Bochner-integrals are much more restrictive than Aumann-integration.

Theorem 8. *Let* $\widetilde{Y} : \Omega \to F^{no}_{coc\infty}(\mathbb{R}^k)$ *denote a random fuzzy set over some probability space* $(\Omega, \mathcal{F}, \mathrm{P})$. *Then the following statements are equivalent:*

.1 $\widetilde{Y}$ *is* $\mathrm{P}-$*Bochner-integrable.*

.2 $\widetilde{Y}$ *is integrably bounded with* $\widetilde{E}^A\widetilde{Y} \in F^{no}_{coc\infty}(\mathbb{R}^k)$, *and* $j_{F^{no}_{coc\infty}(\mathbb{R}^k)} \circ \widetilde{Y}$ *is* $\mathrm{P}-$*Bochner-integrable with* $j_{F^{no}_{coc\infty}(\mathbb{R}^k)}(\widetilde{E}^A\widetilde{Y})$ *being identical with the Bochner-integral of* $j_{F^{no}_{coc\infty}(\mathbb{R}^k)} \circ \widetilde{Y}$.

.3 $\widetilde{Y}$ *is a separably-valued random element in* $F^{no}_{coc\infty}(\mathbb{R}^k)$ *w.r.t.* $\rho_\infty = d_\infty$ *over* $(\Omega, \mathcal{F}, \mathrm{P})$, *and* $\rho_\infty(\widetilde{Y},\widetilde{0}) = d_\infty(\widetilde{Y},\widetilde{0})$ *is* $\mathrm{P}-$*integrable.*

.4 There exists some sequence $(\widetilde{Y}_n)_n$ *of simple* $F^{no}_{coc\infty}(\mathbb{R}^k)-$*valued random fuzzy sets over* $(\Omega, \mathcal{F}, \mathrm{P})$ *which satisfy*

(i) $\lim_{n\longrightarrow\infty} \rho_\infty(\widetilde{Y}_n(\omega),\widetilde{Y}(\omega)) = 0$ *for all* $\omega \in \Omega$

(ii) $\lim_{n\longrightarrow\infty} E\rho_\infty(\widetilde{Y}_n,\widetilde{Y}) = 0$

If any of the statements .1 - .4 is fulfilled, then $\widetilde{Y}$ *is integrably bounded with* $\widetilde{E}^A\widetilde{Y} \in F^{no}_{coc\infty}(\mathbb{R}^k)$, *and*

$$\widetilde{E}^A\widetilde{Y} = \widetilde{E}^B\widetilde{Y} \text{ as well as } \lim_{n\longrightarrow\infty} \rho_\infty(\widetilde{E}^A\widetilde{Y}_n,\widetilde{E}^A\widetilde{Y}) = 0$$

whenever $(\widetilde{Y}_n)_n$ *is a sequence of simple* $F^{no}_{coc\infty}(\mathbb{R}^k)-$*valued random fuzzy sets as in statement .4.* ♦

Every Bochner-integrable random fuzzy set is Pettis-integrable, and the Bochner- coincides with the Pettis-integral. This is a result in accordance with the integration of random elements in Banach spaces.

Theorem 9. *Let* $p \in [1,\infty]$ *and let* $\widetilde{Y} : \Omega \to F^{no}_{cocp}(\mathbb{R}^k)$ *be a random fuzzy set over some probability space* $(\Omega,\mathcal{F},\mathrm{P})$.

If $\widetilde{Y}$ *is* $\mathrm{P}-$*Bochner-integrable w.r.t.* ρ_p, *then it is also* $\mathrm{P}-$*Pettis-integrable w.r.t.* ρ_p *with* $\widetilde{E}^B\widetilde{Y} = \widetilde{E}^P\widetilde{Y}$.

♦

6 Dominated convergence theorems for random fuzzy sets

As a quite easy application of the discussion on integrability of random fuzzy sets we can derive several dominated convergence theorems, dependent on the sample space.

Theorem 10. *Let* $\{\widetilde{Y},\widetilde{Y}_n \mid n \in \mathbb{N}\}$ *be a set of integrably bounded random fuzzy sets over some probability space* $(\Omega,\mathcal{F},\mathrm{P})$ *such that* $\sup_n \delta_\infty([\widetilde{Y}_n]^\alpha,\{0\})$ *is* $\mathrm{P}-$*integrable for every* $\alpha \in]0,1] \cap \mathbb{Q}$.

If $\lim_{n\longrightarrow\infty} d(\widetilde{Y}_n,\widetilde{Y}) = 0$ *a.s. holds for any metric d which induces* $\tau_{F^{no}_{coc}}$, *then* $\lim_{n\longrightarrow\infty} d(\widetilde{E}^A\widetilde{Y}_n,\widetilde{E}^A\widetilde{Y}) = 0$.
♦

Dealing with $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy sets ($p \in [1,\infty]$), one obtains the following dominated convergence theorem.

Theorem 11. *Let* $p \in [1,\infty]$ *be fixed and let* $\{\widetilde{Y},\widetilde{Y}_n \mid n \in \mathbb{N}\}$ *be a set of* $F^{no}_{cocp}(\mathbb{R}^k)-$*valued random fuzzy sets over some probability space* $(\Omega,\mathcal{F},\mathrm{P})$ *which satisfy*

(i) $\rho_p(\widetilde{Y},\widetilde{0})$ *is* $\mathrm{P}-$*integrable.*

(ii) $\sup_n \rho_p(\widetilde{Y}_n,\widetilde{0})$ *is* $\mathrm{P}-$*integrable.*

.1 If $p \in [1,\infty[$, *and if* $\lim_{n\longrightarrow\infty} \rho_p(\widetilde{Y}_n,\widetilde{Y}) = 0$ *a.s., then*

$$\begin{aligned} & \lim_{n\longrightarrow\infty} \rho_p(\widetilde{E}^A\widetilde{Y}_n,\widetilde{E}^A\widetilde{Y}) = \lim_{n\longrightarrow\infty} \rho_p(\widetilde{E}^P\widetilde{Y}_n,\widetilde{E}^P\widetilde{Y}) = \lim_{n\longrightarrow\infty} \rho_p(\widetilde{E}^B\widetilde{Y}_n,\widetilde{E}^B\widetilde{Y}) \\ = & \lim_{n\longrightarrow\infty} d_p(\widetilde{E}^A\widetilde{Y}_n,\widetilde{E}^A\widetilde{Y}) = \lim_{n\longrightarrow\infty} d_p(\widetilde{E}^P\widetilde{Y}_n,\widetilde{E}^P\widetilde{Y}) = \lim_{n\longrightarrow\infty} d_p(\widetilde{E}^B\widetilde{Y}_n,\widetilde{E}^B\widetilde{Y}) \\ = & \ 0 \end{aligned}$$

.2 *If* $\lim_{n\longrightarrow\infty} \rho_\infty(\widetilde{Y}_n,\widetilde{Y}) = 0$ *a.s., then* $\lim_{n\longrightarrow\infty} d_\infty(\widetilde{E}^A\widetilde{Y}_n,\widetilde{E}^A\widetilde{Y}) = \lim_{n\longrightarrow\infty} \rho_\infty(\widetilde{E}^A\widetilde{Y}_n,\widetilde{E}^A\widetilde{Y}) = 0$

♦

Remark:

Statement .2 of Theorem 11 is known from [13], whereas statement .2 generalizes and improves already known dominated convergence theorems:

- All the previous results are formulated w.r.t. the Aumann-integrals only.
- The respective theorems in [8] and [11] are restricted to $F^{no}_{coc\infty}(\mathbb{R}^k)-$valued random fuzzy sets assuming that the mappings $d_\infty(\widetilde{Y}_n,\widetilde{0})$ and $d_\infty(\widetilde{Y},\widetilde{0})$ are integrable.
- In [13] the results from [8] and [11] have been extended to arbitrary $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy sets $\widetilde{Y},\widetilde{Y}_n$ under the quite unsatisfactory condition that the mappings $\rho_p(\widetilde{Y},\widetilde{0})$ and $\rho_p(\widetilde{Y}_n,\widetilde{0})$ are integrable of order p for $p \in [1,\infty[$.

7 Strong law of large numbers and central limit theorems for random fuzzy sets

Since $F^{no}_{cocp}(\mathbb{R}^k)-$valued integrably bounded random fuzzy sets are closely related with Bochner-integrable random elements in $L_p([0,1]\times S^{k-1})$ for $p \in [1,\infty[$, we can make use of limit theorems for random elements in real Banach spaces to obtain strong laws of large numbers and central limit theorems for random fuzzy sets.

Considering pairwise independent, identically distributed random fuzzy sets, Etemadi's strong law of large numbers ([5]) may be applied since $(L_p([0,1]\times S^{k-1}), \|\cdot\|_p)$ is a real separable Banach space for $p \in [1,\infty[$.

Theorem 12. *Let* $p \in [1,\infty[$, *and let* $(\widetilde{Y}_n)_n$ *denote a sequence of pairwise independent, identically distributed* $F^{no}_{cocp}(\mathbb{R}^k)-$*valued random fuzzy sets over some probability space* $(\Omega,\mathcal{F},\mathrm{P})$.

If $\rho_p(\widetilde{Y}_1,\widetilde{0})$ *is* $\mathrm{P}-$*integrable, then* $\lim_{n\longrightarrow\infty} \rho_p(\frac{1}{n}\odot_F(\widetilde{Y}_1\oplus_F\ldots\oplus_F\widetilde{Y}_n),\widetilde{E}^A\widetilde{Y}_1) = 0$ *a.s.*.

Conversely, if there exists some $\widetilde{A}_0 \in F^{no}_{cocp}(\mathbb{R}^k)$ *such that* $\lim_{n\longrightarrow\infty} \rho_p(\frac{1}{n}\odot_F(\widetilde{Y}_1\oplus_F\ldots\oplus_F\widetilde{Y}_n),\widetilde{A}_0) = 0$ *a.s. holds, then* $\rho_p(\widetilde{Y}_1,\widetilde{0})$ *is* $\mathrm{P}-$*integrable, in particular* $\widetilde{Y}_1$ *is integrably boundend as well as* $\mathrm{P}-$*Bochner- and* $\mathrm{P}-$*Pettis-integrable with* $\widetilde{E}^A\widetilde{Y}_1 = \widetilde{E}^B\widetilde{Y}_1 = \widetilde{E}^P\widetilde{Y}_1 = \widetilde{A}_0 \in F^{no}_{cocp}(\mathbb{R}^k)$. ♦

Remark:

Theorem 12 completes a result for $F^{no}_{coc\infty}(\mathbb{R}^k)-$valued random fuzzy sets by Colubi, Lopez-Diaz and Gil (cf. [2]). ♦

Hoffmann-Jorgensen and Pisier introduced a classification of Banach spaces, where classical strong laws of large numbers and central limit theorems can be extended immediately. Their investigations led to the concept of types of Banach spaces. The type of a Banach space is directly linked with the validity of certain limit theorems (cf. [6]). Since $(L_p([0,1]\times S^{k-1}), \|\cdot\|_p)$ is a real separable Banach space of type p in the case of $p \in [1,2]$ and of type 2 if $p \in [2,\infty[$, one can draw on the limit theorems for random elements in these classes of real Banach spaces.

Theorem 13. *Let* $p \in [1,\infty[$ *and* $(\widetilde{Y}_n)_n$ *be a sequence of independent* $F^{no}_{cocp}(\mathbb{R}^k)-$*valued random fuzzy sets over some probability space* $(\Omega,\mathcal{F},\mathrm{P})$ *such that*

(i) $\rho_p(\widetilde{Y}_n,\widetilde{0})$ *is* $\mathrm{P}-$*integrable for all* n,

(ii) $\sum_{n=1}^{\infty} \frac{E\rho_p(\widetilde{Y}_n,\widetilde{E}^A\widetilde{Y}_n)^q}{n^q} < \infty$ *for* $q=p$ *if* $p \in [1,2]$ *and* $q=2$ *if* $p\in[2,\infty[$.

Then

$$\lim_{n\longrightarrow\infty} \rho_p\Big(\frac{1}{n}\odot_F(\widetilde{Y}_1\oplus_F \ldots \oplus_F \widetilde{Y}_n),\frac{1}{n}\odot_F(\widetilde{E}^A\widetilde{Y}_1\oplus_F\ldots\oplus_F\widetilde{E}^A\widetilde{Y}_n)\Big)=0 \ a.s.$$

♦

Theorem 14. *Let* $p\in[2,\infty[$, *and let* $(\widetilde{Y}_n)_n$ *denote a sequence of independent and identically distributed* $F^{no}_{cocp}(\mathbb{R}^k)-$*valued random fuzzy sets over some probability space* $(\Omega,\mathcal{F},\mathrm{P})$ *such that* $\rho_p(\widetilde{Y}_1,\widetilde{0})$ *is* $\mathrm{P}-$ *integrable of order* 2.

Then there exists a Gaussian element Z *in* $L_p([0,1]\times S^{k-1})$ *with vanishing Bochner-integral and covariance operator as* $j_{F^{no}_{cocp}(\mathbb{R}^k)}\circ\widetilde{Y}_1$ *such that the sequence* $\Big(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} j_{F^{no}_{cocp}(\mathbb{R}^k)}\circ\widetilde{Y}_i-\sqrt{n}\,j_{F^{no}_{cocp}(\mathbb{R}^k)}(\widetilde{E}^A\widetilde{Y}_1)\Big)_n$ *converges weakly to* Z. ♦

Remark:

Theorems 13, 14 improve corresponding previous results in [11] (Theorems 5.1, 5.2) and [13] (Theorems 5, 6) where sequences of independent $F^{no}_{cocp}(\mathbb{R}^k)-$valued random fuzzy sets are considered with $\rho_p(\widetilde{Y}_n,\widetilde{0})$ being integrable of order p for all n. ♦

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Copulas and characterization of T-product possibility measures

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1 Introduction

The aim of this contribution is a partial characterization of T-product possibility measures, where T is a t-norm satisfying the Lipschitz property with the constant 1. Any possibility measure can be assigned a set of distribution functions which are dominated by this possibility measure. It will be demonstrated that the set of all joint distribution functions dominated by a T-product possibility measure contains each joint distribution function obtained by an application of a copula $C \leq T$ to some marginal distribution functions dominated by marginal possibility measures.

2 Basic Notions

2.1 Possibility Theory and t-norms

See [1] for a thorough theoretical exposition. Let X be a non-empty set and $\mathcal{A}$ be a *complete Boolean algebra* of its subsets: $\mathcal{A}$ contains X and it is closed under complementation and arbitrary unions. Consequently, $\mathcal{A}$ is closed under arbitrary intersections. A *possibility measure* Π on X is a set function $\Pi : \mathcal{A} \to [0,1]$ such that for any family $(A_i)_{i\in I}$ of elements of $\mathcal{A}$ the condition

$\Pi(\bigcup_{i\in I} A_i) = \sup_{i\in I} \Pi(A_i)$ is satisfied and $\Pi(X) = 1$. The last condition means that only *normal* possibility measures are considered. A *possibility distribution* π is a mapping $\pi : X \to [0,1]$ such that $\Pi(A) = \sup_{x\in A} \pi(x)$ for any $A \in \mathcal{A}$ and $\pi^{-1}(\{a \in [0,1] : a \leq a'\}) \in \mathcal{A}$ for any $a' \in [0,1]$. A purely technical requirement is that X always contains an element $\underline{x}$ such that $\pi(\underline{x}) = 0$. Assume that $\Pi_{X\times Y}$ is a possibility measure on a Cartesian product $X \times Y$. Then its *marginal possibility measure* Π_X is uniquely determined by the formula $\Pi_X(A) := \Pi_{X\times Y}(A \times Y)$, $A \in \mathcal{A}$.

The *t-norm* T is a commutative, associative and monotone binary operation on $[0,1]$ with the neutral element 1. Significant examples of (continuous) t-norms are these: the t-norm *minimum* $T_M(a,b) = \min(a,b)$, the *product t-norm* $T_P(a,b) = ab$ and the *Łukasiewicz' t-norm* $T_L(a,b) = \max(0, a+b-1)$. If Π_X, Π_Y are possibility measures on X, Y, respectively, and T is a continuous t-norm, then we say that $\Pi^T_{X\times Y} : \mathcal{A}_X \otimes \mathcal{A}_Y \to [0,1]$ is a *T-product possibility measure* on $X \times Y$, where $\mathcal{A}_X \otimes \mathcal{A}_Y \subseteq 2^{X\times Y}$ is a *product algebra* of $\mathcal{A}_X$ and $\mathcal{A}_Y$, if

$$\Pi^T_{X\times Y}(A \times B) = T(\Pi_X(A), \Pi_Y(B)), \quad A \in \mathcal{A}_X, B \in \mathcal{A}_Y. \tag{1}$$

A notion of the T-product possibility measure was introduced in [1] and it is evidently a more general analog of the product probability measure used in classical probability theory.

2.2 Copulas

A *copula* C is a binary operation on $[0,1]$ such that

1. for every $a,b \in [0,1]$,
$$C(a,0) = C(0,b) = 0,$$
and
$$C(a,1) = a \quad \text{and} \quad C(1,b) = b;$$
2. for every $a_1,a_2,b_1,b_2 \in [0,1]$ such that $a_1 \leq a_2$ and $b_1 \leq b_2$,
$$C(a_2,b_2) - C(a_2,b_1) - C(a_1,b_2) + C(a_1,b_1) \geq 0.$$

The t-norms T_M, T_P, T_L are all copulas. Moreover, for every copula C and $(a,b) \in [0,1]^2$,

$$T_L(a,b) \leq C(a,b) \leq T_M(a,b). \tag{2}$$

Any t-norm T is a copula if and only if T satisfies Lipschitz property with the constant 1 [3].

3 Characterization of T-product Possibilities

Each possibility measure Π_X on X can be assigned a set $\mathcal{P}_{\Pi_X}$ of finitely-additive probability measures P_X on X dominated by Π_X:

$$\mathcal{P}_{\Pi_X} := \Big\{ P_X : P_X(A) \leq \Pi_X(A),\ A \in \mathcal{A}_X \}. \tag{3}$$

It was proven in [2] that Π_X is even an *upper envelope* of $\mathcal{P}_{\Pi_X}$, i.e.

$$\Pi_X(A) = \sup_{P_X \in \mathcal{P}_{\Pi_X}} P_X(A), \quad A \in \mathcal{A}_X. \tag{4}$$

Instead of probability measures, distribution functions can be considered. Let $\preceq$ be the total ordering on X agreeing with the one given by values of the possibility distribution π_X, that is $x_1 \preceq x_2$ iff $\pi_X(x_1) \leq \pi_X(x_2)$. Let then $\overline{x}$ and $\underline{x}$ denote the greatest and the lowest element of X in this ordering, respectively. Consequently, the mapping $F_X : X \to [0,1]$ defined by

$$F_X(x) := P_X(\{x' \in X : x' \preceq x\}), \quad x \in X, \tag{5}$$

is a *distribution function* since F_X is non-decreasing and $F_X(\overline{x}) = 1$, $F_X(\underline{x}) = 0$. We can define

$$\mathcal{F}_{\Pi_X} := \left\{ F_X : F(x) \leq \Pi_X(\{x' \in X : x' \preceq x\}),\, x \in X \right\}. \tag{6}$$

If F_X and F_Y are distribution functions on X and Y, respectively, and C is a copula, then the mapping

$$F^C_{X\times Y}(x,y) := C(F_X(x), F_Y(y)), \quad (x,y) \in X \times Y, \tag{7}$$

is the *joint distribution function* of F_X and F_Y on $X \times Y$.

Let us consider a T-product possibility measure $\Pi^T_{X\times Y}$, where the t-norm T satisfies Lipschitz property with the constant 1. Under this assumption, we can partially characterize the set of joint distribution functions dominated by $\Pi^T_{X\times Y}$.

Proposition 1. *Let Π_X, Π_Y be possibility measures on X, Y, respectively, $\Pi^T_{X\times Y}$ be a T-product possibility measure, where T is a t-norm satisfying Lipschitz property with the constant* 1*. Consider the set of copulas*

$$\mathcal{C}_T = \{C : C \leq T\}.$$

Then any $F_{X\times Y} \in \mathcal{F}_{\Pi^T_{X\times Y}}$ has marginal distribution functions $F_X \in \mathcal{F}_{\Pi_X}$, $F_Y \in \mathcal{F}_{\Pi_Y}$ and

$$\mathcal{F}_{\Pi^T_{X\times Y}} \supseteq \bigcup_{(F_X,F_Y)\in\mathcal{F}_{\Pi_X}\times\mathcal{F}_{\Pi_Y}} \left\{ F^C_{X\times Y} : F^C_{X\times Y} = C(F_X, F_Y),\, C \in \mathcal{C}_T \right\}. \tag{8}$$

Proof. For the sake of further brevity, let us stipulate that

$$\{x' \preceq x\} := \{x' \in X : x' \preceq x\}.$$

Let $F_{X\times Y} \in \mathcal{F}_{\Pi^T_{X\times Y}}$. Then

$$F_X(x) = F_{X\times Y}(x, \overline{y}) \leq \Pi^T_{X\times Y}(\{x' \preceq x\} \times Y) = \Pi_X(\{x' \preceq x\}),$$

and analogously for F_Y. Consequently, $F_X \in \mathcal{F}_{\Pi_X}$ and $F_Y \in \mathcal{F}_{\Pi_Y}$. Notice that the set $\mathcal{C}_T$ is always non-empty since T is also a copula and, according to (2), there is always at least one copula which is lower or equal to T. To prove the second part of the proposition, let us consider an arbitrary pair of marginal distribution functions $(F_X, F_Y) \in \mathcal{F}_{\Pi_X} \times \mathcal{F}_{\Pi_Y}$ and any copula $C \in \mathcal{C}_T$. Then for any pair $(x,y) \in X \times Y$,

$$F^C_{X\times Y}(x,y) = C(F_X(x), F_Y(y)) \leq T(F_X(x), F_Y(y)) \leq T(\Pi_X(\{x' \leq x\}), \Pi_Y(\{y' \leq y\})),$$

and thus $F^C_{X\times Y} \in \mathcal{F}_{\Pi^T_{X\times Y}}$. □

The previous proposition provides merely a partial characterization of T-product possibility measures: it can be demonstrated that in general case the set of joint distribution functions on the right-hand side of (8) is a proper subset of $\mathcal{F}_{\Pi^T_{X\times Y}}$. Nevertheless, a complete characterization is obtained in some special cases as the following example demonstrates.

Example 2. *If* $T = T_M$*, then* $\mathcal{C}_{T_M}$ *consists of all copulas since* T_M *is the greatest copula. Due to Proposition 1, any* $F_{X\times Y} \in \mathcal{F}_{\Pi^{T_M}_{X\times Y}}$ *has marginal distribution functions* F_X*,* F_Y *belonging to* $\mathcal{F}_{\Pi_X}$ *and* $\mathcal{F}_{\Pi_Y}$*, respectively. Sklar's theorem [4] now implies that there is a copula* C *such that* $F_{X\times Y} = C(F_X, F_Y)$ *and* $\mathcal{F}_{\Pi^{T_M}_{X\times Y}}$ *thus consists of the joint distribution functions obtained by an application of all copulas to all pairs* $(F_X, F_Y) \in \mathcal{F}_{\Pi_X} \times \mathcal{F}_{\Pi_Y}$*.*

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Semicontinuous L-real valued functions

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Recall that when X is a topological space, the lower and upper limit functions f_* and f^* of a given $f : X \to \mathbb{R}$ (or to $\overline{\mathbb{R}}$) are defined as follows:

$$f_*(x) = \bigvee\{\bigwedge f(U) : U \text{ is an open nbhd of } x\}$$

and

$$f^*(x) = \bigwedge\{\bigvee f(U) : U \text{ is an open nbhd of } x\}.$$

In this report we study lower and upper limits of L-real valued functions by extending the operators $(\cdot)_*$ and $(\cdot)^*$ to the framework of L-topological spaces. The approach we are taking involves scales of L-sets.

More specifically, let $(L,')$ be a complete lattice with an order-reversing involution and let X be a set. Each order-preserving family $\mathcal{A} = \{a_r : r \in \mathbb{Q}\cap\mathbb{R}\} \subset L^X$ is called an *extended scale* of L, and it is called a *scale* whenever $\bigvee \mathcal{A} = 1_X$ and $\bigwedge \mathcal{A} = 1_\emptyset$. It is well known that for every $x \in X$ and $t \in \mathbb{R}$, the function $\alpha_x : \mathbb{R} \to L$, defined by $\alpha_x(t) = \bigwedge_{r<t} a'_r(x)$, is order-reversing and, when $\mathcal{A}$ is a scale, then $\bigvee \alpha_x(\mathbb{R}) = 1$ and $\bigwedge \alpha_x(\mathbb{R}) = 0$. The function $f : X \to \mathbb{R}(L)$ (respectively, $\overline{\mathbb{R}}(L)$), defined by $f(x) = [\alpha_x]$, is said to be genrated by the scale (respectively, extended scale) $\mathcal{A}$. For X an L-topological space and an (extended) scale $\mathcal{A}$, we let

$$\mathcal{A}_* = \{\mathrm{Cl}_X\, a_r : r \in \mathbb{Q}\cap\mathbb{R}\} \quad \text{and} \quad \mathcal{A}^* = \{\mathrm{Int}_X\, a_r : r \in \mathbb{Q}\cap\mathbb{R}\}.$$

Then the functions f_* and f^* generated by, respectively, $\mathcal{A}_*$ and $\mathcal{A}^*$, are called *lower* and *limit functions* of f. When the L-real line $\mathbb{R}(L)$ is endowed with the right L-topology $\mathcal{R}_L = \{R_t : t \in \mathbb{R}\} \cup \{1_\emptyset, 1_{\mathbb{R}(L)}\}$ or with the left L-topology $\mathcal{L}_L = \{L_t : t \in \mathbb{R}\} \cup \{1_\emptyset, 1_{\mathbb{R}(L)}\}$, then members of

$$LSC(X,\mathbb{R}(L)) = \{f \in \mathbb{R}(L)^X : u \circ f \text{ is open in } X \text{ for all } u \in \mathcal{R}_L\}$$

and

$$USC(X,\mathbb{R}(L)) = \{f \in \mathbb{R}(L)^X : u \circ f \text{ is open in } X \text{ for all } u \in \mathcal{L}_L\}$$

are called *lower* and *uper semicontinuous* function on X.

A detailed study of the operators $(\cdot)_*$ and $(\cdot)^*$ will be presented. In particular, one has $LSC(X) = \{f : f = f_*\}$ and $USC(X) = \{f : f = f^*\}$. Also, the concept of an epigraph of an L-real valued function will be defined and the classical relationship between the closedness of the epigraph and lower semicontinuity will be shown to hold for the case of stratified L-topological space. Specifically, when $(L,')$ is a frame, then $f \in LSC(X,\mathbb{R}(L))$ if and only if

$$G_f = \bigvee_{t \in \mathbb{R}} (R_t \circ f) \times L_t$$

is open in the product L-topological space $X \times \mathbb{R}(L)$.

For $\mathbb{I}(L)$-valued function, the assignment $\mathbb{I}(L)^X \ni f \mapsto G_f \in L^{X \times [0,1)(L)}$ will be used to construct the hypergraph functor form the category $\mathbf{TOP}(\mathbb{I}(L))$ of $\mathbb{I}(L)$-topological spaces into the category $\mathbf{TOP}(L)$ of L-topological spaces, which with L the two-pont chain reduces to the classical hypergraph functor. With L a meet-continuous lattice, the link between that hypergraph functor and the functors $\Omega_L : \mathbf{TOP}(L) \to \mathbf{TOP}(\mathbb{I}(L))$ and $\mathrm{I}_L : \mathbf{TOP}(\mathbb{I}(L)) \to \mathbf{TOP}(L)$ continues to hold.

Semicontinuous L-real valued functions are well-known to play an important role in characterizing some important classes of L-topological spaces, including L-normal and completely L-regular spaces. Some of those results will be restated (and sometimes reproved) in trems of lower and upper limit functions.

Topological locally finite MV-algebra and the Riemann Surface

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1 Preliminaries

For *MV*-algebras, we refer to [1], [2] and for bounded commutative *BCK*-algebras to [3]. In [3] refered to [7], it is showed that an *MV*- algebra defines a bounded commutative *BCK*-algebra and conversely. In fact, let $\cdot$ and + be the binary operations and $^-$ the unary operation of an *MV*-algebra. If $\star$ is the operation of a bounded commutative *BCK*-algebra, then $x \star y = x \cdot \bar{y}$. On the other hand, we have $\bar{x} = 1 \star x$, $x \cdot y = x \star \bar{y}$, $x + y = (\bar{x} \star y)^-$. The partial ordering $\leq$ in a bounded commutative *BCK*-algebra is defined as follows: $x \leq y$ iff $x \star y = 0$. By [3], I is an ideal of an *MV*-algebra iff I is an ideal of a *BCK*-algebra. For *BCK*-algebras and ideals of a *BCK*-algebra and an *MV*-algebra see [5], [1] and [2]. For the Riemann surfaces we refer to [6].

2 Topological locally finite *MV*-algebra

2.1 Linear Topology induced by Ideals

In [4], it is constructed a topology for an *MV*-algebra A considered as a bounded commutative *BCK*-algebra.

Let Λ be a directed set i.e. a partially ordered set such that for $\lambda, \mu \in \Lambda$ there is $\nu \in \Lambda$ for which $\lambda \leq \nu$ and $\mu \leq \nu$.

Let $\mathcal{F} = \{I_\lambda \mid \lambda \in \Lambda\}$ be a family of ideals of A such that if $\lambda < \mu$ then $I_\mu \subset I_\lambda$. Define a relation $\sim$ in the following way [5]:

$$x \sim y \bmod I_\lambda \qquad \text{iff} \qquad x \star y \in I_\lambda \qquad \text{and} \qquad y \star x \in I_\lambda, \tag{1}$$

and let

$$U(x, \lambda) = \{y \in A \mid x \sim y \bmod I_\lambda\} \tag{2}$$

The neighborhoods $U(x, \lambda)$ defines a topology in A called the linear topology induced by $\mathcal{F}$. Further, $(x, y) \to x \star y$ and $x \to \bar{x}$ are continuous. Therefore, A is a topological *MV*-algebra.

2.2 Locally finite MV-algebras as Topological MV-algebras

The following proposition is proved by C.S.Hoo:

Proposition 1. *[4] The topology on a locally finite MV-algebra is one of the following types:*

1. *Hausdorff and connected,*
2. *Hausdorff and totally disconnected,*
3. *the trivial topology.*

It is known that a locally finite MV-algebra A is isomorphic to a subset of the unit interval [0,1], [1], with a Lukasiewicz structure. Without loss of generality we suppose that the smallest and the greatest elements of this subset are 0 and 1. We keep A just as this subset and obtain

$$x \star y \;=\; x \cdot \overline{y} \;=\; \max(0, x+1-y-1) \;=\; \max(0, x-y). \tag{3}$$

$$y \star x \;=\; y \cdot \overline{x} \;=\; \max(0, y+1-x-1) \;=\; \max(0, y-x). \tag{4}$$

Therefore, if

$$x \geq y, \qquad x \star y \;=\; x-y \;=\; |x-y| \tag{5}$$

$$y \star x \;=\; 0 \tag{6}$$

$$y \geq x, \qquad y \star x \;=\; y-x \;=\; |y-x| \tag{7}$$

$$x \star y \;=\; 0 \tag{8}$$

Let I be an ideal of A. By relation $\sim$ modulo I

$$x \sim y \bmod I \qquad \text{iff} \qquad x \star y \in I \qquad \text{and} \qquad y \star x \in I \qquad \text{iff} \qquad |x-y| \in I. \tag{9}$$

and so

$$U(x) \;=\; \{y \in A \mid x \sim y \bmod I\} \;=\; \{y \in A \mid |x-y| \in I\}. \tag{10}$$

Since the only ideals of A are $\{0\}$ and the whole A [1], we analyse the results of Proposition 1 with the neighborhoods $U(x)$:

1. Let $I = \{0\}$. Then $x = y$ and $U(x) = \{x\}$ for every $x \in A$. In this case every singleton $\{x\}$ is open and the topology is discrete and so totally disconnected. Conversely, if A is totally disconnected, then for every $x \in A$, the component of x is $\{x\}$. Especially, the component of 0 is $\{0\}$. Since the component is an ideal [4], $I = \{0\}$. It is proved that $I = \{0\}$ iff A is totally disconnected.

2. Let $I = A$. The topology is trivial iff $U(x) = A$ for every $x \in A$.

3. Let $I = A$ such that $U(x) \neq A$ for some $x \in A$. Neither A is totally disconnected nor the topology on A is trivial, by Proposition 1, A is Hausdorff and connected. In this case we choose the relative usual topology on A.

3 Topological locally finite MV-algebra as the Riemann Surface

The theory of the Riemann surface which is used is found from [6]. Let A be a topological locally finite MV-algebra. Consider the case where A is Hausdorff and connected.

Proposition 2. *Let A be a topological locally finite MV-algebra. If A is Hausdorff and connected, then $A \times A$ is a compact Riemann surface which is topologically a torus.*

Proof. The theory of the Riemann surface which is used is found from [6]. Let S be a Riemann surface which has the complex plane as its universal covering surface D. Assume the covering group G has two generators $z \to z+1$ and $z \to z+i$ (translations). A fundamental domain is now the interior of the square with vertices at $0, 1, 1+i, i$, which is (isomorphic to) the interior of $[0,1] \times [0,1]$. In this case the Riemann surface $S = D/G$ (modulo conformal equivalence) is compact. Since the opposite sides of the square $0, 1, 1+i, i$ are equivalent under G, topologically $S = A \times A$ is a torus.

□

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Dequantization of mathematics, idempotent semirings and fuzzy sets

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1. Introduction. The traditional mathematics over numerical fields can be dequantized as the Planck constant $\hbar$ tends to zero taking pure imaginary values. This dequantization leads to the so-called Idempotent Mathematics based on replacing the usual arithmetic operations by a new set of basic operations (e.g., such as maximum or minimum), that is on the concepts of idempotent semifield and semiring. Typical examples are given by the so-called $(\max, +)$ algebra $\mathbf{R}_{\max}$ and $(\min, +)$ algebra $\mathbf{R}_{\min}$. Let $\mathbf{R}$ be the field of real numbers. Then $\mathbf{R}_{\max} = \mathbf{R} \cup \{-\infty\}$ with operations $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$. Similarly $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with the operations $\oplus = \min$, $\odot = +$. The new addition $\oplus$ is idempotent, i.e., $x \oplus x = x$ for all elements x. Some problems that are nonlinear in the traditional sense turn out to be linear over a suitable idempotent semiring (idempotent superposition principle [1]). For example, the Hamilton-Jacobi equation (which is an idempotent version of the Schrödinger equation) is linear over $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$.

The basic paradigm is expressed in terms of an *idempotent correspondence principle* [2].This principle is similar to the well-known correspondence principle of N. Bohr in quantum theory (and closely related to it). Actually, there exists a heuristic correspondence between important, interesting and useful constructions and results of the traditional mathematics over fields and analogous constructions and results over idempotent semirings and semifields (i.e., semirings and semifields with idempotent addition). For example, the well-known Legendre transform can be treated as an $\mathbf{R}_{\max}$-version of the traditional Fourier transform (this observation is due to V. P. Maslov).

A systematic and consistent application of the idempotent correspondence principle leads to a variety of results, often quite unexpected. As a result, in parallel with the traditional mathematics over rings, its "shadow", the Idempotent Mathematics, appears. This "shadow" stands approximately in the same relation to the traditional mathematics as classical physics to quantum theory. In many respects Idempotent Mathematics is simpler than the traditional one. However, the transition from traditional concepts and results to their idempotent analogs is often nontrivial.

In this talk a brief survey of basic ideas of Idempotent Mathematics is presented. Relations between this theory and the theory of fuzzy sets as well as the possibility theory and some applications (including computer applications) are discussed. Hystorical surveys and the corresponding references can be found in [2]–[5].

2. Semirings, semifields, and idempotent dequantization. Consider a set S equipped with two algebraic operations: *addition* $\oplus$ and *multiplication* $\odot$. It is a *semiring* if the following conditions are satisfied:

- the addition $\oplus$ and the multiplication $\odot$ are associative;
- the addition $\oplus$ is commutative;

- the multiplication $\odot$ is distributive with respect to the addition $\oplus$: $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ and $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ for all $x, y, z \in S$.

The semiring is *commutative* if the multiplication $\odot$ is commutative. A *unity* of a semiring S is an element $\mathbf{1} \in S$ such that $\mathbf{1} \odot x = x \odot \mathbf{1} = x$ for all $x \in S$. A *zero* of a semiring S is an element $\mathbf{0} \in S$ such that $\mathbf{0} \neq \mathbf{1}$ and $\mathbf{0} \oplus x = x$, $\mathbf{0} \odot x = x \odot \mathbf{0} = \mathbf{0}$ for all $x \in S$. A semiring S is called an *idempotent semiring* if $x \oplus x = x$ for all $x \in S$. A semiring S with neutral elements $\mathbf{0}$ and $\mathbf{1}$ is called a *semifield* if every nonzero element of S is invertible.

The following examples are important. Let $\mathbf{P}$ be the segment $[0,1]$ equipped with the operations $x \oplus y = \max\{x, y\}$ and $x \odot y = \min\{x, y\}$; then $\mathbf{P}$ is a commutative idempotent semiring (but not a semifield). The subset $\mathbf{B} = \{0,1\}$ in M equipped with the same operations is the well-known Boolean algebra which is an idempotent semifield. In this case $\oplus$ and $\odot$ are the usual Boolean operations (disjunction and conjunction). In the general case the semiring addition and multiplication could be treated as generalized logical (Boolean) operations.

Let $\mathbf{R}$ be the field of real numbers and $\mathbf{R}_+$ the semiring of all nonnegative real numbers (with respect to the usual addition and multiplication). The change of variables $x \mapsto u = h \ln x$, $h > 0$, defines a map $\Phi_h \colon \mathbf{R}_+ \longrightarrow S = \mathbf{R} \cup \{-\infty\}$. Let the addition and multiplication operations be mapped from $\mathbf{R}$ to S by Φ_h, i.e., let $u \oplus_h v = h \ln(\exp(u/h) + \exp(v/h))$, $u \odot v = u + v$, $\mathbf{0} = -\infty = \Phi_h(0)$, $\mathbf{1} = 0 = \Phi_h(1)$. It can easily be checked that $u \oplus_h v \longrightarrow \max\{u, v\}$ as $h \longrightarrow 0$ and S forms a semiring with respect to addition $u \oplus v = \max\{u, v\}$ and multiplication $u \odot v = u + v$ with zero $\mathbf{0} = -\infty$ and unit $\mathbf{1} = 0$. Denote this semiring by $\mathbf{R}_{\max}$; it is idempotent. The semiring $\mathbf{R}_{\max}$ is actually a commutative semifield. This construction is due to V.P. Maslov [1]; now it is known as *Maslov's dequantization*.

The analogy with quantization is obvious; the parameter h plays the rôle of the Planck constant, so $\mathbf{R}_+$ (or $\mathbf{R}$) can be viewed as a "quantum object" and $\mathbf{R}_{\max}$ as the result of its "dequantization". A similar procedure gives the semiring $\mathbf{R}_{\min} = \mathbf{R} \cup \{+\infty\}$ with the operations $\oplus = \min$, $\odot = +$; in this case $\mathbf{0} = +\infty$, $\mathbf{1} = 0$. The semirings $\mathbf{R}_{\max}$ and $\mathbf{R}_{\min}$ are isomorphic. Connections with physics and imaginary values of the Planck constant are discussed in [4]. The commutative idempotent semiring $\mathbf{R} \cup \{-\infty\} \cup \{+\infty\}$ with the operations $\oplus = \max$, $\odot = \min$ can be obtained as a result of a "second dequantization" of $\mathbf{R}$ (or $\mathbf{R}_+$). Dozens of interesting examples of nonisomorphic idempotent semirings may be cited as well as a number of standard methods of deriving new semirings from these (see, e.g., [2]–[5]).

Idempotent dequantization is a generalization of Maslov's dequantization. This is a passage from fields to idempotent semifields and semirings in mathematical constructions and results. The idempotent correspondence principle (see Introduction and [2, 4]) often works for this idempotent dequantization.

3. Idempotent Analysis. Let S be an arbitrary semiring with idempotent addition $\oplus$ (which is always assumed to be commutative), multiplication $\odot$, zero $\mathbf{0}$, and unit $\mathbf{1}$. The set S is supplied with the *standard partial order* $\preccurlyeq$: by definition, $a \preccurlyeq b$ if and only if $a \oplus b = b$. Thus all elements of S are positive: $\mathbf{0} \preccurlyeq a$ for all $a \in S$. Due to the existence of this order, Idempotent Analysis is closely related to the lattice theory, the theory of vector lattices, and the theory of ordered spaces. Moreover, this partial order allows to simulate a number of basic notions and results of Idempotent Analysis at the purely algebraic level.

Calculus deals mainly with functions whose values are numbers. The idempotent analog of a numerical function is a map $X \longrightarrow S$, where X is an arbitrary set and S is an idempotent semiring. Functions with values in S can be added, multiplied by each other, and multiplied by elements of S.

The idempotent analog of a linear functional space is a set of S-valued functions that is closed under addition of functions and multiplication of functions by elements of S, or an S-semimodule. Consider, e.g., the S-semimodule $\mathcal{B}(X,S)$ of functions $X \longrightarrow S$ that are bounded in the sense of the standard order on S.

If $S = \mathbf{R}_{\max}$, then the idempotent analog of integration is defined by the formula

$$I(\varphi) = \int_X^{\oplus} \varphi(x)\,dx = \sup_{x\in X} \varphi(x), \tag{1}$$

where $\varphi \in \mathcal{B}(X,S)$. Indeed, a Riemann sum of the form $\sum_i \varphi(x_i)\cdot\sigma_i$ corresponds to the expression $\bigoplus_i \varphi(x_i)\odot\sigma_i = \max_i\{\varphi(x_i)+\sigma_i\}$, which tends to the right-hand side of (1) as $\sigma_i \longrightarrow 0$. Of course, this is a purely heuristic argument. Formula (1) defines the idempotent integral not only for functions taking values in $\mathbf{R}_{\max}$, but also in the general case when any of bounded (from above) subsets of S has the least upper bound.

An idempotent measure on X is defined by $m_\psi(Y) = \sup_{x\in Y}\psi(x)$, where $\psi \in \mathcal{B}(X,S)$. The integral with respect to this measure is defined by

$$I_\psi(\varphi) = \int_X^{\oplus} \varphi(x)\,dm_\psi = \int_X^{\oplus} \varphi(x)\odot\psi(x)\,dx = \sup_{x\in X}(\varphi(x)\odot\psi(x)). \tag{2}$$

Obviously, if $S = \mathbf{R}_{\min}$, then the standard order $\preccurlyeq$ is opposite to the conventional order $\leqslant$, so in this case equation (2) assumes the form

$$\int_X^{\oplus} \varphi(x)\,dm_\psi = \int_X^{\oplus} \varphi(x)\odot\psi(x)\,dx = \inf_{x\in X}(\varphi(x)\odot\psi(x)), \tag{3}$$

where inf is understood in the sense of the conventional order $\leqslant$.

The functionals $I(\varphi)$ and $I_\psi(\varphi)$ are linear over S; their values correspond to limits of Lebesgue (or Riemann) sums. The formula for $I_\psi(\varphi)$ defines the idempotent scalar product of the functions ψ and φ. Various idempotent functional spaces and an idempotent version of the theory of distributions can be constructed on the basis of the idempotent integration, see, e.g., [1], [3]–[5]. The analogy of idempotent and probability measures leads to spectacular parallels between optimization theory and probability theory. For example, the Chapman–Kolmogorov equation corresponds to the Bellman equation (see, e.g., [6, 5]). Many other idempotent analogs may be cited (in particular, for basic constructions and theorems of functional analysis [4]).

4. The superposition principle and linear problems. Basic equations of quantum theory are linear (the superposition principle). The Hamilton–Jacobi equation, the basic equation of classical mechanics, is nonlinear in the conventional sense. However it is linear over the semirings $\mathbf{R}_{\min}$ and $\mathbf{R}_{\max}$. Also, different versions of the Bellman equation, the basic equation of optimization theory, are linear over suitable idempotent semirings (V. P. Maslov's idempotent superposition principle), see [1, 3]. For instance, the finite-dimensional stationary Bellman equation can be written in the form $X = H\odot X\oplus F$, where X, H, F are matrices with coefficients in an idempotent semiring S and the unknown matrix X is determined by H and F [7]. In particular, standard problems of dynamic programming and the well-known shortest path problem correspond to the cases $S = \mathbf{R}_{\max}$ and $S = \mathbf{R}_{\min}$, respectively. In [7], it was shown that main optimization algorithms for finite graphs correspond to standard methods for solving systems of linear equations of this type (i.e., over semirings). Specifically, Bellman's shortest path algorithm corresponds to a version of Jacobi's algorithm, Ford's algorithm corresponds to the Gauss–Seidel iterative scheme, etc.

Linearity of the Hamilton–Jacobi equation over $\mathbf{R}_{\min}$ (and $\mathbf{R}_{\max}$) is closely related to the (conventional) linearity of the Schrödinger equation, see [4] for details.

5. Correspondence principle for algorithms and their computer implementations. The idempotent correspondence principle is valid for algorithms as well as for their software and hardware implementations [2]. In particular, according to the superposition principle, analogs of linear algebra algorithms are especially important. It is well-known that algorithms of linear algebra are convenient for parallel computations; so their idempotent analogs accept a parallelization. This is a regular way to use parallel computations for many problems including basic optimization problems. It is convenient to use universal algorithms which do not depend on a concrete semiring and its concrete computer model. Software implementations for universal semiring algorithms are based on object-oriented and generic programming; program modules can deal with abstract (and variable) operations and data types, see [2, 8] for details.

The most important and standard algorithms have many hardware realizations in the form of technical devices or special processors. These devices often can be used as prototypes for new hardware units generated by substitution of the usual arithmetic operations for its semiring analogs, see [2] for details. Good and efficient technical ideas and decisions can be transposed from prototypes into new hardware units. Thus the correspondence principle generates a regular heuristic method for hardware design. Note that to get a patent it is necessary to present the so-called "invention formula", that is to indicate a prototype for the suggested device and the difference between these devices.

6. Idempotent interval analysis. An idempotent version of the traditional interval analysis is presented in [9]. Let S be an idempotent semiring equipped with the standard partial order (see the beginning of Section 3). A *closed interval* in S is a subset of the form $\mathbf{x} = [\underline{\mathbf{x}}, \overline{\mathbf{x}}] = \{x \in S | \underline{\mathbf{x}} \preccurlyeq x \preccurlyeq \overline{\mathbf{x}}\}$, where the elements $\underline{\mathbf{x}} \preccurlyeq \overline{\mathbf{x}}$ are called *lower* and *upper bounds* of the interval $\mathbf{x}$. A *weak interval extension* $I(S)$ of the semiring S is the set of all closed intervals in S endowed with operations $\oplus$ and $\odot$ defined as $\mathbf{x} \oplus \mathbf{y} = [\underline{\mathbf{x}} \oplus \underline{\mathbf{y}}, \overline{\mathbf{x}} \oplus \overline{\mathbf{y}}]$, $\mathbf{x} \odot \mathbf{y} = [\underline{\mathbf{x}} \odot \underline{\mathbf{y}}, \overline{\mathbf{x}} \odot \overline{\mathbf{y}}]$; the set $I(S)$ is a new idempotent semiring with respect to these operations. It is proved that basic problems of idempotent linear algebra are polynomial, whereas in the traditional interval analysis problems of this kind are generally NP-hard. Exact interval solutions for the discrete stationary Bellman equation (this is the matrix equation discussed in Section 4) and for the corresponding optimization problems are constructed and examined.

7. Generalized fuzzy sets. Let Ω be the so-called universe consisting of "elementary events" and S an idempotent semiring. Denote by $\mathcal{F}(S)$ the set of functions defined on Ω and taking their values in S; then $\mathcal{F}(S)$ is an idempotent semiring with respect to the pointwise addition and multiplication of functions. We shall say that elements of $\mathcal{F}(S)$ are *generalized fuzzy sets*. We have the well known classical definition of fuzzy sets (L.A. Zadeh [10]) if $S = \mathbf{P}$, where $\mathbf{P}$ is the segment $[0,1]$ with the semiring operations $\oplus = \max$ and $\odot = \min$, see Section 2. Of course, functions from $\mathcal{F}(\mathbf{P})$ taking their values in the Boolean algebra $\mathbf{B} = \{0,1\} \subset \mathbf{P}$ correspond to traditional sets from Ω and semiring operations correspond to standard operations for sets. In the general case if S has neutral elements $\mathbf{0}$ and $\mathbf{1}$ (and $\mathbf{0} \neq \mathbf{1}$), then functions from $\mathcal{F}(S)$ taking their values in $\mathbf{B} = \{\mathbf{0}, \mathbf{1}\} \subset \mathbf{S}$ can be treated as traditional subsets in Ω. If S is a lattice (i.e. $x \odot y = \inf\{x,y\}$ and $x \oplus y = \sup\{x,y\}$), then generalized fuzzy sets coincide with L-fuzzy sets in the sense of J.A. Goguen [11]. The set $I(S)$ of intervals is an idempotent semiring (see Section 6), so elements of $\mathcal{F}(I(S))$ can be treated as interval (generalized) fuzzy sets.

It is well known that the classical theory of fuzzy sets is a basis for the theory of possibility [12]. Of course, it is possible to develop a similar generalized theory of possibility starting from generalized fuzzy sets. In general the generalized theories are noncommutative; they seem to be more qualitative and less quantitative with respect to the classical theories presented in [10, 12]. We see that Idempotent Analysis and the theory of (generalized) fuzzy sets and possibility have the same objects, i.e. functions taking their values in semirings. However, basic problems and methods could be different for these theories (like for the measure theory and the probability theory).

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De Morgan triplets in the theory of fuzzified normal forms

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Abstract

The purpose of normal forms is to provide a standard representation or approximation of various kinds of functions. Boolean functions, for instance, have both a disjunctive and conjunctive normal form representation. Interpreting these normal forms in a suitable t-norm-based logic leads to some interval-valued fuzzification of the original Boolean function. We will deal with two mathematical questions: first, in which t-norm-based logic do we actually obtain intervals and second, if so, to what extent does the length of the intervals depend on the original Boolean function.

1 Introduction

A Boolean expression is an expression involving variables each of which can take either the value true or false. These variables are combined using Boolean operations such as conjunction ($\wedge$), disjunction ($\vee$) and negation ($'$). It is common knowledge that each Boolean function can be represented by a well-formed formula in Boolean propositional logic. Moreover, there are two special forms, the disjunctive and conjunctive normal form, which are of great interest, for each of these forms defines the Boolean function in a unique way.

In many cases, crisp models are too 'poor' to represent the 'human way of thinking'. Fuzzy sets provide a widely accepted solution to that end. Typical to fuzzy set theory is the large set of options (logical operations, shapes of membership functions, parameters) that are available to the user. A unique and definite definition of the intersection of two fuzzy sets, for instance, cannot be expected. However, fuzzifying the disjunctive and conjunctive normal form representation of a Boolean expression results in two standard fuzzifications of the original Boolean function. All attention so far has focused on the comparability of these fuzzified normal forms, in particular for binary Boolean functions [1, 3, 11, 12, 13, 14]. We contribute to the existing knowledge on this comparability. Because of their theoretical importance, special attention will be drawn to left-continuous t-norms.

Before we start we fix some notations. Let ϕ be an $[0,1]$-automorphism and $\mathcal{N}$ the standard negator, then the De Morgan triplets $\langle (T_{\mathbf{M}})_\phi, (S_{\mathbf{M}})_\phi, \mathcal{N}_\phi \rangle$, $\langle (T_{\mathbf{P}})_\phi, (S_{\mathbf{P}})_\phi, \mathcal{N}_\phi \rangle$, $\langle (T_{\mathbf{L}})_\phi, (S_{\mathbf{L}})_\phi, \mathcal{N}_\phi \rangle$, $\langle (T_{\mathbf{D}})_\phi, (S_{\mathbf{D}})_\phi, \mathcal{N}_\phi \rangle$ and $\langle (T^{\mathbf{nM}})_\phi, (S^{\mathbf{nM}})_\phi, \mathcal{N}_\phi \rangle$ will be called respectively (M,ϕ)-, (P,ϕ)-, (L,ϕ)-, (D,ϕ)- and (nM,ϕ)-triplets. In case ϕ is the identity mapping, we talk about the M-, P-, L-, D- and nM-triplet.

2 Fuzzified normal forms of n-ary Boolean functions

Consider the Boolean algebra $(\{0,1\},\vee,\wedge,',0,1)$. The disjunctive and conjunctive normal forms of an n-ary Boolean function f are given by

$$D_{\mathcal{B}}(f)(x_1,\dots,x_n) = \bigvee_{f(e_1,\dots,e_n)=1} x_1^{e_1} \wedge \dots \wedge x_n^{e_n} \tag{1}$$

and

$$C_{\mathcal{B}}(f)(x_1,\dots,x_n) = \bigwedge_{f(e_1,\dots,e_n)=0} x_1^{e_1'} \vee \dots \vee x_n^{e_n'}, \tag{2}$$

where $x^e = x$ if $e=1$ and $x^e = x'$ if $e=0$. One can fuzzify expressions (1) and (2) by replacing $(\wedge,\vee,')$ by a triplet (T,S,N), with N an involutive negation. The corresponding disjunctive and conjunctive fuzzified normal forms are denoted by D_F and C_F. For each n-ary Boolean function f we obtain two $[0,1]^n \longrightarrow [0,1]$ mappings $D_F(f)$ and $C_F(f)$:

$$\begin{aligned} D_F(f)(\mathbf{x}) &= S\{f(\mathbf{e})\ T(\mathbf{x}^{\mathbf{e}}) \mid \mathbf{e}\in\{0,1\}^n\}, \\ C_F(f)(\mathbf{x}) &= T\left\{\left[(1-f(\mathbf{e}))\ S\left(\mathbf{x}^{(\mathbf{e}^{\mathbf{0}})}\right)^N\right]^N \mid \mathbf{e}\in\{0,1\}^n\right\}, \end{aligned}$$

where $\mathbf{x}\in[0,1]^n$, $\mathbf{0}=(0,\dots,0)$ and $\mathbf{x}^{\mathbf{e}} = (x_1^{e_1},\dots,x_n^{e_n})$.

While $D_F \le C_F$[4] does not hold for all continuous De Morgan triplets [14, 10], we wonder whether $D_F \le C_F$ is true for the basic continuous De Morgan triplets (M,ϕ), (P,ϕ), and (L,ϕ). Remark that, in case we work with the M-triplet, the inequality $D_F \le C_F$ also follows from [3].

Theorem 1. [10] *For any (M,ϕ)-, (P,ϕ)- and (L,ϕ)-triplet it holds that $D_F(f) \le C_F(f)$, for all n-ary Boolean functions f.*

Because a full characterization of non-continuous t-norms, in particular left-continuous ones, is still lacking, we restrict ourselves in the non-continuous case to the basic triplets (D,ϕ) and (nM,ϕ). We obtain a similar result as for the three prototypical continuous triplets.

Theorem 2. [10] *For any (D,ϕ)-, (nM,ϕ)-triplet it holds that $D_F(f) \le C_F(f)$, for all n-ary Boolean functions f.*

3 Independence of the n-ary Boolean function: a system of functional equations

Knowing that $D_F \le C_F$ holds for a triplet (T,S,N), it remains an intriguing problem, from a mathematical point of view, to understand to what extent $C_F(f) - D_F(f)$ depends on the n-ary Boolean function f. More specifically, we wonder for which triplets $C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x})$ is only a function of the variable $\mathbf{x}\in[0,1]^n$ (i.e. independent of the Boolean function f). In [10], we have already encountered three solutions: the L-triplet, the nM-triplet and all (D,ϕ)-triplets.

[4] $D_F \le C_F$ is a shorthand to express that $D_F(f)(\mathbf{x}) \le C_F(f)(\mathbf{x})$, for every $\mathbf{x}\in[0,1]^n$

As shown in the following lemma, although there are 2^{2^n} different n-ary Boolean functions f, imposing that $C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x})$ must be independent of the Boolean function f, is equivalent to a system of 3 functional equations.

Theorem 3. *Consider a triplet (T,S,N), with N an involutive negation with fixpoint a_N. Then $C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x})$ is independent of the Boolean function f if and only if for all $\mathbf{x} \in [0,a_N]^n$, $x_1 \leq x_2 \leq \ldots \leq x_n$, the following expressions are equal to each other*

$$S(x_1,\ldots,x_{n-1},x_n^N) - T(x_1^N,\ldots,x_{n-1}^N,x_n^N)\,, \tag{3}$$

$$S(x_1,\ldots,x_{n-1},x_n) - T(x_1^N,\ldots,x_{n-1}^N,x_n)\,, \tag{4}$$

$$T(S(x_1,\ldots,x_{n-1},x_n^N),S(x_1,\ldots,x_{n-1},x_n))\,, \tag{5}$$

$$1 - S(T(x_1^N,\ldots,x_{n-1}^N,x_n),T(x_1^N,\ldots,x_{n-1}^N,x_n^N))\,. \tag{6}$$

When considering a De Morgan triplet $\langle T,S,\mathcal{N}\rangle$, this system of functional equations reduces to a single functional equation in two dimensions:

Theorem 4. *Consider a De Morgan triplet $\langle T,S,\mathcal{N}\rangle$. Then $C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x})$ is independent of the Boolean function f if and only if*

$$S(T(x^{\mathcal{N}},y),T(x^{\mathcal{N}},y^{\mathcal{N}})) = T(x^{\mathcal{N}},y) + T(x^{\mathcal{N}},y^{\mathcal{N}})\,,$$

for any $(x,y) \in [0,1/2]^2$, $x \leq y$.

We have shown that the L-triplet is the only continuous De Morgan triplet for which the difference between both normal forms is independent of the n-ary Boolean function f.

Theorem 5. *Consider a triplet (T,S,N), with T a left-continuous t-norm, S a right-continuous t-conorm and N an involutive negation with fixpoint a_N. Suppose that $T(x,a_N)$ and $S(x,a_N)$ are continuous and that (T,S,N) is a De Morgan triplet or $N = \mathcal{N}$. Then $C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x})$ is independent of the Boolean function f if and only if (T,S,N) is the* L*-triplet.*

Further, we characterize the De Morgan triplets $\langle T,S,N\rangle$, with T a left-continuous t-norm that fulfills some additional continuity conditions, for which $C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x})$ is only a function of the variable $\mathbf{x} \in [0,1]^n$. We obtain a unique De Morgan triplet that is based on a t-norm T_λ, with $\lambda \in [0,1/2[$, defined by

$$T_\lambda(x,y) = \begin{cases} 0 & \text{, if } x+y \leq 1\,, \\ \min(x,y) & \text{, if } x+y > 1 \;\wedge\; \min(x,y) \in]\lambda, 1-\lambda]\,, \\ x+y-1 & \text{, if } x+y > 1 \\ & \qquad \wedge\; (x+y \geq 2-\lambda \;\vee\; \min(x,y) \in [0,\lambda])\,, \\ 1-\lambda & \text{, if } x+y \leq 2-\lambda \;\wedge\; \min(x,y) \in]1-\lambda, 1]\,. \end{cases}$$

Every t-norm in this family can be obtained by applying the rotation construction of Jenei [4, 5, 6, 7] on a suitable ordinal sum [8]. The nilpotent minimum $T^{\mathbf{nM}}$ [2] corresponds to $\lambda = 0$.

Theorem 6. *Consider a De Morgan triplet $\langle T,S,N\rangle$ based on a left-continuous t-norm T and an involutive negation N with fixpoint a_N. Suppose that $T_y(x) := T(x,y)$ is continuous on $]y^N,1]$ for any $y \in [0,a_N]$ and is continuous on $[y,1]$ for any $y \in]a_N,1]$. Suppose also that*

$$\lim_{x \overset{\geq}{\to} a_N} T(x,a_N) > 0\,.$$

Then $C_F(f)(\mathbf{x}) - D_F(f)(\mathbf{x})$ is independent of the Boolean function f if and only if $N = \mathcal{N}$ and $T = T_\lambda$, for some $\lambda \in [0,1/2[$.

4 Further research

In future work, it would be worthwhile to try once again to get rid of the extra conditions ($N = \mathcal{N}$, (T,S,N) is a De Morgan triplet, ...) in the characterization theorems. Moreover, the new insights in the treated system of functional equations force us to review the inequality $D_F \leq C_F$ more closely. It would be preferable to establish a necessary condition on (T,S,N) for $D_F \leq C_F$ to hold, when working with *n*-ary Boolean functions, and which covers all the known suitable triplets (T,S,N). Finally, we are challenged to lay bare all connections between interval-valued preference structures, based on fuzzified normal forms [1], and those based on a pair of generators [9].

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k-intolerant capacities and Choquet integrals

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Abstract

We define an aggregation function to be (at most) k-intolerant if it is bounded from above by its kth lowest input value. Applying this definition to the discrete Choquet integral and its underlying capacity, we introduce the concept of k-intolerant capacities which, when varying k from 1 to n, cover all the possible capacities on n objects. Just as the concepts of k-additive capacities and p-symmetric capacities have been previously introduced essentially to overcome the problem of computational complexity of capacities, k-intolerant capacities are proposed here for the same purpose but also for dealing with intolerant or tolerant behaviors of aggregation.

Keywords: multi-criteria analysis, interacting criteria; capacities; Choquet integral.

1 Introduction

In a previous work [7] the author investigated the intolerant behavior of the discrete Choquet integral when used to aggregate interacting criteria. Roughly speaking, the Choquet integral $\mathcal{C}_v$, or equivalently its associated capacity v, has an intolerant behavior if its output (aggregated) value is often close to the lowest of its input values. More precisely, consider the domain $[0,1]^n$ of $\mathcal{C}_v$ as a probability space, with uniform distribution, and the mathematical expectation of $\mathcal{C}_v$, which expresses the typical position of $\mathcal{C}_v$ within the unit interval. A low expectation then means that the Choquet integral is rather intolerant and behaves nearly like the minimum on average. Similarly, a high expectation means that the Choquet integral is rather tolerant and behaves nearly like the maximum on average. Note that such an analysis is meaningless when criteria are independent since, in that case, the Choquet integral boils down to a weighted arithmetic mean whose expectation is always one half (neither tolerant nor intolerant.)

In this paper we pursue this idea by defining k-intolerant Choquet integrals [5]. The case $k = 1$ corresponds to the unique most intolerant Choquet integral, namely the minimum. The case $k = 2$ corresponds to the subclass of n-variable Choquet integrals that are bounded from above by their second lowest input values. Those Choquet integrals are more or less intolerant but not as much as the minimum. As an example, the following 3-variable Choquet integral

$$\mathcal{C}_v(x_1,x_2,x_3) = \frac{1}{2}\min(x_1,x_2) + \frac{1}{2}\min(x_1,x_3)$$

[5]Equivalently, we define k-intolerant capacities since there is a one-to-one correspondence between n-variable Choquet integrals and capacities defined on n objects.

is clearly 2-intolerant, while being different from the minimum.

More generally, denoting by $x_{(1)},\ldots,x_{(n)}$ the order statistics resulting from reordering $x_1,\ldots,x_n$ in the nondecreasing order, we say that an n-variable Choquet integral $\mathcal{C}_v$, or equivalently its underlying capacity v, is at most k-intolerant if

$$\mathcal{C}_v(x) \leq x_{(k)} \qquad (x \in [0,1]^n) \tag{1}$$

and it is exactly k-intolerant if, in addition, there is $x^* \in [0,1]^n$ such that $\mathcal{C}_v(x^*) > x^*_{(k-1)}$, with convention that $x_{(0)} := 0$.

Interestingly, condition (1) clearly implies that the output value of $\mathcal{C}_v$ is zero whenever at least k input values are zeros. We will see in Section 3 that the converse holds true as well.

At first glance, defining k-intolerant aggregation functions may appear as a pure mathematical exercise without any real application behind. In fact, in many real-life decision problems, experts or decision-makers are or must be intolerant. This is often the case when, in a given selection problem, we search for most qualified candidates among a wide population of potential alternatives. It is then sensible to reject every candidate which fails at least k criteria.

Example 1. Consider a (simplified) problem of selecting candidates applying for a university permanent position and suppose that the evaluation procedure is handled by appointed expert-consultants on the basis of the following academic selection criteria:

1. Scientific value of curriculum vitae,
2. Teaching effectiveness,
3. Ability to supervise staff and work in a team environment,
4. Ability to communicate easily in English,
5. Work experience in the industry,
6. Recommendations by faculty and other individuals.

Assume also that one of the rules of the evaluation procedure states that the complete failure of any two of these criteria results in automatic rejection of the applicant. This quite reasonable rule forces the Choquet integral, when used for the aggregation procedure, to be 2-intolerant, thus restricting the class of possible Choquet integrals for such a selection problem.

On the other hand, there are real-life situations where it is recommended to be tolerant, especially if the criteria are hard to meet simultaneously and if the potential alternatives are not numerous. To deal with such situations, we introduce k-tolerant aggregation functions and we will say that an n-variable Choquet integral $\mathcal{C}_v$, or equivalently its underlying capacity v, is at most k-tolerant if

$$\mathcal{C}_v(x) \geq x_{(n-k+1)} \qquad (x \in [0,1]^n).$$

In that case, the output value of $\mathcal{C}_v$ is one whenever at least k input values are ones.

Example 2. Consider a family who consults a Real Estate agent to buy a house. The parents propose the following house buying criteria:

1. Close to a school,
2. With parks for their children to play in,
3. With safe neighborhood for children to grow up in,
4. At least 100 meters from the closest major road,
5. At a fair distance from the nearest shopping mall,
6. Within reasonable distance of the airport.

Feeling that it is likely unrealistic to satisfy all six criteria simultaneously, the parents are ready to accept a house that would fully succeed any five over the six criteria. If a 6-variable Choquet integral is used in this selection problem, it must be 5-tolerant.

Considering k-intolerant and k-tolerant capacities can also be viewed as a way to make real applications easier to model from a computational viewpoint. Those "simplified" capacities indeed require less parameters than classical capacities (actually $O(n^{k-1})$ parameters instead of $O(2^n)$; see Section 3). Moreover, when varying k from 1 to n, we clearly recover all the possible capacities on n objects.

Notice however that this idea of partitioning capacities into subclasses is not new. Grabisch [3] proposed the k-additive capacities, which gradually cover all the possible capacities starting from additive capacities ($k = 1$). Later, Miranda et al. [8] introduced the p-symmetric capacities, also covering the possible capacities but starting from symmetric capacities ($p = 1$). Note also that other approaches to overcome the exponential complexity of capacities have also been previously proposed in the literature: Sugeno λ-measures [10], $\perp$-decomposable measures (see e.g. [5]), hierarchically decomposable measures [11], distorted probabilities (see e.g. [9]) to name a few.

2 Basic definitions

Let $F : [0,1]^n \longrightarrow [0,1]$ be an aggregation function. By considering the cube $[0,1]^n$ as a probability space with uniform distribution, we can compute the mathematical expectation of F, that is,

$$E(F) := \int_{[0,1]^n} F(x)\,dx. \tag{2}$$

This value gives the average position of F within the interval $[0,1]$.

When F is *internal* (i.e., $\min \leq F \leq \max$) then it is convenient to rescale $E(F)$ within the interval $[E(\min), E(\max)]$. This leads to the following normalized and mutually complementary values [1, 7]:

$$\text{andness}(F) := \frac{E(\max) - E(F)}{E(\max) - E(\min)} \tag{3}$$

$$\text{orness}(F) := \frac{E(F) - E(\min)}{E(\max) - E(\min)} \tag{4}$$

Thus defined, the degree of *andness* (resp. *orness*) of F represents the degree or intensity (between 0 and 1) to which the average value of F is close to that of "min" (resp. "max"). In some sense, it also reflects the extent to which F behaves like the minimum (resp. the maximum) on average.

Define the kth order statistic function $\mathrm{OS}_k : [0,1]^n \longrightarrow [0,1]$ as

$$\mathrm{OS}_k(x) = x_{(k)} \qquad (x \in [0,1]^n),$$

where $x_{(k)}$ is the kth lowest coordinate of x. It can be proved [7] that

$$E(\mathrm{OS}_k) = \frac{k}{n+1} \qquad (k \in \{1,\ldots,n\})$$

and hence the set $\{E(\mathrm{OS}_k) \mid k = 1,\ldots,n\}$ partitions the unit interval $[0,1]$ into $n+1$ equal-length subintervals.

Now, as mentioned in the introduction, when a function $F : [0,1]^n \longrightarrow [0,1]$ is used to aggregate decision criteria, it is clear that the more $E(F)$ is low, the more F has an intolerant behavior. This suggests the following definition:

Definition 3. Let $k \in \{1,\ldots,n\}$. An aggregation function $F : [0,1]^n \longrightarrow [0,1]$ is *at most k-intolerant* if $F \leq \mathrm{OS}_k$. It is *k-intolerant* if, in addition, $F \nleq \mathrm{OS}_{k-1}$, where $\mathrm{OS}_0 := 0$ by convention.

It follows immediately from this definition that, for any k-intolerant function F, we have $E(F) \leq E(\mathrm{OS}_k)$ and, if F is internal, we have $\mathrm{andness}(F) \geq \mathrm{andness}(\mathrm{OS}_k)$ and $\mathrm{orness}(F) \leq \mathrm{orness}(\mathrm{OS}_k)$.

Example 4. The product $F(x) = \prod_i x_i$, defined on $[0,1]^n$, is 1-intolerant and we have $E(F) = 1/2^n$.

By duality, we can also introduce k-tolerant functions as follows:

Definition 5. Let $k \in \{1,\ldots,n\}$. An aggregation function $F : [0,1]^n \longrightarrow [0,1]$ is *at most k-tolerant* if $F \geq \mathrm{OS}_{n-k+1}$. It is *k-tolerant* if, in addition, $F \ngeq \mathrm{OS}_{n-k+2}$, where $\mathrm{OS}_{n+1} := 1$ by convention.

It is immediate to see that when a function $F : [0,1]^n \longrightarrow [0,1]$ is k-intolerant, its *dual* $F^* : [0,1]^n \longrightarrow [0,1]$, defined by

$$F^*(x_1,\ldots,x_n) := 1 - F(1-x_1,\ldots,1-x_n) \qquad (x \in [0,1]^n) \tag{5}$$

is k-tolerant and vice versa.

In the next section we investigate the particular case where F is the Choquet integral and we define the concepts of k-intolerant and k-tolerant capacities.

3 Case of Choquet integrals and capacities

The use of the Choquet integral has been proposed by many authors as an adequate substitute to the weighted arithmetic mean to aggregate interacting criteria; see e.g. [2, 6]. In the weighted arithmetic mean model, each criterion is given a weight representing the importance of this criterion in the decision. In the Choquet integral model, where criteria can be dependent, a capacity is used to define a weight on each combination of criteria, thus making it possible to model the interaction existing among criteria.

Let us first recall the formal definitions of these concepts. Throughout, we will use the notation $N := \{1,\ldots,n\}$ for the set of criteria.

Definition 6. A *capacity* on N is a set function $v : 2^N \longrightarrow [0,1]$, that is nondecreasing with respect to set inclusion and such that $v(\varnothing) = 0$ and $v(N) = 1$.

Definition 7. Let v be a capacity on N. The *Choquet integral* of $x : N \longrightarrow \mathbb{R}$ with respect to v is defined by

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_{(i)} \,[v(A_{(i)}) - v(A_{(i+1)})], \tag{6}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq \ldots \leq x_{(n)}$. Furthermore $A_{(i)} := \{(i), \ldots, (n)\}$ and $A_{(n+1)} := \varnothing$.

In this section we apply the ideas of k-intolerance and k-tolerance (cf. Definitions 3 and 5) to the Choquet integral. Since this integral is internal, it can be seen as a function from $[0,1]^n$ to $[0,1]$.

Let us denote by $\mathcal{F}_N$ the set of all capacities on N. The following proposition, inspired from [7, S4], gives equivalent conditions for a Choquet integral to be at most k-intolerant.

Proposition 8. *Let $k \in \{1, \ldots, n\}$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

i) $\mathcal{C}_v(x) \leq x_{(k)} \;\; \forall x \in [0,1]^n$,

ii) $v(T) = 0 \; \forall T \subseteq N$ *such that* $|T| \leq n-k$,

iii) $\mathcal{C}_v(x) = 0 \; \forall x \in [0,1]^n$ *such that* $x_{(k)} = 0$,

iv) $\mathcal{C}_v(x)$ *is independent of* $x_{(k+1)}, \ldots, x_{(n)}$,

v) $\exists \lambda \in [0,1)$ *such that* $\forall x \in [0,1]^n$ *we have* $x_{(k)} \leq \lambda \;\Rightarrow\; \mathcal{C}_v(x) \leq \lambda$,

As we can see, some assertions of Proposition 8 are natural and can be interpreted easily. Some others are more surprising and show that the Choquet integral may have an unexpected behavior.

First, assertion (ii) enables us to define k-intolerant capacities as follows:

Definition 9. Let $k \in \{1, \ldots, n\}$. A capacity $v \in \mathcal{F}_N$ is *k-intolerant* if $v(T) = 0$ for all $T \subseteq N$ such that $|T| \leq n-k$ and there is $T^* \subseteq N$, with $|T^*| = n-k+1$, such that $v(T^*) \neq 0$.

Assertion (iii) says that the output value of the Choquet integral is zero whenever at least k input values are zeros. This is actually a straightforward consequence of k-intolerance.

Assertion (iv) is more surprising. It says that the output value of the Choquet integral does not take into account the values of $x_{(k+1)}, \ldots, x_{(n)}$. Back to Example 1, only the two lowest scores are taken into account to provide a global evaluation, regardless of the other scores.

Assertion (v) is also of interest. By imposing that $\mathcal{C}_v(x) \leq \lambda$ whenever $x_{(k)} \leq \lambda$ for a given threshold $\lambda \in [0,1)$, we necessarily force $\mathcal{C}_v$ to be at most k-intolerant. For instance, consider the problem of evaluating students with respect to different courses and suppose that it is decided that if the lowest k marks obtained by a student are less than 18/20 then his/her global mark must be less than 18/20. In this case, the Choquet integral utilized is at most k-intolerant.

Proposition 8 can be easily rewritten for k-tolerance by considering the dual $\mathcal{C}_v^*$ of the Choquet integral $\mathcal{C}_v$ as defined in Eq. (5). On this issue, Grabisch et al. [4, S4] showed that the dual $\mathcal{C}_v^*$ of $\mathcal{C}_v$ is the Choquet integral $\mathcal{C}_{v^*}$ defined from the *dual capacity* v^*, which is constructed from v by

$$v(T) = 1 - v(N \setminus T) \qquad (T \subseteq N).$$

We then have

$$\mathcal{C}_v \geq \mathrm{OS}_{n-k+1} \quad \Leftrightarrow \quad \mathcal{C}_{v^*} \leq \mathrm{OS}_k.$$

Proposition 10. *Let $k \in \{1,\ldots,n\}$ and $v \in \mathcal{F}_N$. Then the following assertions are equivalent:*

i) $\mathcal{C}_v(x) \geq x_{(n-k+1)} \ \ \forall x \in [0,1]^n$,

ii) $v(T) = 1 \ \forall T \subseteq N$ *such that* $|T| \geq k$,

iii) $\mathcal{C}_v(x) = 1 \ \forall x \in [0,1]^n$ *such that* $x_{(n-k+1)} = 1$,

iv) $\mathcal{C}_v(x)$ *is independent of* $x_{(1)},\ldots,x_{(n-k)}$,

v) $\exists \lambda \in (0,1]$ *such that* $\forall x \in [0,1]^n$ *we have* $x_{(n-k+1)} \geq \lambda \ \Rightarrow \ \mathcal{C}_v(x) \geq \lambda$,

Here again, some assertions are of interest. First, assertion (ii) enables us to define k-tolerant capacities as follows:

Definition 11. Let $k \in \{1,\ldots,n\}$. A capacity $v \in \mathcal{F}_N$ is *k-tolerant* if $v(T) = 1$ for all $T \subseteq N$ such that $|T| \geq k$ and there is $T^* \subseteq N$, with $|T^*| = k-1$, such that $v(T^*) \neq 1$.

Assertion (iii) says that the output value of the Choquet integral is one whenever at least k input values are ones.

Assertion (iv) says that the output value of the Choquet integral does not take into account the values of $x_{(1)},\ldots,x_{(n-k)}$. As an application, consider students who are evaluated according to n homework assignments and assume that the evaluation procedure states that the two lowest homework scores of each student are dropped, which implies that each student can miss two homework assignments without affecting his/her final grade. If a n-variable Choquet integral is used to aggregate the homework scores, it should not take $x_{(1)}$ and $x_{(2)}$ into consideration and hence it is at most $(n-2)$-tolerant.

4 Conclusion

In this paper, which can be considered as the sequel of [7], we have proposed the concepts of k-intolerant and k-tolerant Choquet integrals and capacities. Besides the obvious computational advantage of these concepts (comparable to that of k-additive and p-symmetric capacities), they can be easily interpreted in practical decision problems where the decision makers must be intolerant or tolerant. In an extended version of this paper, we also introduce axiomatically intolerance and tolerance indices which measure the degree to which the Choquet integral is k-intolerant and k-tolerant. These indices, when varying k from 1 to $n-1$, make it possible to identify and measure the intolerant or tolerant character of the decision maker.

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Choquet measures, Shapley values, and inconsistent pairwise comparison matrices: an extension of Saaty's A.H.P.

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The Analytic Hierarchy Process (AHP), developed by Thomas L. Saaty [6] [7] [8] [9], is a well-known multicriteria aggregation model. It is based on pairwise comparison matrices at two fundamental levels: the lower level encodes pairwise comparison matrices between alternatives (one such matrix for each criterion) and the higher level encodes a single pairwise comparison matrix between criteria. In its most general form, the higher level of the AHP can be structured hierarchically, with several layers of criteria, but in this paper we focus on the single layer case, with a single matrix of pairwise comparisons between criteria.

Pairwise comparison matrices are typically inconsistent. However, the AHP extracts from each pairwise comparison matrix a vector of importance weights (also called priorities) given by the principal eigenvector or, alternatively, by the geometric mean vector. In both cases the priority vectors have positive components normalized to unit sum. In this paper we consider only the geometric mean method, because its structural properties are more suited for our study. Once the priority vectors are obtained, the AHP uses the priority vector at the higher level to aggregate (by means of a weighted average) the lower level priority vectors.

In this paper we propose an extension of Saaty's AHP based on Choquet measures. In our model, inconsistency is explicitly used in the aggregation process in order to attenuate the importance values of those criteria that (on average) are more inconsistent with the others. Accordingly, our model emphasizes the importance values of those criteria that (on average) are more consistent with the remaining ones.

Consider a finite set of interacting criteria $N = \{1,2,\ldots,n\}$.
A *Choquet measure* [2] on the set N is a set function $\mu : \mathcal{P}(N) \longrightarrow [0,1]$ satisfying

(i) $\mu(\emptyset) = 0$, $\mu(N) = 1$

(ii) $S \subseteq T \subseteq N \quad \Rightarrow \quad \mu(S) \leq \mu(T)$.

Given a Choquet measure μ we can define the *Choquet integral* [2] [3] [4] of a vector $\mathbf{x} = (x_1,\ldots,x_n) \in [0,1]^n$ with respect to μ as

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} \tag{1}$$

where $(\cdot)$ indicates a permutation on N such that $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$. Also $A_{(i)} = \{(i),\ldots,(n)\}$ and $A_{(n+1)} = \emptyset$.

Notice that the Choquet integral with respect to an additive measure μ reduces to a weighted arithmetic mean, whose weights w_i are given by the $\mu(i)$ values,

$$\mu(A_{(i)}) = \mu((i)) + \mu((i+1)) + \ldots + \mu((n))$$

$$\mathcal{C}_\mu(\mathbf{x}) = \sum_{i=1}^{n} [\mu(A_{(i)}) - \mu(A_{(i+1)})]\, x_{(i)} = \sum_{i=1}^{n} \mu((i)) x_{(i)} = \sum_{i=1}^{n} w_i x_i \,. \qquad (2)$$

The *importance index* or *Shapley value* [5] [10] of criterion $i \in N$ with respect to μ is defined as

$$\phi_\mu(i) = \sum_{T \subseteq N \setminus i} \frac{(n-1-t)!t!}{n!} [\mu(T \cup i) - \mu(T)]\,, \qquad \sum_{i=1}^{n} \phi_\mu(i) = 1\,. \qquad (3)$$

It amounts to a weighted average of the marginal contribution of element i with respect to all coalitions $T \subseteq N \setminus i$ and it can be interpreted as an effective importance weight.
Consider now a positive reciprocal $n \times n$ matrix $\mathbf{A} = [a_{ij}]$,

$$a_{ij} > 0 \qquad a_{ji} = 1/a_{ij} \qquad i, j = 1, \ldots, n \qquad (4)$$

All pairwise comparison matrices in Saaty's AHP are of this form. However, our model regards only the single pairwise comparison matrix between criteria at the higher level of the AHP. This is because that matrix is the one that controls the aggregation process: in Saaty's AHP, the aggregation is performed through a weighted average whose weights are the components of the higher level priority vector.

In general, the positive reciprocal matrix $\mathbf{A}$ above is inconsistent, where consistency means $a_{ij} = a_{ik} a_{kj}$ for all $i, j, k = 1, \ldots, n$. However, we can associate to the matrix $\mathbf{A}$ a consistent matrix $\tilde{\mathbf{A}} = [\tilde{a}_{ij}]$ in the following way,

$$\tilde{a}_{ij} = w_i / w_j \qquad w_i = u_i / \Sigma_{j=1}^{n} u_j \qquad i, j = 1, \ldots, n \qquad (5)$$

where u_i is the geometric mean of the elements of the row i,

$$u_i = \sqrt[n]{\Pi_{j=1}^{n} a_{ij}} \qquad i, j = 1, \ldots, n \qquad (6)$$

and the weights $w_i > 0$ are normalized $\Sigma_{j=1}^{n} w_i = 1$.

The positive reciprocal matrix $\tilde{\mathbf{A}}$ is in fact consistent, since

$$\tilde{a}_{ij} = w_i / w_j = (w_i / w_k)(w_k / w_j) = \tilde{a}_{ik} \tilde{a}_{kj} \qquad i, j, k = 1, \ldots, n\,. \qquad (7)$$

Moreover, $\tilde{u}_i = \sqrt[n]{\Pi_{j=1}^{n} \tilde{a}_{ij}} = w_i / \sqrt[n]{\Pi_{j=1}^{n} w_j}$ and thus $\tilde{w}_i = \tilde{u}_i / \Sigma_{j=1}^{n} \tilde{u}_j = w_i$, which means that the consistent matrix associated to $\tilde{\mathbf{A}}$ is again $\tilde{\mathbf{A}}$ itself.

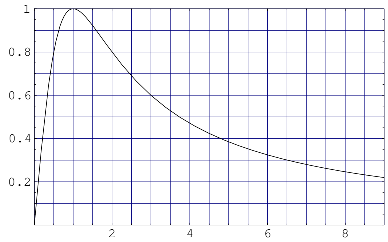
Given an element a_{ij} of the matrix $\mathbf{A}$ we define the *neighbourhood* $U(a_{ij})$ as the set of the elements of the row i and the column j of $\mathbf{A}$,

$$U(a_{ij}) = \{a_{ik}, a_{lj} \,|\, k, l = 1, \ldots, n\}\,. \qquad (8)$$

We say that a_{ij} is *locally consistent* if, on average, it is consistent with the elements in its neighbourhood,

$$a_{ij} = \tilde{a}_{ij} = \sqrt[n]{\Pi_{k=1}^{n} a_{ik} a_{kj}} \qquad i, j = 1, \ldots, n\,. \qquad (9)$$

We now define the *scaling function* $f : (0, \infty) \to (0, 1)$ as $f(x) = 2/(x + x^{-1})$, for $x > 0$, whose graph is shown below.



Notice that the scaling function f has a single critical point at $x = 1$, where it reaches the maximum value $f(1) = 1$. Moreover, the scaling function f has the important property $f(x) = f(x^{-1})$, for all $x > 0$.

By means of the scaling function f, we can associate a positive symmetric $n \times n$ matrix $\mathbf{V} = [v_{ij}]$ to the matrix $\mathbf{A} = [a_{ij}]$ in the following way,

$$v_{ij} = f(a_{ij}/\tilde{a}_{ij}) \qquad i, j = 1, \ldots, n \tag{10}$$

We have

$$v_{ij} \in (0, 1] \qquad v_{ij} = v_{ji} \qquad i, j = 1, \ldots, n\,. \tag{11}$$

The fact that the $n \times n$ matrix $\mathbf{V} = [v_{ij}]$ is symmetric is due to the reciprocity of the positive matrix $\mathbf{A}$ plus the fact that $f(x) = f(x^{-1})$, for $x > 0$, since

$$v_{ji} = f(a_{ji}/\tilde{a}_{ji}) = f(\tilde{a}_{ij}/a_{ij}) = f(a_{ij}/\tilde{a}_{ij}) = v_{ij} \qquad i, j = 1, \ldots, n\,. \tag{12}$$

Notice that $v_{ij} = 1$ if and only if $a_{ij} = \tilde{a}_{ij}$, otherwise $0 < v_{ij} < 1$: the more $a_{ij}/\tilde{a}_{ij}$ differs from 1 the more v_{ij} gets closer to 0. Therefore we can consider the matrix $\mathbf{V} = [v_{ij}]$ as a measure of local consistency. Moreover, we note that our matrix $\mathbf{V} = [v_{ij}]$ can be regarded as a $[0, 1]$-scaled version of the so-called totally inconsistent matrix associated with the original pairwise comparison matrix $\mathbf{A} = [a_{ij}]$, see [1].

Given a general (typically inconsistent) positive reciprocal $n \times n$ matrix $\mathbf{A} = [a_{ij}]$, one can define a 2-additive Choquet measure $\mu : 2^N \longrightarrow [0, 1]$ in the following way: making use of the Möbius transform m of the measure μ, we define $m(i) = w_i/\mathcal{N}$ for each singlet $\{i\}$ and $m(ij) = -w_i(1 - v_{ij})w_j/\mathcal{N}$ for each doublet $\{i, j\}$, with null higher order terms. Then, we define the value of the 2-additive measure μ on a coalition S as the sum of the singlets and doublets contained in the coalition S, as given by the Möbius transform m,

$$\mu(S) = \sum_{\{i\} \subseteq S} w_i/\mathcal{N} + \sum_{\{i,j\} \subseteq S} (-w_i(1 - v_{ij})w_j)/\mathcal{N} \tag{13}$$

where the normalization factor $\mathcal{N}$ is the sum of all singlets and doublets in the set N,

$$\begin{aligned} \mathcal{N} &= \sum_{\{i\} \subseteq N} w_i + \sum_{\{i,j\} \subseteq N} -w_i(1 - v_{ij})w_j = 1 - \frac{1}{2} \sum_{i,j=1}^{n} w_i(1 - v_{ij})w_j \\ &= \frac{1}{2}(1 + \sum_{i,j=1}^{n} w_i v_{ij} w_j) = \frac{1}{2}(1 + \sum_{i=1}^{n} w_i v_i) = \frac{1}{2}(1 + v) \end{aligned} \tag{14}$$

where $v_i = \sum_{j=1}^{n} v_{ij} w_j$ and $v = \sum_{i=1}^{n} w_i v_i$ denote weighted averages of local consistency values, with $w_i < v_i \leq 1$ for $i = 1, \ldots, n$ and $\sum_{i=1}^{n} w_i^2 < v \leq 1$. In particular, we have

$$\mu(i) = w_i/\mathcal{N} \qquad i, j = 1, \ldots, n$$

$$\mu(ij) = (w_i + w_j - w_i(1 - v_{ij})w_j)/\mathcal{N} . \tag{15}$$

The graph interpretation of this definition, in which singlets correspond to nodes and doublets correspond to edges between nodes, is that the value of the 2-additive measure μ on a coalition S is the sum of the nodes and edges contained in the subgraph associated with the coalition S.

The measure μ satisfies the boundary conditions $\mu(\emptyset) = 0$ and $\mu(N) = 1$, and is monotonic and subadditive. The (strict) monotonicity of the measure is guaranteed by the fact that the positive value w_i associated to each node of the graph dominates (in absolute value) the sum of the negative values $-w_i(1 - v_{ij})w_j$ associated to the $n-1$ edges connecting that node with the other nodes in the graph,

$$w_i - \sum_{j=1}^{n} w_i(1 - v_{ij})w_j = w_i - w_i(1 - v_i) = w_i v_i > w_i^2 > 0 \qquad i = 1, \ldots, n . \tag{16}$$

Notice that this model is an extension of Saaty's AHP: if the matrix **A** is consistent then the Choquet measure μ is additive and the Choquet integral coincides with a weighted arithmetic mean whose weights are w_i as in Saaty's AHP.
The Shapley values ϕ_i, $i = 1, \ldots, n$ associated with the measure μ defined above are given by

$$\phi_i = \frac{\varphi_i}{\Sigma_{j=1}^{n} \varphi_j} \qquad i = 1, \ldots, n \tag{17}$$

where the unnormalized values $\varphi_i > 0$, $i = 1, \ldots, n$ are given by

$$\varphi_i = w_i - \frac{1}{2} \sum_{j=1}^{n} w_i(1 - v_{ij})w_j = \frac{1}{2} w_i(1 + v_i) \qquad i = 1, \ldots, n \tag{18}$$

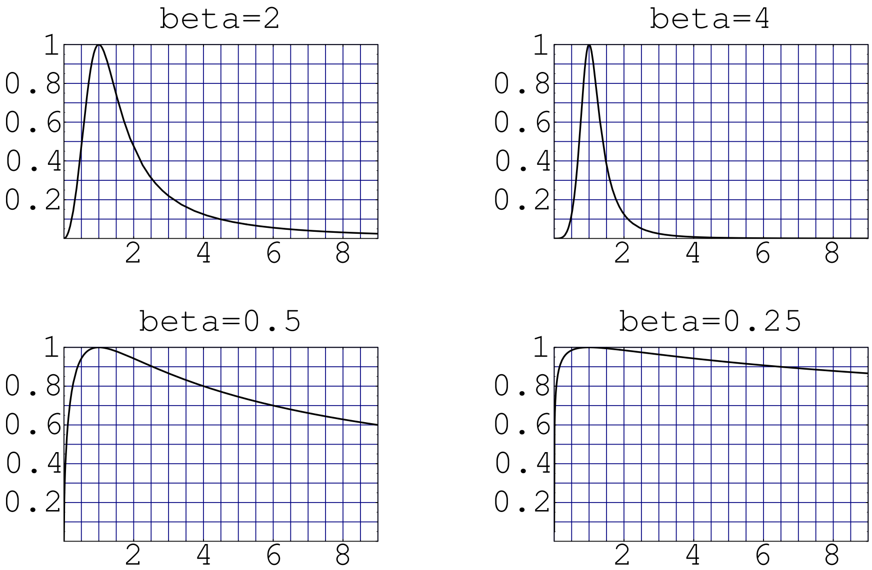
which means that $\Sigma_{j=1}^{n} \varphi_j = \frac{1}{2}(1 + v) = \mathcal{N}$.
In our multicriteria aggregation model the Shapley values encode the effective importance weights of the various criteria. When the matrix **A** is consistent, we have $v_{ij} = 1$ for all $i, j = 1, \ldots, n$ and equation (18) implies that the Shapley values are $\phi_i = w_i$. In general, the fact that **A** is inconsistent changes the original distribution of weights, attenuating the importance values of those criteria that on average are more inconsistent with the others and emphasizing those criteria that on average are more consistent with the others.

In fact, if we compute the second order Taylor expansion of the Shapley values $\phi_i = w_i(1 + v_i)/(1 + v)$, $i = 1, \ldots, n$, around the consistency condition we get

$$\phi_i \approx w_i(1 + \frac{1}{2}(v_i - v)(1 + \frac{1}{2}(1 - v))) \qquad i = 1, \ldots, n \tag{19}$$

Notice that the second order approximation of the Shapley values is still normalized to unit sum, since $\sum_{i=1}^{n} w_i(v_i - v) = 0$. Moreover, the Taylor expansion above shows clearly that, in the small inconsistency approximation, the Shapley value ϕ_i increases if $v_i > v$ and decreases if $v_i < v$. In other words, the Shapley value increases (decreases) if the single consistency measure v_i is greater (smaller) than the overall consistency measure v, in a compensatory mechanism typical of weighted averaging schemes.

Finally, we note that the definition of the scaling function can easily be extended in order to accommodate a free parameter $\beta \geq 0$. We define the parametrized *scaling function* $f : (0, \infty) \to (0, 1)$ as $f_\beta(x) = 2/(x^\beta + x^{-\beta})$, for $x > 0$. Clearly, $f_{\beta=0} = 1$. The graphs of the scaling function f_β for $\beta = 2, 4$ and $\beta = \frac{1}{2}, \frac{1}{4}$ are shown below.



As before, the scaling function f_β has a single critical point at $x = 1$, where it reaches the maximum value $f_\beta(1) = 1$. Moreover, the scaling function f_β has the important property $f_\beta(x) = f_\beta(x^{-1})$, for all $x > 0$.

The scaling function f_β has two different asymptotic behaviours close to the origin in relation with the parameter ranges $0 < \beta < 1$ (vertical asymptote at the origin) and $\beta > 1$ (horizontal asymptote at the origin), as can be easily derived from the expressions below,

$$f_\beta(x) = \frac{2x^\beta}{1+x^{2\beta}} \qquad f'_\beta(x) = \frac{2\beta x^{\beta-1}(1-x^{2\beta})}{(1+x^{2\beta})^2} \qquad x > 0\,. \tag{20}$$

Moreover, it is straightforward to show that the consistency measure provided by the scaling function becomes stricter for increasing values of β. In other words, as β increases, all the local consistency measures $v_{ij}(\beta)$ decrease, with the exception of those associated with exact consistency $v_{ij} = 1$. Accordingly, the inconsistency effects in the context of our model can be attenuated or emphasized, relatively to the original case $\beta = 1$, by means of appropriate choices of the parameter β: higher values of the parameter lead to stronger inconsistency effects.

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Construction of compositional modifiers generated by n-ary functions

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1 Introduction

We consider an idea, how to generate modifiers by *n*-placed functions defined on $[0,1]^n$. The subject matter of modifiers are fuzzy sets, i.e. membership functions defined on interval $I = [0,1]$. We give the definition of modifiers and the case, how *n*-placed generator functions fit together with this definition. We consider some few properties of modifiers. Some examples of generator functions and modifiers generated by them are given. The examples illustrate how graded modifier systems can be created. The place number *n* of a generator function can take effect to the strength of a modifier. It is also possible to keep the place number *n* constant and use numbers 1 or 0 in some places of the variables of the function.

The concept of 'modifier' appears in many ways in the scope of fuzzy logic. For example, Prof. L. A. Zadeh used this term already in the early theory of fuzzy logic. The author has studied modifiers and their logics from modal logical point of view and created some logical systems for modifiers basing on relational Kripke structures of graded modalities (see e.g. Mattila [9]). Kortelainen's [3] concept *modified sets* is one example about the use of this term. In the linguistic view, a modifier can be an adjective, or an adverb, or a phase or clause acting as an adjective or adverb. In every case, the basic principle is the same: the modifier adds information to another element in the sentence (Frances Peck, '*Terms of use*', University of Ottawa). Also some fuzzy logic blocs altering the behavior of PID controllers are called modifiers, too.

The author has considered modifiers and modifier logics in several situations (see e.g. J. K. Mattila [6, 7, 8]). Some considerations about modifiers generated by t-norms and t-conorms are done in [8]. After this work Dr. József Dombi suggested the author to study modifiers generated by n-placed functions in the way to use some t-norms and t-conorms generalized for several variables. Some results from these studies are [10] and [11].

We refer to Kortelainen's concept *modified sets*. His operators are set functions modifying at first hand ordinary sets. We concentrate here upon modifiers generated by n-placed functions, n = 2, 3, ... Entities *M* considered here are so-called *compositional modifiers* because the result of modifying a fuzzy set μ by a modifier *M* is the composition of *M* and μ, $M \circ \mu$. The aim is to consider modifiers as operators for modifying fuzzy sets. One important example is to use especially t-norms and t-conorms and their generalizations for more than two-placed cases as basic tools. (This case is considered in [10].)

2 Basic Concepts

We choose the range of fuzzy sets (i.e. membership functions) to be the unit interval $I = [0,1]$, as usually. Thus the set of all fuzzy sets of a non-empty set X is the set I^X (including the usual power-set 2^X (i.e. the set of all characteristic functions of the usual subsets of X) as a special case). It is also a well-known fact that I and I^X are partially ordered sets. In fact, they are also completely distributed complete lattices and Brouwerian lattices (see e.g. Lowen [5]).

These modifiers we consider here are *compositional*, because when we apply a modifier to a fuzzy set we form a composition of two functions.

Definition 1. (Modifier). We say that a mapping $M : I^X \longrightarrow I^X$ is (i) a *substantiating modifier* if for any fuzzy set $\mu \in I^X$,

$$\forall x \in X, M(\mu(x)) \leq \mu(x), \tag{1}$$

(ii) a *weakening modifier* if for any $\mu \in I^X$,

$$\forall x \in X, \mu(x) \leq M(\mu(x)), \tag{2}$$

(iii) an identity modifier if for any $\mu \in I^X$,

$$\forall x \in X, M(\mu(x)) = \mu(x). \tag{3}$$

Identity modifiers are identity mappings on I^X. They are sometimes needed as links between substantiating and weakening modifiers in some logical structures of modifiers.

A given modifier we can associate with the dual modifier according to the following

Definition 2. (Dual Modifier). Let M and M^* be modifiers. We say that M^* is the *dual modifier* associated with M, if for any fuzzy set $\mu \in I^X$,

$$\forall x \in X, M^*(\mu(x)) = n(M(n(\mu(x)))), \tag{4}$$

where n is a strong negation.

Proposition 3. *If M is a substantiating modifier then its dual M^* is a* weakening modifier *and vice versa.*

Proof. (See also [10]) Suppose $\mu \in I^X$, and M is a substantiating modifier. Thus $\forall x \in X, M(\mu(x)) \leq \mu(x)$. We have to show that $\forall x \in X, \mu(x) = M^*(\mu(x))$. Let n be a strong negation function. Thus $\forall x \in X, M^*((x)) = n(M(n(\mu(x))))$. Clearly

$$M(n(\mu(x))) \leq n(\mu(x))$$

by Def.1. From this it follows by the properties of membership functions that

$$\forall x \in X, n(M(n(\mu(x)))) \geq n(n(\mu(x))) = \mu(x),$$

i.e.

$$M^*(\mu(x)) \geq \mu(x). \tag{5}$$

Conversely, the result follows in the similar way. ■

The condition

$$\forall x \in X, M^*(\mu(x)) = n(M(n(\mu(x)))) \tag{6}$$

in the previous proof says that the operators M, M^*, and n satisfy DeMorganńs law. Thus dual pairs of modifiers with strong negation form classes called DeMorgan triples of operators ([1]). Originally, DeMorgan triples used to consist of a t-norm, corresponding t-conorm, and negation.

We denote α-level set of a fuzzy set μ, as usually,

$$\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha,\ \alpha \in I\}$$

Thus the α-level set of $M(\mu)$ is

$$(M \circ \mu)_\alpha = \{x \in X \mid M(\mu(x)) \geq \alpha,\ \alpha \in I\}. \tag{7}$$

It is easy to see that modifiers have following properties. Suppose M is a substantiating modifier. Then we have

$$M(\mathbf{0}_X) = \mathbf{0}_X, \tag{8}$$
$$M^*(\mathbf{1}_X) = \mathbf{1}_X, \tag{9}$$
$$(M^*)^*(\mu(x)) = M(\mu(x)), \tag{10}$$

where $\mathbf{0_X}$ and $\mathbf{1}_X$ are the constant functions $\mathbf{0}_X(x) = 0$ and $\mathbf{1}_X(x) = 1$ for all $x \in X$.

3 The idea of generating modifiers using n-ary functions

The simplest idea using n-ary functions for generating modifiers for fuzzy sets is to replace every variable with the membership function of a fuzzy set to be modified. To illustrate the idea, we proceed in the following way. We put the same argument x in every place in the n-tuple of arguments in the function f. Thus we have the generating formulas for substantiating, weakening and identity modifiers. A substantiating modifier is generated by any function f, such that

$$\forall\, x \in I, \quad f(x, x, \ldots, x) \leq x. \tag{11}$$

A weakening modifier is generated by any function f, such that

$$\forall\, x \in I, \quad f(x, x, \ldots, x) \geq x. \tag{12}$$

An identity operator is generated by any function f, such that

$$\forall\, x \in I, \quad f(x, x, \ldots, x) = x. \tag{13}$$

According to the formulas (11), (12), and (13), we prove some results concerning modifiers generated by those n-ary functions. For this task, suppose that a modifier M is generated by a function $f(x_1, x_2, \ldots, x_n)$, $x_i \in I$ $(i = 1, \ldots, n)$, such that

$$M(\mu) = M \circ \mu = f(\mu, \mu, \ldots, \mu), \tag{14}$$

where μ is any fuzzy set. The function f is (at least piecewise) continuous on the interval I.

Proposition 4. *Let $f : I^n \longrightarrow I$ be an n-ary function, then f generates a* substantiating modifier

$$F(\mu(x)) = (F \circ \mu)(x) = f(\mu(x), \mu(x), \ldots, \mu(x)) \tag{15}$$

if for all $t_1, t_2, \ldots, t_n \in [0,1]$ the condition

$$f(t_1, t_2, \ldots, t_n) \leq \min(t_1, t_2, \ldots, t_n) \tag{16}$$

holds.

Proof. Suppose the formula (16) holds and denote $min(t_1, t_2, \ldots, t_n) = t_{min}$. Especially, if $t_i = a$ for all $t_i \in I$ then $t_{min} = a$, and thus $f(a, \ldots, a) \leq a$ for any $a \in I$ by (16). Let μ be any fuzzy set and $x_0 \in X$ arbitrarily chosen. Thus

$$f(\mu(x_0), \mu(x_0), \ldots, \mu(x_0)) \leq \mu(x_0).$$

Because x_0 is arbitrarily chosen from X, the same holds for other x's, too. Thus we have

$$f(\mu(x), \mu(x), \ldots, \mu(x)) = F(\mu(x)) \leq \mu(x)$$

for any $x \in X$. Thus the formula (15) holds, and f generates a substantiating modifier F by means of Def.1. ■

Proposition 5. *Let $f : I^n \longrightarrow I$ be an n-ary function, then f generates a* weakening modifier

$$H(\mu(x)) = (H \circ \mu)(x) = f(\mu(x), \mu(x), \ldots, \mu(x)) \tag{17}$$

if for all $t_1, t_2, \ldots, t_n \in [0,1]$ the condition

$$f(t_1, t_2, \ldots, t_n) \geq \max(t_1, t_2, \ldots, t_n) \tag{18}$$

holds.

Proof. Suppose the formula (18) holds. From this it follows that for all $a \in I, \quad H(a) = f(a, \ldots, a) \geq a$ by (14). Let μ be any fuzzy set, and $x_0 \in X$ is arbitrarily chosen. Thus

$$H(\mu(x_0)) = f(\mu(x_0), \ldots, \mu(x_0)) \geq \mu(x_0).$$

Because x_0 is an arbitray element of X, the formula

$$\forall\, x \in X, \quad H(\mu(x), \ldots, \mu(x)) \geq \mu(x)$$

holds. Thus the formula (17) holds, and f generates a weakening modifier. ■

Proposition 6. *Let $f : I^n \longrightarrow I$ be an n-ary function, then f generates an* identity modifier

$$F_0(\mu(x)) = (F_0 \circ \mu) = f(\mu(x), \mu(x), \ldots, \mu(x)) \tag{19}$$

if for all $t_1, t_2, \ldots, t_n \in [0,1]$ the condition

$$\min(t_1, t_2, \ldots, t_n) \leq f(t_1, t_2, \ldots, t_n) \leq \max(t_1, t_2, \ldots, t_n) \tag{20}$$

holds.

Proof. Suppose the formula (20) holds. From this it follows that for all $a \in I, \min(a,\dots,a) \leq f(a,\dots,a) \leq \max(a,\dots,a)$. This is equivalent to $a \leq f(a,\dots,a) \leq a$ which is equivalent to $f(a,\dots,a) = a$. Let μ be any fuzzy set and $x_0 \in X$ is arbitrarily chosen. Thus we have $F_0(\mu(x_0)) = f(\mu(x_0),\dots,\mu(x_0)) = \mu(x_0)$. Because x_0 is arbitrarily chosen from X, this means that the formula

$$\forall\, x \in X, \quad F_0(\mu(x))\, f(\mu(x),\dots,\mu(x)) = \mu(x).$$

Thus the formula (19) holds, and f generates an identity modifier. ■

We see that the Definition1 and the Propositions 4, 5 and 6 correspond to each others.

According to the Propositions 4, 5, and 6, we can use the formulas (16), (18), and (20) as the conditions for *n*-ary functions generating modifiers.

We can have the inverse results of the Propositions 4, 5, and 6. For this we need the following lemma.

Lemma 7. *Let $f : I^n \longrightarrow I$ be a (at least piecewise) continuous n-ary function.*

(a) If f generates a substantiating modifier then this implies the formula (16).

(b) If f generates a weakening modifier then this implies the formula (18).

(c) If f generates an identity modifier then this implies the formula (20).

Proof. (a) Let us give the counter-hypothesis: $f(t_1,\dots,t_n) > \min(t_1,\dots,t_n)$. From this it follows that f generates either an identity modifier by Proposition 6 or a weakening modifier by Proposition 5. This contradicts the supposition that f generates a substantiating modifier. Thus the counter-hypothesis is not correct.

The cases (b) and (c) can be proved in similar ways. ■.

After collecting the results from Propositions 4, 5, 6, and Lemma 7 we have proved the following

Theorem 8. *Let $f : I^n \longrightarrow I$ be (at least piecewise) continuous n-ary function. f generates a modifier $F \circ \mu$ which is*

(a) substantiating iff $f(t_1,\dots,t_n) \leq \min(t_1,\dots,t_n)$,

(b) weakening iff $f(t_1,\dots,t_n) \geq \max(t_1,\dots,t_n)$,

(c) an identity modifier iff $\min(t_1,\dots,t_n) \leq f(t_1,\dots,t_n) \leq \max(t_1,\dots,t_n)$,

where the compositions are calculated by means of (14.

Using Theorem 8 we can prove the following

Theorem 9. *Let $f : I^n \longrightarrow I$ be a function generating a substantiating modifier F. Then the function $f_{co} : I^n \longrightarrow I \,:\, f_{co}(x_1,\dots,x_n) = 1 - f(1-x_1,\dots,1-x_n)$ generates a weakening modifier being the dual of F.*

Proof. It follows from the supposition, that f generates a substantiating modifier, that $f(x_1,\ldots,x_n) \leq \min(x_1,\ldots,x_n)$, and $\forall\, x \in X,\ F(\mu(x) = f(\mu(x),\ldots,\mu(x)) \leq \mu(x)$ by Theorem 8. This is equivalent to $1-F(\mu(x)) \geq 1-\mu(x)$. Replace $\mu(x)$ by $1-\mu(x)$, then we have

$$1-F(1-\mu(x)) \geq 1-(1-\mu(x)).$$

On the other hand, $1-F(1-\mu(x)) = 1-f(1-\mu(x),\ldots,1-\mu(x))$. Thus

$$1-f(1-\mu(x),\ldots,1-\mu(x)) \geq 1-(1-\mu(x)).$$

Thus the conclusion is that the function $f_{co}(x_1,\ldots,x_n) = 1-f(1-x_1,\ldots,1-x_n)$ generates a weakening modifier by Theorem 8. Clearly this modifier is the dual of F. ■

Example 10. The formula

$$f(x_1,x_2,\ldots,x_n) = \prod_{i=1}^{n} x_i, \qquad \forall i, x_i \in [0,1], \tag{21}$$

generates a substantiating modifier

$$F_{n-1}(\mu(x)) = (\mu(x))^n, \qquad \forall x \in X, \tag{22}$$

because it clearly satisfies the condition (16), i.e. $f(x_1,x_2,\ldots,x_n) \leq min(x_1,x_2,\ldots,x_n)$. The bigger n is the more substantiating modifier we have. Thus we can have a graded system of modifiers. Especially, if $n=1$, we have the identity modifier $F_0 = \mu$, that have no substantiating effect.

Example 11. The formula

$$f(x_1,x_2,\ldots,x_n) = 1-\prod_{i=1}^{n}(1-x_i), \qquad \forall i, x_i \in [0,1], \tag{23}$$

generates a weakening modifier

$$H_{n-1}(\mu(x)) = 1-(1-\mu(x))^n, \qquad \forall x \in X \tag{24}$$

because it clearly satisfies the condition (18). To see this, let a delivery of values from the interval $[0,1]$ be such that $x_k, 1 \leq k \leq n$, has the greatest value. In this situation we can write $max(x_1,x_2,\ldots,x_n) = x_k$. Thus we have $1-x_k \geq 1-x_i, 1 \leq i \leq n$, and this implies

$$(1-x_k)^n \geq \prod_{i=1}^{n}(1-x_i)$$

which implies

$$1-\prod_{i=1}^{n}(1-x_i \geq 1-(1-x_k)^n.$$

From this it clearly follows that $1-(1-x_k)^n \geq 1-(1-x_k) = x_k = max(x_1,x_2,\ldots,x_n)$ by the suppoaition of the delivery of values. The special case $n=1$ gives the idetity modifier $H_0(\mu(x)) = \mu(x) \forall x \in X$, as it should be.

Example 12. In addition to the special cases of previous examples, consider some generators for identity modifier. The formula

$$f(x_1,x_2,\ldots,x_n) = \frac{1}{n}(x_1+x_2+\ldots+x_n), \tag{25}$$

generates identity modifier, because (18) holds clearly.

Another way for generating identity modifier is to use the function

$$f(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} \lambda_i x_i, \tag{26}$$

where $\sum_{i=1}^{n} \lambda_i = 1$. It is easy to show that this function satisfies the condition (18).

Also $max(x_1,x_2,\ldots,x_n)$ and $min(x_1,x_2,\ldots,x_n)$ generate identity modifiers, because the operators max and min do not have any modifying effect.

According to Def.2, the *dual* of a modifier F is defined by the condition

$$F^*(x) = n(F(n(x))) \tag{27}$$

where n is a strong negation function. This also means that if F is substantiating then F^* is weakening, and if H is weakening then H^* is substantiating, by Proposition 3.

Example 13. The modifiers given in Examples 10 and 11 are duals of each others when $\forall x \in X, n(\mu(x)) = 1-\mu(x)$. The modifiers (22) and (24) are basing on extensions of the t-norm *algebraic product* and the t-conorm *algebraic sum*, respectively.

4 Some Concluding Remarks

One purpose for studying modifiers is to create some concrete tools for manipulating fuzzy numbers so that we can have arithmetic operations to be easily used. However, these operations should be in accordance with the original definition where extension principle is used. Also the study of logical systems of modifiers is very interesting. From this study we can draw connections to topological properties of fuzzy systems (see e.g. Kortelainen's paper [3] and his other papers, too).

According to the substance itself, n-ary functions being extensions of some Archimedean t-norms and t-conorms are very interesting for generators of modifiers, as we already had a short view in the form of Examples 10 – 13 above. It is well known that Archimedean t-norms and corresponding t-conorms have modifying effects (see e.g. Mattila [8]).

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Ordinal sorting in the presence of interacting points of view: TOMASO

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1 Introduction

This paper presents an ordered sorting procedure based on the Choquet integral as a discriminant function. It uses information provided by the Decision Maker (DM) in terms of a set of prototypes (alternatives well-known to the DM). The capacities of the Choquet integral are assessed through the solving of a linear program or a quadratic program. An interpretation of the results is provided by means of importance and interaction indexes of the points of view.

We analyze a sorting procedure for ordinal data in a very general case, where the points of view can have interactions. Its name, TOMASO stands for **T**ool for **O**rdinal **M**ulti**A**ttribute **S**orting and **O**rdering. The first version of this method has been described in [7] and [9]. Later, in [6] the authors present further evolutions to the first ideas, and describe a software which is directly inspired from the sorting procedure.

Three important features differentiate this procedure from other multiple criteria sorting methods. First of all, the possibility to treat purely ordinal data. Secondly, the use of a Choquet integral [1] as a discriminant function. And finally, the way the capacities ("weights") of the aggregator are learnt from a reference set of alternatives called prototypes. These three key features allow to treat a quite large set of problems. In particular, the learning feature of the method is interesting as it allows to ask the Decision Maker (DM) a minimal set of technical details. In order to allow a more effective and objective analysis of the problem, we think that it is useful to have a permanent interaction with the DM. But this questioning should mainly be restricted to his expertise domain and not to technical parameters of the method. The use of the prototypes fits to this philosophy.

The method works in two steps. First of all, the ordinal data is transformed into partial net scores, where each alternative is compared to all the other ones for each point of view. Then, the Choquet integral is used to aggregate these partial net scores. As already mentioned earlier, the capacities of the aggregator are learnt from the reference set of prototypes. Here, two options appear: either the prototypes don't violate the axioms ([11]) for the use of a Choquet integral as a discriminant function, or the structure of the prototypes does not allow its use as an aggregator. In the first case, the capacities are learnt by solving a linear constraints satisfaction problem. This procedure is briefly recalled in section 3.1. In the second case, the capacities are learnt by trying to be as close as possible to the original sorting imposed by the prototypes. This part is described in section 3.2.

This paper is organized as follows. First of all, general concepts are introduced in section 2. Then, in section 3.1 we recall the first ideas of TOMASO already published in [6]. In section 3.2 we present how to work in case the classical way fails. Finally, in 4 we draw some conclusions, and discuss further improvements.

2 Preliminary considerations

Let A be a set of q potential alternatives which are to be assigned to disjoint ordered classes. Let $F = \{g_1, \dots, g_n\}$ be a set of points of view. For each index of point of view $j \in \mathcal{J} = \{1, \dots, n\}$, the alternatives are evaluated according to a s_j-point ordinal performance scale represented by a totally ordered set

$$X_j := \{g_1^j \prec_j \dots \prec_j g_{s_j}^j\}.$$

Therefore, an alternative $x \in A$ can be identified with its corresponding profile

$$(x_1, \dots, x_n) \in \prod_{j=1}^{n} X_j =: X,$$

where for any $j \in \mathcal{J}$, x_j is the partial evaluation of x on point of view j.

Let us consider a partition of $X := \Pi_{j=1}^n X_j$ into m nonempty increasingly ordered classes $\{Cl_t\}_{t=1}^m$. This means that for any $r, s \in \{1, \dots, m\}$, with $r > s$, the elements of Cl_r are considered as better than the elements of Cl_s. The sorting problem we are dealing with consists in partitioning assigning the alternatives of A to the classes $\{Cl_t\}_{t=1}^m$.

In Roubens [9] it is justified how an n-place Choquet integral as a discriminant function and normalised scores as criteria function can be used to solve this problem. Hereafter we present the sorting procedure derived from this particular case.

3 The TOMASO method

The TOMASO method (**T**echnique for **O**rdinal **M**ulti**a**ttribute **S**orting and **O**rdering) is mainly based on two techniques (which can lead to the same results under certain conditions). The original method has first been described in [9]. In the following Subsection, we present its basics. In Subsection 3.2 we show how it is possible to deal with a larger set of problems.

3.1 The classical way

The different stages of the original TOMASO are listed below:

1. Modification of the criteria evaluations into normalised scores;
2. Use of a Choquet integral as a discriminant function;

3. Assessment of fuzzy measures by questionning the DM and by solving a linear constraint satisfaction problem;

4. Calculation of the borders of the classes and assignment of the alternatives to the classes;

5. Analysis of the results (interaction, importance, leave one out, visualisation).

In this Section we roughly present these different elements.

First of all, concerning the scales on the points of view, two natural approaches can be considered: either the score of each alternative is built on the basis of all the alternatives in A or this score is constructed in a context-free manner, that is, independently of the other alternatives. The decision maker must be aware that the final results may significantly differ according to the considered approach. Therefore, a prior analysis of the problem is recommended to choose the scores appropriately.

In the first approach, one possible way to build the scores is to consider comparisons of the alternatives on each of the points of view. We consider $S_j(x)$, the jth partial net score of alternative $x \in A$ along point of view $j \in \mathcal{J}$, as the number of times that x is preferred to any other alternative of A minus the number of times that any other alternative of A is preferred to x for point of view j. We furthermore normalize these scores so that they range in the unit interval, i.e.,

$$S_j^N(x) := \frac{S_j(x) + (q-1)}{2(q-1)} \in [0,1] \qquad (j \in \mathcal{J},$$

where $q = |A|$. Clearly, this normalized score is not a utility, and should not be considered as such. Indeed, observing an extreme value (close to 0 or 1) means that x is rather “atypical” compared to the other alternatives along point of view j. Thus, the resulting evaluations strongly depend on the alternatives which have been chosen to build A.

Consider now the second approach, that is, where the score of each alternative does not depend on the other alternatives in A. In this case, we suggest the decision maker provides the score functions as utility functions. Alternatively, we can approximate these utility functions by the following linear formula:

$$S_j^N(x) := \frac{\mathrm{ord}_j(x) - 1}{s_j - 1} \in [0,1] \qquad (j \in \mathcal{J}),$$

where $\mathrm{ord}_j : A \to \{1, \ldots, s_j\}$ is a mapping defined by $\mathrm{ord}_j(x) = r$ if and only if $x_j = g_r^j$. In this latter case, S_j^N does not necessarily represent a real utility and probably does not correspond to the utility the decision maker has in mind. We therefore continue to call it a score.

We now come to the crucial part of the aggregation of the normalised partial net scores of a given alternative x by means of a Choquet integral [1]. The advantage of this aggregator is mainly that it allows to deal with interacting (depending) points of view. According to the general definition of the Choquet integral, we have:

$$\mathcal{C}_v(S^N(x)) := \sum_{j=1}^{n} S_{(j)}^N(x)[v(A_{(j)}) - v(A_{(j+1)})]$$

where v is a fuzzy measure on $\mathcal{J}$; that is a monotone set function $v : 2^{\mathcal{J}} \to [0,1]$ fulfilling $v(\emptyset) = 0$ and $v(\mathcal{J}) = 1$. The parentheses used for indexes stand for a permutation on $\mathcal{J}$ such that

$$S_{(1)}^N(x) \leq \ldots \leq S_{(n)}^N(x),$$

and for any $j \in \mathcal{J}$, $A_{(j)}$ represents the subset $\{(j),\ldots,(n)\}$. The characterisation of the Choquet integral by Marichal ([4], [5]) clearly justifies the way the partial scores are aggregated.

The next step of this method is to assess the fuzzy measures in order to classify the alternatives of A. One can easily understand that it is impossible to ask the DM to give values for the $2^n - 2$ free parameters of the fuzzy measure v. Practically, the assessment of the fuzzy measures is done by asking the DM to provide a set of prototypes $P \subseteq A$ and their assignments to the given classes; that is a partition of P into prototypic classes $\{P_t\}_{t=1}^m$ where $P_t := P \cap Cl_t$ for $t \in \{1,\ldots,m\}$. The values of the fuzzy measure are then derived from this information as described hereafter.

We would like the Choquet integral to strictly separate the classes Cl_t. Therefore, the following necessary condition is imposed

$$\mathcal{C}_v(S^N(x)) - \mathcal{C}_v(S^N(x')) \geq \varepsilon \tag{1}$$

for each ordered pair $(x,x') \in P_t \times P_{t-1}$ and each $t \in \{2,\ldots,m\}$, where ε is a given strictly positive thershold.

Due to the increasing monotonicity of the Choquet integral, the number of separation constraints 1 can be reduced significantly. Thus, it is enough to consider *border elements* of the classes. To formalise this concept, we first define a dominance relation D (partial order) on X by

$$xDy \quad \text{iff} \quad x_j \succeq_j y_j, \text{ for all } j \in \mathcal{J}.$$

As *upper border* of the prototypic class P_t we use the set of non-dominated alternatives of P_t defined by

$$ND_t := \{x \in P_t \text{ s.t. } \nexists x' \in P_t \setminus \{x\} : x'Dx\}.$$

Similarly, the *lower border* of the prototypic class is given by the set of non-dominating alternatives of P_t which is defined by

$$Nd_t := \{x \in P_t \text{ s.t. } \nexists x' \in P_t \setminus \{x\} : xDx'\}.$$

The separation conditions restricted to the prototypes of the subsets $ND_t \cup Nd_t$, $t \in \{1,\ldots,m\}$ put together with the monotonicity constraints on the fuzzy measure, form a linear program [7] whose unknowns are the capacities $v(S)$, $S \subset \mathcal{J}$ and where ε is a non-negative variable to be maximised in order to deliver well separated classes.

We use the principle of parsimony for the resolution of this problem. If there exists a k-additive fuzzy measure v^* ([3]), k being kept as low as possible, then we determine the boundaries of the classes as follows:

- lower boundary of Cl_t: $z(t) := \min_{x \in Nd_t} \mathcal{C}_{v^*}(S^N(x))$;
- upper boundary of Cl_t: $Z(t) := \max_{x \in ND_t} \mathcal{C}_{v^*}(S^N(x))$.

At this point, any alternative $x \in A$ can be classified in the following way:

- x is assigned to class Cl_t if $z_t \leq \mathcal{C}_{v^*}(S^N(x)) \leq Z_t$;
- x is assigned to class $Cl_t \cup Cl_{t-1}$ if $Z_{t-1} < \mathcal{C}_{v^*}(S^N(x)) < z_t$.

A final step of the classical TOMASO method concerns the evaluation of the results and the interpretation of the behavior of the Choquet integral. The meaning of the values $v(T)$ is not clear to the DM. They don't immediatly indicate the global importance of the points of view, nor their degree of interaction. It is possible to derive some indexes from the fuzzy measure which are helpful to interpret its behavior. Among them, the TOMASO method proposes to have a closer look at the importance indexes [10] and the interaction indexes [8].

3.2 An alternate way

It may happen that the linear program described in Subsection 3.1 has no solution. This occurs when the prototypic elements violate the axioms that are imposed to produce a discriminant function of Choquet type ([5] [11]), in particular the triple cancellation axiom.

In such a case, and in order to present a solution to the DM, we suggest to find a fuzzy measure by solving the following quadratic program

$$\min \sum_{x \in \cup_{t \in \{1,\dots,m\}} \{ND_t \cup ND_t\}} [\mathcal{C}_v(S^N(x)) - y(x)]^2,$$

where the unkowns are

- the capacities $v(S)$ which determine the fuzzy measure;
- some global evaluations $y(x)$ for each $x \in \cup_{t \in \{1,\dots,m\}} \{ND_t \cup Nd_t\}$.

The capacities $v(S)$ are constrained by the monotonicity conditions (as previously shown in Section 3.1). The global evaluations $y(x)$ must verify the classification imposed by the DM. In other words, for every ordered pair $(x, x') \in Nd_t \times ND_{t-1}$, $t \in \{2, \dots, m\}$ the condition $y(x) - y(x') \geq \varepsilon'$, $\varepsilon' > 0$ must be satisfied.

Intuitively, for a given alternative $x \in A$, its Choquet integral $\mathcal{C}_v(S^N(x))$ should be as close as possible to the global evaluation $y(x)$, without being constrained by monotonicity conditions which might violate the triple cancellation axiom for example. On the other hand, the evaluation $y(x)$ is constrained by the conditions derived from the original classification given by the DM on the prototypes.

Unlike the method described in Section 3.1, in this case, ε' plays the role of a parameter, which needs to be fixed by the DM. As previously, we use the principle of parsimony when searching for a solution (keep k as low as possible; at worst k equals the number of points of view). A correct choice of ε' remains one of the main challenges of our future research. It is clear that ε' has to be chosen in $]0, 1/n[$.

As in the classical method, the next step is to determine the structure of the classes. We determine an assignment for every alternative of X in terms of intervals of contiguous classes on the basis of the information provided by the Choquet integrals related to the prototypes of $P \subseteq A$.

First of all, let us suppose that $S^N(x^-) := (0, \dots, 0)$ is classified to the worst class, Cl_1 and that $S^N(x^+) := (1, \dots, 1)$ is classified to the best class, Cl_m.

To each assignment $I(x)$ correspond a lower class label $\underline{l}(x)$ and an upper class label $\bar{l}(x)$, $\underline{l}, \bar{l} \in \mathcal{J}$. We say that the alternative $x \in X$ is *precisely assigned* to $Cl_{l(x)}$ if for the assignment $I(x)$ we have $\underline{l}(x) = \bar{l}(x) =: l(x)$. Else, the alternative x is said to be *ambiguously assigned* to the interval of labels

$I(x) = [\underline{l}(x), \bar{l}(x)]$. The *degree of the assignment* corresponds to the number of contiguous classes contained in $I(x)$, $d(x) = \bar{l}(x) - \underline{l}(x) + 1$.

The assignments are done according to the procedure described hereafter. Starting from the prototypes $x \in P$, their Choquet integrals $\mathcal{C}_v(S^N(x))$ and their original classification label $Cl(x)$ (according to the DM's choice), we define for every $u \in [0,1]$,

$$m(u) = \max_{x \in P: \mathcal{C}_v(S^N(x)) \leq u} Cl(x), \text{and}$$

$$M(u) = \min_{x \in P: \mathcal{C}_v(S^N(x)) \geq u} Cl(x).$$

m (resp. M) is a right (resp. left) continuous stepwise function of argument u with values belonging to the discrete finite set $\mathcal{J}$.

We now define for each $u \in [0,1]$ an interval of contiguous classes $I(u) = [\underline{l}(u), \bar{l}(u)]$ where

$$\underline{l}(u) = \min\{m(u), M(u)\}$$

$$\bar{l}(u) = \max\{m(u), M(u)\}.$$

Obviously $\underline{l}(u) \leq \bar{l}(u)$ and due to monotonicity of m and M we have: $\underline{l}(u) \leq \underline{l}(v), \bar{l}(u) \leq \bar{l}(v), \forall u, v \in [0,1]$ with $u \leq v$.

The interval $[0,1]$ is partitioned into (closed, semi-open or open) intervals $I_s, s = 1, \ldots, S$, and each of those intervals of $[0,1]$ receives an assignment of the type $[\underline{l}(s), \bar{l}(s)]$ (or semi-open or open) in such a way that: if $u, v \in [0,1], u \leq v$ and if u is assigned to $I_r := [\underline{l}(r), \bar{l}(r)]$ and v is assigned to $I_{r'} := [\underline{l}(r'), \bar{l}(r')]$ then $\underline{l}(r) \leq \underline{l}(r')$ and $\bar{l}(r) \leq \bar{l}(r')$.

Moreover if $u = \mathcal{C}_v(S^N(x)), x \in P$ then $\underline{l}(u) \leq Cl(x) \leq \bar{l}(u)$. This means that each prototype is *correctly classified*, possibly with ambiguity if $d(x) \geq 1$.

The assignment of a prototype a to the intervals of classes leads now to two scenarios:

- a is assigned to a single class ($d(a) = 1$) which corresponds to the original class decided by the DM
- a is assigned to an interval of classes and the original class decided by the DM belongs to this interval.

The quality of a model (classifier) depends on different ratios. A good model has the following *natural* properties:

- a simple model according to parsimony (low k);
- a high number of precise assignments of the elements of P;
- a low number of ambiguous assignments of the elements of P (and the lower the degree of the assignment, the better the model)

For a given ε', the DM has to select a model (k) which seems the best compromise to him in terms of the previously described assignments. The simplest additive model ($k = 1$) can in certain situations be this ideal compromise between simplicity and quality. But in more complex problems, k has to be increased in order to obtain a satisfying number of precisely assigned prototypes.

3.3 Behavioral analysis of aggregation

Now that we have a sorting model for assigning alternatives to classes (based on the linear program or the quadratic program), an important question arises: How can we interpret the behavior of the Choquet integral or that of its associated fuzzy measure? Of course the meaning of the values $v(T)$ is not always clear for the DM. These values do not give immediately the global importance of the points of view, nor the degree of interaction among them.

In fact, from a given fuzzy measure, it is possible to derive some indexes or parameters that will enable us to interpret the behavior of the fuzzy measure. These indexes constitute a kind of *id card* of the fuzzy measure. The TOMASO method presently allows to analyse both the importance of points of view (Shapley indexes [10]), and their interactions ([8]).

3.4 Interpretation of the behaviour of the fuzzy measure

In this Section we briefly show the main advantage to use a Choquet integral rather than the weighted sum as a discriminant function. We therefore take the simple case of two points of view, which can be represented in a plane. Figure 4 presents 5 possible ranges of values for the weights v and the corresponding structures of the limits of the classes. One can see that the main difference between the classical weighted sum and the Choquet integral is the greater flexibility of the borders of the classes. The Choquet integral creates piecewise linear borders, which allows to build more precise classes. The different possibilities are summarised by the following list:

- I: $v(1)+v(2) < v(12)$: synergy
- II: $v(1)+v(2) > v(12)$: redundancy
- III: $v(1)+v(2) = v(12) = 1$: additivity
- IV: $v(1) = v(2) = 0$: limit case; maximal synergy
- V: $v(1) = v(2) = 1$: limit case; maximal redundancy

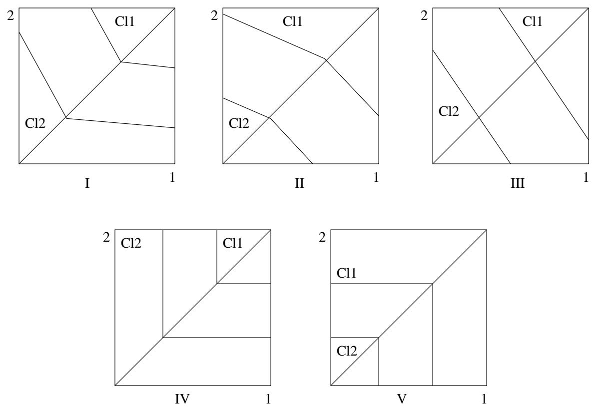

Figure 4: Interpretation of the discriminant functions

In [2] the authors give an interpretation to the first two cases. In case of synergy, although the importance of a single criterion for the decision is rather low, the importance of the pair is large. The criteria are said to be *complementary*. In case of redundance, or negative synergy, the union of criteria does not bring much, and the importance of the pair might be roughly the same as the importance of a single criterion.

The limit case (IV) occurs for maximal synergy. In that case, the Choquet integral corresponds to the aggregation by the min function. Maximal redundancy occurs for case (V), where the Choquet integral is the max function.

In case the number of points of view is larger than two, it becomes quite hard to represent the problem. Nevertheless, the previous short example helps to understand how the borders of the classes are built in such more general examples.

3.5 The software TOMASO

In this short part of the paper we briefly present the key-characteristics of the software TOMASO . It can be downloaded on http://patrickmeyer.tripod.com. It is an implementation of the algorithms which were presented previously. Its name stands for "Tool for Ordinal MultiAttribute Sorting and Ordering". It is written in Visual Basic and uses two external solvers: a free linear program solver (lp_solve 3.0, ftp://ftp.ics.ele.tue.nl/pub/lp_solve/, released under the LGPL license), and a non free quadratic program solver (bpmpd, free trial version at http://www.sztaki.hu/ meszaros/bpmpd/).

It is still under development and many improvements are added on a regular basis. The general steps of the software are outlined hereafter:

- Loading of the ordinal data;
- Choice of a scoring method according to the problem's specificities and calculation of the normalised partial net scores;
- Definition of the prototypes by the DM;
- Search for a fuzzy measure (either by the linear program, or the quadratic program)
- Analysis of the results (classes, Shapley indexes, interaction indexes, accuracies, ...)

A detailed description of the software can be obtained from the author.

4 Concluding remarks

We have presented a procedure for ordinal sorting in the presence of interacting points of view. It has already been applied to real life cases (in particular to a noise annoyance problem) quite successfully. Future work will concern the simplification of the software in order to make it even more user-friendly. Furthermore, the automatic determination of ε' will also be one of our main concerns. The implementation of other indexes (veto, favour, ...) is also planned.

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Regular measures on tribes of fuzzy sets

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Abstract

The classical measure and probability theory is based on the notion of σ-algebra of subsets of a set. Butnariu and Klement [3] generalized it to fuzzy sets by considering collections of fuzzy sets called T-tribes (where T denotes a fixed triangular norm). Their concept of T-measure is fundamental in the fuzzification of classical measure theory. However, it has been successfully applied elsewhere, too (e.g., in finding solutions of games with fuzzy coalitions, see [4]). Here we summarize results about characterization of measures on tribes. Unlike preceding papers, we put emphasis on *regular* measures which were introduced in [21]. We argue that this notion could be considered as a promising alternative to the original notion of Butnariu and Klement.

1 Introduction

The notion of "fuzzy measure theory" is used in different meanings (see [10] and the overview in [23]). Here we try to define *real-valued* measures on collections of *fuzzy sets*. Thus, we want to fuzzify the *domain* but not the *range* of a measure. When the generalized notions are restricted to systems of crisp sets, we expect them to coincide with the classical ones. A certain work in this direction was initiated by Butnariu and Klement [3, 4, 11]. They introduced T-tribes of fuzzy sets with T-measures as a natural generalization of a measure space. They made the first steps towards a characterization of monotone real-valued T-measures for a Frank triangular norm T. This project has been completed by Mesiar and Navara in [16]. Detailed summaries of this approach, together with a thorough analysis of Jordan decomposition, Lyapunov theorem, etc., may be found in [5, 6].

Later on, Barbieri and H. Weber and independently Navara found two generalizations, one for vector-valued T-measures with respect to Frank t-norms (in particular for non-monotone ones) [2], the other for monotone real-valued T-measures with respect to general strict t-norms [20]. A common generalization of these two results was proved by Barbieri, Navara, and H. Weber in[1]— a characterization of non-monotone (even vector-valued) T-measures with respect to an arbitrary strict t-norm.

All these results assumed a special structure of the tribes. Recently in [7] it was found that these assumptions are satisfied for many, but not all strict t-norms. The measure-theoretical consequences of this fact, as well as a new approach to proofs of all preceding results, form the subject of the paper [21]; here we summarize its main conclusions. Unless specified otherwise, we use the terminology and notation of [12].

2 Tribes

The notion of tribe was suggested by Butnariu and Klement [3, 4] as a fuzzification of a σ-algebra. In order to define measures on fuzzy subsets of some set, we need the underlying collections of measurable fuzzy sets (tribes) to be closed under fuzzy operations corresponding to those used in a σ-algebra. In particular, we need a fuzzy complement and a fuzzy union or a fuzzy intersection.

Assumption 1. Throughout this paper we assume that a *fuzzy complement*, f', of a fuzzy set f is obtained by the pointwise application of a *(strong) fuzzy negation*, i.e., an involutive decreasing bijection $': [0,1] \longrightarrow [0,1]$. A *fuzzy intersection*, resp. a *fuzzy union*, is obtained by a pointwise application of a t-norm T, resp. the t-conorm S dual to T with respect to $'$. (We use the same symbols for fuzzy operations on truth values from $[0,1]$ and operations on fuzzy sets induced by them.) The symbol $\leq$ denotes the usual ordering of fuzzy sets as real-valued functions (fuzzy inclusion), and $f_n \nearrow f$ (resp. $f_n \searrow f$) stands for the pointwise convergence of an increasing (resp. decreasing) sequence of functions $(f_n)_{n\in\mathbb{N}}$.

Definition 2. Let X be a non-empty set. A *tribe* on X is a pentuple $(\mathcal{T}, T, ', 0, \leq)$, where $\mathcal{T} \subseteq [0,1]^X$, T is a t-norm, $'$ is a fuzzy negation, 0 is the constant zero function on X, $\leq$ is the fuzzy inclusion, and the following conditions are satisfied:

(T1) $\quad 0 \in \mathcal{T}$,

(T2) $\quad f \in \mathcal{T} \implies f' \in \mathcal{T}$,

(T3) $\quad f, g \in \mathcal{T} \implies T(f,g) \in \mathcal{T}$,

(T4) $\quad (f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \implies f \in \mathcal{T}$.

We refer to X as the *domain* of the tribe $(\mathcal{T}, T, ', 0, \leq)$. By *$T$-tribe operations* we mean the following operations: nulary 0, unary $'$, binary T, and the limit of increasing sequences.

Assumption 3. From now on, we shall consider only tribes with the *standard negation* $a' = 1 - a$.

Remark 4. The latter assumption is not much restrictive, because every tribe is isomorphic to a tribe in which $'$ is the standard negation. (All preceding papers—including [3, 4]—admitted only the standard negation in the definition of a tribe. In this aspect, our definition is more general.)

Using a multiplicative generator, also any strict t-norm may be considered equivalent to the product t-norm. However, this does not mean that any tribe is isomorphic to a tribe with the product t-norm and the standard negation. The problem is that the multiplicative generator does not have to preserve the standard negation. Thus only one of the operations—the t-norm or the fuzzy negation—can be standardized using an isomorphism of tribes.

We have already fixed the standard fuzzy negation $'$. Also 0 and $\leq$ have their stable meaning. On the other hand, the choice of the t-norm T is crucial and we shall always need to specify it. When there is no risk of confusion, we shall speak briefly of a tribe $(\mathcal{T}, T)$ (as in [1]), resp. of a T*-tribe* $\mathcal{T}$. (The latter is the original terminology of [3, 4]. The full notation $(\mathcal{T}, T, ', 0, \leq)$ was used in [23].) We also speak of a T-tribe when we need to refer to the t-norm T, but not to the tribe itself.

Condition (T2) allows us to use duality, hence every T-tribe contains the constant function 1 and it is closed under the t-conorm S dual to T and under limits of decreasing sequences. Thus every T-tribe is closed also under the application of t-norm T to infinite sequences:

$$\text{(T3+)} \qquad (f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}} \implies \mathsf{T}_{n\in\mathbb{N}} f_n \in \mathcal{T},$$

because $\mathsf{T}_{n\in\mathbb{N}} f_n$ is the limit of the decreasing sequence $(\mathsf{T}_{n=1}^{k} f_n)_{k\in\mathbb{N}}$. In the original definition of a T-tribe by Butnariu and Klement [3, 4], conditions (T3), (T4) were replaced by (T3+). In this aspect, our definition is slightly less general. However, this difference is not essential. In fact, in many important cases the two definitions coincide. In particular, all results found in the literature were obtained for tribes which satisfy also our definition. We shall see that the definition presented here is quite natural and advantageous for introducing measures on tribes.

Let T be a t-norm and $(\mathcal{T}, T)$ be a tribe on X. The elements of $\mathcal{T} \cap \{0,1\}^X$ are called *Boolean elements*.

Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. Let $\mathcal{S}$ be the corresponding collection of characteristic functions,

$$\mathcal{S} = \{\chi_A \mid A \in \mathcal{A}\},$$

and

$$\mathcal{T} = \{f \in [0,1]^X \mid f \text{ is } \mathcal{A}\text{-measurable}\}.$$

For any t-norm T, $(\mathcal{S}, T)$ is a tribe called the *Boolean tribe* induced by $\mathcal{A}$. For any measurable t-norm T, $(\mathcal{T}, T)$, is a tribe called the *full tribe* induced by $\mathcal{A}$. (Full tribes were first studied in [3], where they were called *generated tribes*. Here we use the terminology from [22].)

3 Measures on tribes

In [3, 4], Butnariu and Klement introduced T*-measures* as a natural generalization of σ-additive measures on σ-algebras. Here we call them only *measures* because the t-norm T is specified with the tribe. By $\mathbb{R}_+$ we denote the set of all non-negative reals.

Definition 5. Let $(\mathcal{T}, T, ', 0, \leq)$ be a tribe. A functional $\mu\colon \mathcal{T} \longrightarrow \mathbb{R}_+$ is called a *measure* if it satisfies the following axioms:

$$\text{(M1)} \qquad \mu(0) = 0,$$

$$\text{(M2)} \qquad f, g \in \mathcal{T} \implies \mu(T(f,g)) + \mu(S(f,g)) = \mu(f) + \mu(g),$$

$$\text{(M3)} \qquad (f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \implies \lim_{n\in\mathbb{N}} \mu(f_n) = \mu(f).$$

Remark 6. Condition (T4) ensures that $f \in \mathcal{T}$ in (M3). In the original definition of a T-measure [3], (T4) was not required and (M3) was replaced by a weaker condition which applies only to sequences whose limits are in $\mathcal{T}$:

$$\text{(M3–)} \qquad f \in \mathcal{T},\ (f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ f_n \nearrow f \implies \lim_{n\in\mathbb{N}} \mu(f_n) = \mu(f)\,.$$

Although using a more general condition (M3–), all previous papers on this topic dealt with special cases of tribes satisfying (T4) and measures satisfying (M3).

Condition (M3) is the left continuity of the measure. In fact, in a Boolean tribe it implies also the right continuity. However, this is not generally true for tribes. Therefore the following more specific notion has been introduced in [21]:

Definition 7. A measure μ on a tribe $(\mathcal{T}, T)$ is called *regular* if it satisfies (M1), (M2), and

$$\text{(M3+)} \qquad (f_n)_{n\in\mathbb{N}} \in \mathcal{T}^{\mathbb{N}},\ (f_n \nearrow f \text{ or } f_n \searrow f) \implies \lim_{n\in\mathbb{N}} \mu(f_n) = \mu(f)\,.$$

Proposition 8. *Let T be a t-norm and $(\mathcal{T}, T, ', 0, \leq)$ be a tribe satisfying the law of contradiction, i.e., $T(f, f') = 0$ for all $f \in \mathcal{T}$. Then every measure on $(\mathcal{T}, T, ', 0, \leq)$ is regular. In particular, every measure on a Boolean tribe or on a $T_{\mathbf{L}}$-tribe (where $T_{\mathbf{L}}$ is the Łukasiewicz t-norm) is regular.*

For a tribe $(\mathcal{T}, T)$ on X, we define

$$\check{\mathcal{T}} = \{A \subseteq X \mid \chi_A \in \mathcal{T}\}\,.$$

It is a σ-algebra of subsets of X. A measure μ on $(\mathcal{T}, T)$ induces a measure $\check{\mu}$ on $\check{\mathcal{T}}$ (introduced in [3])

$$\check{\mu}(A) = \mu(\chi_A)\,.$$

4 Frank and nearly Frank t-norms

Frank t-norms $T^{\mathbf{F}}_{\lambda}$, $\lambda \in [0, \infty]$, were defined in [9] by

$$T^{\mathbf{F}}_{\lambda}(x,y) := \begin{cases} \log_{\lambda}\left(1 + \frac{(\lambda^x-1)(\lambda^y-1)}{\lambda-1}\right) & \text{if } \lambda \in \,]0,\infty[\, \setminus \{1\}\,, \\ \min(x,y) & \text{if } \lambda = 0\,, \\ x \cdot y & \text{if } \lambda = 1\,, \\ \max(x+y-1, 0) & \text{if } \lambda = \infty\,. \end{cases}$$

(The t-norms $T_{\mathbf{M}} = T^{\mathbf{F}}_0, T_{\mathbf{P}} = T^{\mathbf{F}}_1, T_{\mathbf{L}} = T^{\mathbf{F}}_{\infty}$ are the minimum, the product, and the Łukasiewicz t-norm, respectively.) Frank t-norms $T^{\mathbf{F}}_{\lambda}$ are strict iff $\lambda \in \,]0, \infty[$. They play a special role in the characterization of measures due to the following property [9]:

Theorem 9. *Let T be a Frank t-norm and S its dual t-conorm. Then*

$$\forall a, b \in [0,1] : T(a,b) + S(a,b) = a + b\,. \tag{1}$$

Conversely, if a continuous Archimedean t-norm T and its dual S satisfy (1)*, then they are Frank.*

Let us recall the definition of nearly Frank t-norms [20]. We say that an increasing bijection $h\colon [0,1] \longrightarrow [0,1]$ *commutes with the standard negation* if

$$\forall a \in [0,1] : h(a') = h(a)' .$$

(Then h is called a *negation preserving automorphism* [20].)

Definition 10. A t-norm T is called *nearly Frank* if there is an increasing bijection $h\colon [0,1] \longrightarrow [0,1]$ which commutes with the standard negation and a Frank t-norm T^* satisfying

$$T^*(a,b) = h(T(h^{-1}(a), h^{-1}(b))) \tag{2}$$

for all $a, b \in [0,1]$.

Proposition 11 (see [20]**).** *If T is a nearly Frank t-norm different from $T_{\mathbf{M}}$, then the bijection h and the Frank t-norm T^* satisfying* (2) *are unique.*

The question of how to recognize whether or not a given t-norm is nearly Frank has been solved in [15].

5 Characterization of regular measures

Measures on T-tribes, where T is a nearly Frank t-norm, were characterized in [16]. For regular measures, we obtain the following consequence:

Theorem 12. *Let T be a strict nearly Frank t-norm with h satisfying* (2) *and $(\mathcal{T}, T)$ be a tribe. Then regular measures on $(\mathcal{T}, T)$ are exactly all functionals of the form*

$$\mu(f) = \int h \circ f \, d\nu , \qquad f \in \mathcal{T} , \tag{3}$$

where $\nu = \check{\mu}$ is a measure on $\check{\mathcal{T}}$.

For Frank t-norms, $h = \mathrm{id}$ and we obtain the following:

Corollary 13. *Let $T^{\mathbf{F}}_{\lambda}$, $\lambda \in \,]0,\infty[$, be a strict Frank t-norm and $(\mathcal{T}, T^{\mathbf{F}}_{\lambda})$ be a tribe. Then also $(\mathcal{T}, T_{\mathbf{L}})$ is a tribe and regular measures on $(\mathcal{T}, T^{\mathbf{F}}_{\lambda})$ are exactly (regular) measures on $(\mathcal{T}, T_{\mathbf{L}})$. They are of the form*

$$\mu(f) = \int f \, d\nu , \qquad f \in \mathcal{T} , \tag{4}$$

where $\nu = \check{\mu}$ is a measure on $\check{\mathcal{T}}$.

Following [20], a regular measure μ of the form (3) is called a *(generalized) integral measure*. The particular form (4) obtained for Frank t-norms is called a *linear integral measure*. It coincides with measures on σ-complete MV-algebras studied in [8, 22].

If the t-norm T is not nearly Frank, the characterization of measures is different. For the special case of a full tribe, it follows from [1]:

Theorem 14. *Let T be a strict t-norm which is not nearly Frank. Then there is no non-zero regular measure on any full T-tribe.*

To analyze tribes which are not full, we introduce several notions. Let $(\mathcal{T},T)$ be a tribe on X and Y be a non-empty subset of X. Let

$$\mathcal{T}_Y = \{f \upharpoonright Y \mid f \in \mathcal{T}\} \subseteq [0,1]^Y .$$

Then $(\mathcal{T}_Y, T)$ is a tribe on Y called the *restriction* of $(\mathcal{T},T)$ to Y. Suppose, moreover, that $Y \in \check{\mathcal{T}}$ and μ is a measure on $(\mathcal{T},T)$. Then $\mu_Y : \mathcal{T}_Y \longrightarrow \mathbb{R}_+$ defined by

$$\mu_Y(f \upharpoonright Y) = \mu(f \cdot \chi_Y) \tag{5}$$

is a measure on $(\mathcal{T}_Y, T)$ called the *restriction* of μ to Y.

Remark 15. In fact, the restriction μ_Y of a measure μ may be understood as a measure *conditioned* by a (crisp) event Y. A probabilistic interpretation is straightforward. Nevertheless, attempts to introduce conditional probability which is conditioned by *fuzzy* events lead to difficulties even in the special case of $T_{\mathbf{L}}$-tribes (see [22]).

Let $(\mathcal{T},T)$ be a tribe. For $f \in \mathcal{T}$, we denote the following subsets:

$$\begin{aligned} Uf &= f^{-1}(1)\,, \\ Ff &= f^{-1}(]0,1[)\,, \\ \operatorname{supp} f &= Uf \cup Ff = f^{-1}(]0,1]) \qquad \text{(the } \textit{support} \text{ of } f). \end{aligned}$$

They all belong to $\check{\mathcal{T}}$.

Proposition 16. *Let $(\mathcal{T},T)$ be a tribe and let μ be a measure on $(\mathcal{T},T)$. Then*

$$\mu(f) = \mu(\chi_{Uf}) + \mu(f \cdot \chi_{Ff}) = \check{\mu}(Uf) + \mu_{Ff}(f \upharpoonright Ff)\,. \tag{6}$$

If $Ff = \emptyset$, then f is Boolean and $\mu(f) = \check{\mu}(Uf)$. It only remains to determine the summand $\mu_{Ff}(f \upharpoonright Ff)$ for $Ff \neq \emptyset$. We have its characterization if the restriction $(\mathcal{T}_{Ff}, T)$ is a full tribe. As we shall see, this is often the case (not only for strict nearly Frank t-norms). Even if $(\mathcal{T}_{Ff}, T)$ is not a full tribe, we can characterize regular measures [21]. For this, we define

$$\Delta_{\mathcal{T}} = \{Ff \mid f \in \mathcal{T}\}\,.$$

It is a σ-ideal in the σ-algebra $\check{\mathcal{T}}$.

Theorem 17. *Let T be a strict t-norm which is not nearly Frank and $(\mathcal{T},T)$ be a tribe. Then regular measures on $(\mathcal{T},T)$ are exactly all functionals of the form* (4)*, where $\nu = \check{\mu}$ is a measure on $\check{\mathcal{T}}$ such that $\nu \upharpoonright \Delta_{\mathcal{T}} = 0$.*

Remark 18. In Theorem 17, $\nu(Ff) = \check{\mu}(Ff) = 0$. Then (4) may be written in many equivalent forms:

$$\mu(f) = \int f \, d\nu = \nu(\operatorname{supp} f) = \nu(Uf)$$

and also as (3), where $h\colon [0,1] \longrightarrow [0,1]$ is any increasing bijection.

According to the above results, any regular measure on a tribe is fully determined by a measure on a σ-algebra. This characterization allows us to use many results derived in the classical measure theory. On the other hand, the context of full tribes is more general and extension to fuzzy subsets brings new phenomena.

6 Characterization of general measures

Now we shall generalize the results from the preceding section to measures which need not be regular (we assume only left continuity in (M3)). A new type of measure occurs:

Proposition 19. *Let T be a t-norm and $(\mathcal{T},T)$ be a tribe. The functional μ on $\mathcal{T}$ defined by*

$$\mu(f) = \check{\mu}(\operatorname{supp} f)$$

is a measure on $(\mathcal{T},T)$ called a support measure.

The characterization from [20] may be reformulated as follows:

Theorem 20. *Let T be a strict nearly Frank t-norm and let $(\mathcal{T},T)$ be a tribe on X. Every measure μ on $(\mathcal{T},T)$ is a linear combination of an integral measure and a support measure.*

As in Remark 18, a measure on a Boolean element may be considered an integral measure as well as a support measure. Therefore the decomposition to an integral measure and a support measure in Theorem 20 is not unique. The coefficients of the linear combination need not be non-negative:

Example 21. Let $T^{\mathbf{F}}_{\lambda}$, $\lambda \in]0,\infty[$ be a strict Frank t-norm. Then each measure μ on $([0,1],T^{\mathbf{F}}_{\lambda})$ (the full $T^{\mathbf{F}}_{\lambda}$-tribe with a singleton domain) is of the form

$$\mu(a) = \begin{cases} p + q\,a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \end{cases}$$

where $p \geq 0$ and $p + q \geq 0$. The measure μ is

- regular iff $p = 0$,
- monotone iff $q \geq 0$.

E.g., if we take $p = 1$, $q = -1$, we obtain

$$\mu(a) = \begin{cases} 1 - a & \text{if } a > 0, \\ 0 & \text{if } a = 0. \end{cases}$$

This is a measure which is not monotone.

As in the case of regular measures, we use Proposition 16. It is helpful if the restriction $(\mathcal{T}_{Ff},T)$ is full. In fact, the proof of Theorem 20 is based on Proposition 19, the characterization of regular measures from Theorem 12, and the following:

Lemma 22. *Let T be a strict nearly Frank t-norm and let $(\mathcal{T},T)$ be a tribe on X. If there is an $f \in \mathcal{T}$ such that $Ff = X$, then the restriction $(\mathcal{T},T)$ is a full tribe.*

Recently in [7] Lemma 22 was generalized to many other strict t-norms which are called *sufficient* because they give rise to sufficient (or *functionally complete*) sets of fuzzy logical connectives (see [7] for details about this notion). In particular, sufficient t-norms include all t-norms from the Aczél–Alsina and Mizumoto eighth and tenth families (see [12, 13] for the definitions and [7] for further examples).

Theorem 23. *Let T be a strict sufficient t-norm which is not nearly Frank and let $(\mathcal{T}, T)$ be a tribe. Every measure μ on $(\mathcal{T}, T)$ is a support measure.*

The question whether Lemma 22 remains valid for all strict t-norms has been open for many years. It is related to problems published, e.g., in [14, 16, 17, 18]. Counterexamples were found recently in [7]; the Hamacher product is one of them. For t-norms which are not sufficient, a characterization of measures on tribes is known only in special cases when it leads again to support measures.

Problem 24. Is there a strict t-norm T which is not nearly Frank, a tribe $(\mathcal{T}, T)$ and a measure μ on $(\mathcal{T}, T)$ which is not a support measure?

For *regular* measures, the characterization is known for *all strict* t-norms.

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The logic and algebra of fuzzy IF-THEN rules

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This paper is an (incomplete) overview of the existing approaches to interpretation of fuzzy IF-THEN rules and derivation of a conclusion on the basis of them.

We will focus especially on two principal interpretations of linguistic description. The first one is called *relational*. The main idea is to find a good approximation of some function known only roughly. Therefore, it is divided into imprecise "parts" using fuzzy relations constructed from fuzzy sets with continuous membership functions of more or less arbitrary shape. Each such membership function is assigned some name to be able to get better orientation in the rules, but without real linguistic meaning. Formally, these are sets of fuzzy IF-THEN rules assigned one of two kinds of normal forms: the disjunctive or conjunctive normal form (see [10]). The resulting fuzzy relation then depends on the choice of the underlying algebra of truth values.

Most interpretations of fuzzy IF-THEN rules found in the literature stick on this interpretation. Then derivation of a conclusion on the basis of them is done on the level of semantics rather than on the level of syntax. However, there are also *logical interpretations*, e.g. those presented in [6, 9] and elsewhere. An important case which, at the same time, belongs to logical interpretation is presented in [9, 7]. Its main goal is to use genuine linguistic expressions interpreted in a way which mimics human understanding to them. The fuzzy IF-THEN rules, which are then interpreted as linguistically characterised logical implications, form special axioms of some formal theory.

There are several other kinds of interpretations which in various degrees can be ranked to the relational one (cf. [5]). In the paper, we will discuss and compare these interpretations from several points of view.

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Solvability and approximate solvability of a system of fuzzy relation equations from functional point of view

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Abstract: The paper summarized the last author's results concerning the problem of solvability and approximate solvability of a system of fuzzy relation equations. A number of new criteria of the so called Mamdani relation to be a solution to the system is suggested. At the same time those criteria are sufficient conditions of a solvability of the system in general. A new, easy to check criterion of a solvability of the system with special fuzzy parameters is found.

With the notion of a fuzzy function as a mapping between universes of fuzzy sets we threw a new light on the problem of solvability and approximate solvability. In this setting, precise and approximate solutions to a system of fuzzy relation equations are considered as the interpolating and approximating fuzzy functions with respect to the given data. Different approximating spaces and different criteria of approximation have been introduced. We have proved that the widely known fuzzy relations introduced by E. Sanchez and E. H. Mamdani are the best approximations in the respective spaces and under the respective criteria.

Keywords: system of fuzzy relation equations, solvability of a fuzzy relation equation system, fuzzy equivalence, fuzzy point, fuzzy function

1 Introduction

Systems of fuzzy relation equations are connected with applications like fuzzy control, identification of fuzzy systems, prediction of fuzzy systems, decision-making, etc. Such systems arise in the process of formalization of fuzzy IF–THEN rules, which well recommend themselves as an approximating instrument for continuous dependencies. In this correspondence, the problem of solvability of a system of fuzzy relation equations relates to a problem of verification of correctness of the chosen formalization of fuzzy IF–THEN rules.

In the proposed overview, we will consider the problem of solvability of a system of fuzzy relation equations in the following aspects:

- criteria of general solvability, i.e. necessary and sufficient and only necessary or only sufficient [4, 10, 16, 18, 19];
- simple criteria of solvability in special cases where original data are fuzzy sets which constitute fuzzy partitions of respective universes [10, 15];
- solvability and interpolation of fuzzy functions [17, 16];

- criteria of solvability in the case of finite universes;
- approximate solvability in different approximating spaces and with respect to different criteria [17, 16];
- approximate solvability and approximation of fuzzy functions [13, 16, 17, 20];
- approximate solvability in special metric spaces induced by t-norm.

For this publication we have chosen only new results recently established by the author.

1.1 Basic algebra of logic operations

We choose a BL-algebra (BL stands for basic fuzzy logic) as a basic algebra of operations. In a certain sense, the BL-algebra generalizes boolean one and occurs when the conjunction is split in two different operations: a pure lattice operation and the other monoidal one (called multiplication) which a pseudo-inverse. The following definition summarizes definitions originally introduced in [9].

Definition 1. A *BL-algebra* is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \tag{1}$$

with four binary operations and two constants such that

(i) $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a lattice with $\mathbf{0}$ and $\mathbf{1}$ as the least and greatest elements w.r.t. the lattice ordering,

(ii) $(L, *, \mathbf{1})$ is a commutative semigroup with unit $\mathbf{1}$, such that the multiplication $*$ is associative, commutative and $1 * x = x$ for all $x \in L$,

(iii) $*$ and $\rightarrow$ form an adjoint pair, i.e.
$z \leq (x \rightarrow y)$ iff $x * z \leq y$ for all $x, y, z \in L$,

(iv) and moreover, for all $x, y \in L$
$x * (x \rightarrow y) = x \wedge y$,
$(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}$.

The well known examples of BL-algebra are Gødel, Łukasiewicz and product algebras.

Another binary operation $\leftrightarrow$ of $\mathcal{L}$ can be defined by:

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x).$$

The following properties will be used in the sequel:

$$\begin{aligned} x \leq y \quad &\text{iff} \quad (x \rightarrow y) = \mathbf{1}, \\ x \leftrightarrow y = \mathbf{1} \quad &\text{iff} \quad x = y. \end{aligned}$$

Note that, in particular, if $L = [0, 1]$ then $*$ is a t-norm.

From now and until the end of this paper, we fix some complete BL-algebra $\mathcal{L}$ with a support L.

1.2 Fuzzy sets and fuzzy relations

We accept here a mathematical definition of a fuzzy set. Let $\mathbf{X}$ be a non-empty set. Then a fuzzy set or better, a fuzzy subset of $\mathbf{X}$ is identified with a function $A : \mathbf{X} \longrightarrow L$. This function is known as a membership function of fuzzy set A. The set of all fuzzy subsets of $\mathbf{X}$ is denoted by $\mathcal{F}(\mathbf{X})$, so that we can write

$$\mathcal{F}(\mathbf{X}) = \{A : X \longrightarrow L\} = L^X.$$

For two fuzzy sets $A, B \in \mathcal{F}(\mathbf{X})$ we let

$$A = B \quad \text{iff} \quad (\forall x) A(x) = B(x)$$

and

$$A \le B \quad \text{iff} \quad (\forall x) A(x) \le B(x).$$

A fuzzy set $A \in \mathcal{F}(\mathbf{X})$ is called *normal* if $A(x_0) = \mathbf{1}$ holds for some $x_0 \in \mathbf{X}$. The algebra of operations over fuzzy subsets of X is introduced as the induced BL-algebra on L^X. This means that each operation from $\mathcal{L}$ is the operation on L^X taken pointwise. For example, the $*$-operation between fuzzy sets A and B is defined by

$$(A * B)(x) = A(x) * B(x).$$

The operations over fuzzy subsets fulfill the same properties as the corresponding operations in the respective BL-algebra.

Let $\mathbf{X}$ and $\mathbf{Y}$ be two universes, not necessary different. A (binary) fuzzy relation on $\mathbf{X} \times \mathbf{Y}$ is a fuzzy subset of this set, i.e. a function $R : \mathbf{X} \times \mathbf{Y} \longrightarrow L$. The set of all fuzzy relations on $\mathbf{X} \times \mathbf{Y}$ is denoted by $\mathcal{F}(\mathbf{X} \times \mathbf{Y})$. An n-ary fuzzy relation can be introduced analogously.

If $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ and $S \in \mathcal{F}(\mathbf{Y} \times \mathbf{Z})$ then the fuzzy relation T on $\mathbf{X} \times \mathbf{Z}$

$$T(x,z) = \bigvee_{y \in \mathbf{Y}} R(x,y) * S(y,z)$$

is called a composition (or $\sup - *$-composition) of R and S and denoted by

$$T = R \circ S.$$

In particular, if A is a unary fuzzy relation on $\mathbf{X}$ or a fuzzy subset of $\mathbf{X}$ then $\sup - *$-composition between A and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ is defined by

$$B(y) = \bigvee_{x \in \mathbf{X}} A(x) * R(x,y),$$

so that $B = A \circ R$ and $B \in \mathcal{F}(\mathbf{Y})$.

1.3 Fuzzy equivalence and fuzzy points

Fuzzy equivalence is a special fuzzy relation on a universe $\mathbf{X}$ which, analogously as the classical equivalence fulfills the properties of reflexivity, symmetry and transitivity, but with the generalized meaning. Namely, we say that $E : \mathbf{X} \times \mathbf{X} \longrightarrow L$ is a fuzzy equivalence on $\mathbf{X}$ if

$$E(x,x) = 1,$$
$$E(x,y) = E(y,x),$$
$$E(x,y) * E(y,z) \le E(x,z)$$

holds true for all $x,y,z \in \mathbf{X}$.

Suppose that some fuzzy equivalence E on $\mathbf{X}$ is given. Then we may fix one argument $x = x_0$ and consider the function $A(x) = E(x_0, x)$ which determines a normal fuzzy subset of $\mathbf{X}$. We say that the fuzzy subset of this type is a fuzzy point of $\mathbf{X}$ with respect to x_0 and fuzzy equivalence E.

It is not difficult to show that each normal fuzzy subset A of $\mathbf{X}$, such that $A(x_0) = \mathbf{1}$, can be considered as a fuzzy point with respect to x_0 and special fuzzy equivalence E given by

$$E(x,y) = A(x) \leftrightarrow A(y).$$

The situation is more difficult if we have a collection of normal fuzzy subsets of $\mathbf{X}$. The following theorem has been proved in [11].

Theorem 2. *Let A_i, $i \in I$, be a family of normal fuzzy subsets of $\mathbf{X}$, such that there exist $x_i \in \mathbf{X}$ which make true the following: $A_i(x_i) = \mathbf{1}$. Then the following two statements are equivalent*

- *there exists a fuzzy equivalence E on $\mathbf{X}$, such that all fuzzy sets A_i are fuzzy points with respect to x_i and E, i.e.*
$$A_i(x) = E(x_i, x) \tag{2}$$
- *for all $i, j \in I$*
$$\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) \leq \bigwedge_{y \in \mathbf{X}} (A_i(y) \leftrightarrow A_j(y)) \tag{3}$$
holds.

Remark 3. From the proof of this theorem it follows that

- if (3) is true then each fuzzy set A_i from the above given family is a fuzzy point with respect to x_i and fuzzy equivalence $\hat{E}$ given by
$$\hat{E}(x,y) = \bigwedge_{i \in I} (A_i(x) \leftrightarrow A_i(y)). \tag{4}$$
- if each fuzzy set A_i from the above given family is a fuzzy point with respect to x_i and some fuzzy equivalence E then it is a fuzzy point with respect to x_i and fuzzy equivalence $\hat{E}$.

The following lemma can be proved as a corollary of Theorem 2.

Lemma 4. *Let A_i, $i \in I$, be a family of normal fuzzy subsets of $\mathbf{X}$, such that there exist $x_i \in \mathbf{X}$ which make true the following: $A_i(x_i) = \mathbf{1}$. Moreover, let inequality (3) hold true. Then inequality (3) turns to the equality*

$$\begin{aligned}\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) = &\bigwedge_{y \in \mathbf{X}} (A_i(y) \leftrightarrow A_j(y)) \\ =&\hat{E}(x_i, x_j)\end{aligned} \tag{5}$$

where $i, j \in I$ and $\hat{E}(x,y)$ is given by (4).

Corollary 5. *Let the conditions of Lemma 4 be fulfilled. Then inequality (3) turns to the equality*

$$\begin{aligned}\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) = &\bigwedge_{y \in \mathbf{X}} (A_i(y) \leftrightarrow A_j(y)) = \\ &= E(x_i, x_j)\end{aligned} \tag{6}$$

where $i, j \in I$ and $E(x,y)$ is any fuzzy equivalence which makes all fuzzy subsets A_i to be fuzzy points with respect to it and x_i.

1.4 System of fuzzy relation equations

Let **X** and **Y** be two universes, not necessary different. A system of fuzzy relation equations

$$A_i \circ R = B_i, \quad 1 \le i \le n, \tag{7}$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ and '$\circ$' is the sup-*-composition, is considered with respect to unknown fuzzy relation R.

Since in general, solution of (7) may not exist, the investigation of necessary and sufficient, or also of only sufficient conditions for solvability becomes necessary. This problem has been widely studied in the literature, and some nice theoretical results have been obtained. Let us point out some of them: [19], [18], [4] with necessary and sufficient conditions, [5], [10] with sufficient conditions.

All of these results have practical importance only in the case when universes of discourse X and Y are finite. If these universes are infinite, then the complexity of verification of these conditions is comparable with the direct checking of solvability. Therefore, the problem of discovering easy to check solvability conditions or criteria is still actual. This paper is a contribution to this topic.

We recall basic facts concerning solvability of system (7) of fuzzy relation equations

$$A_i \circ R = B_i, \quad 1 \le i \le n,$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$.

- If system (7) with respect to unknown fuzzy relation R is solvable then relation

  $$\hat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \tag{8}$$

  is the greatest solution to (7) (see [19]).

- Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \le i \le n$, be normal. Then fuzzy relation

  $$\check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)) \tag{9}$$

  is a solution to (7) if and only if for all $i, j = 1, \ldots, n$

  $$\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) \le \bigwedge_{y \in \mathbf{Y}} (B_i(y) \leftrightarrow B_j(y)) \tag{10}$$

  holds (see [10]).

It is worth notice that fuzzy relation $\check{R}$ is known in literature as Mamdani relation.

2 Sufficient conditions of solvability

As mentioned above, a system of fuzzy relation equations arises on the way of formalization of a set of fuzzy IF–THEN rules. In fact, a fuzzy relation R which solves the system of fuzzy relation equations in the form (7) describes a certain dependence between variables $x \in \mathbf{X}$ and $y \in \mathbf{Y}$. If the variable x is furthermore specified by some value expressed by a fuzzy set $A \in \mathcal{F}(\mathbf{X})$ then the respective (fuzzy) value of variable y can be computed by taking the sup-*-composition

$$B = A \circ R. \tag{11}$$

This procedure is used as an interpretation of so the called Generalized Modus Ponens inference rule in the fuzzy logic in broader sense.

Keeping in mind the computation of sup-*-composition (11), in which R is replaced by a solution to system (7), we may argue that fuzzy relation $\check{R}$ requires less computations that fuzzy relation $\hat{R}$. Therefore, the conditions which guarantee that $\check{R}$ is a solution to (7) are more important than conditions of general solvability. On the other hand, these conditions are sufficient with respect to general solvability of (7).

Therefore, we focus in this section on conditions ensuring that $\check{R}$ is a solution to (7). Of course, the inequality (10) is the first representative of such conditions. The next theorem proved in [1], presents the equivalence between (10) and another inequality, which can be used as the second condition of this type.

Theorem 6. *The inequality (10) is equivalent with*

$$\check{R} \leq \hat{R}. \tag{12}$$

The following corollary immediately follows from Theorem 6 and Klawonn's condition of solvability.

Corollary 7. *Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \leq i \leq n$, be normal. Then the fuzzy relation $\check{R}$ in (9) is a solution to (7) if and only if* $\check{R} \leq \hat{R}$.

Remark 8. If the fuzzy relation $\check{R}$ is a solution to (7) then the system (7) is solvable. Therefore, the condition (12) is a sufficient condition for the solvability of the system (7), provided that fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$ are normal.

Let us investigate a special situation when fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$ are fuzzy points with respect to fuzzy equivalences E on $\mathbf{X}$ and F on $\mathbf{Y}$. The following nice (and easy to check) criterion of solvability of (7) by fuzzy relation $\check{R}$ summarizes almost all the facts discussed above.

Theorem 9. *Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \leq i \leq n$, be normal, so that there exist $x_i \in \mathbf{X}$ and $y_i \in \mathbf{Y}$ which make true the following: $A_i(x_i) = \mathbf{1}$, $B_i(y_i) = \mathbf{1}$. Further, let fuzzy equivalence E on $\mathbf{X}$ and fuzzy equivalence F on $\mathbf{Y}$ exist so that all the fuzzy sets A_i are fuzzy points with respect to x_i and E, and all the fuzzy sets B_i are fuzzy points with respect to y_i and F, i.e.*

$$(\forall x) A_i(x) = E(x_i, x) \tag{13}$$

and

$$(\forall y) B_i(y) = F(y_i, y). \tag{14}$$

Then the fuzzy relation $\check{R}$ in (9) is a solution to (7) if and only if

$$(\forall i)(\forall j) A_i(x_j) \leq B_i(y_j). \tag{15}$$

We can again remark that condition (15) and the assumptions of Theorem 9 give easy to check sufficient condition of solvability of system (7).

3 One useful necessary condition of solvability

Necessary conditions are very useful in verifying the solvability in general. When they are not fulfilled, the system cannot be solvable. We will suggest hear one condition which has easy to understand interpretation.

Theorem 10. *If the system (7) is solvable then for arbitrary $i, j \in \{1, \ldots, n\}$*

$$\bigwedge_{x \in \mathbf{X}} (A_i(x) \leftrightarrow A_j(x)) \leq \bigwedge_{y \in \mathbf{Y}} (B_i(y) \leftrightarrow B_j(y)). \tag{16}$$

The interpretation of the condition (16) is such that the sets A_i, A_j cannot be closer than their respective counterparts B_i and B_j.

4 A new criterion of solvability

In this section we prove even more: the condition (15) is the necessary and sufficient condition for the solvability of system (7) provided that (13) is fulfilled.

Theorem 11. *Let the conditions of Theorem 9 be fulfilled. Then (15) is the necessary and sufficient condition of the general solvability of system (7).*

5 Fuzzy function. Interpolation of a fuzzy function

We will introduce the problem of solvability of fuzzy relation equations in a new framework as the problem of interpolation and approximation of a fuzzy function.

Our idea is to introduce a fuzzy function as a mapping between two universes $\mathcal{F}(\mathbf{X})$ and $\mathcal{F}(\mathbf{Y})$ of fuzzy sets, so that it maps uniquely a fuzzy "point" from one universe to the respective fuzzy "point" from the other universe. Trying to be as much as possible close to the classical case we give the following definition (see also Perfilieva & Gottwald [17]).

Definition 12. Let $\mathcal{F}(\mathbf{X}), \mathcal{F}(\mathbf{Y})$ be the classes of all fuzzy subsets on the universes of discourse $\mathbf{X}$ and $\mathbf{Y}$. A mapping f from $\mathcal{F}(\mathbf{X})$ into $\mathcal{F}(\mathbf{Y})$ is called a *fuzzy function* if for any fuzzy subsets $A, A' \in \mathcal{F}(\mathbf{X})$ and for fuzzy subsets $B, B' \in \mathcal{F}(\mathbf{Y})$ which are f-related with A, A', respectively,

$$A = A' \Rightarrow B = B'. \tag{17}$$

holds true.

Example 13. Any fuzzy relation $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ determines via sup-*-composition a fuzzy function, defined as the mapping f_R from $\mathcal{F}(\mathbf{X})$ to $\mathcal{F}(\mathbf{Y})$ which is described by

$$f_R(A)(y) = (A \circ R)(y) = \bigvee_{x \in \mathbf{X}} (A(x) * R(x, y)).$$

In this example, the fuzzy set $f_R(A) = A \circ R$ is the value of fuzzy function f_R determined by R in the "fuzzy point" determined by A.

Remark 14. As mentioned in Introduction, there is another approach to the notion of a fuzzy function shared by the authors [9, 10]. According to their approach, a fuzzy function is a special kind of a fuzzy relation — which "respects" two given similarities on the universes of discourse. The "fuzzy" constituent in their definitions refers to the uniqueness property, so that they define what may be called a "blurred" mapping. Moreover, they clearly distinguish between partial and total fuzzy function.

Contrary to the definitions cited above, we stress that a fuzzy function is a (ordinary) mapping between two universes of fuzzy sets, so that it maps uniquely a fuzzy "point" from one universe to the respective fuzzy "point" from the other universe. In our opinion, it is not necessary to indicate in the general definition of a function whether it is partially defined or not. It is reasonable to stress this characteristics when we speak, for example about the problem of interpolation. (Below, we formulate this problem and discuss methods of its solution.) However, in general we suppose that a fuzzy function is defined on the whole universe $\mathcal{F}(\mathbf{X})$.

The definition of a (fuzzy) function, in general, does not provide us with a constructive way of its representation (except for finite $\mathcal{F}(\mathbf{X})$). Therefore, the problem of representation of a function is of a primary importance. By this we mean, that having a function as a mapping, we want to find a formula which represents this mapping. However, in practice we know a mapping (between infinite or large universes) only partially, as a finite set of couples and therefore, the problem of representation may be solved also partially. There are two possible approaches to obtain a formulation of, say, partial representation problem: one leads to the interpolation and the other one — to the approximation of a function. We give formulations of both problems in fuzzy setting and then discuss the specificity of these problems in the case when fuzzy function is determined by a fuzzy relation.

Definition 15. Let a list of original data consisting of ordered pairs of fuzzy sets (A_i, B_i) where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$, $i = 1, \ldots, n$, be given. A fuzzy function f defined on $\mathcal{F}(\mathbf{X})$ *interpolates* these data if

$$f(A_i) = B_i, \qquad i = 1, \ldots, n. \tag{18}$$

We will also call f an interpolating fuzzy function.

Very often, the above defined interpolation problem appears in the literature as a problem of finding a fuzzy relation partially described by a list of fuzzy IF–THEN rules

$$\text{IF } x \text{ is } A_i \text{ THEN } y \text{ is } B_i, \qquad i = 1, \ldots, n,$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$. The natural requirement for such a fuzzy relation is that it should "agree" with the original data. This means in our terminology that the required fuzzy relation determines the fuzzy function which interpolates the given data (the details are below in Lemma 16).

As an important remark, we point out that interpolation of a fuzzy function may not exist; if it exists, it need not be unique. In the latter case, this is the reason why the interpolation problem in classical mathematics is solved in a predetermined class of (interpolating) functions, for example in the class of polynomials.

We consider a solution to the fuzzy interpolation problem in the class of fuzzy functions represented by fuzzy relations. It is easy to see that there is a close relation between the existence of an interpolation function and the solvability of the respective system of fuzzy relation equations.

Lemma 16. *Let ordered pairs of fuzzy sets (A_i, B_i) be given where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$, $i = 1, \ldots, n$. A fuzzy relation R determines an interpolating fuzzy function with respect to the given data (A_i, B_i), $i = 1, \ldots, n$, if and only if R is a solution of the corresponding system (7) of relation equations.*

Proof. Obvious. □

As a consequence of this statement, we can assert that not every fuzzy function can be determined by the respective fuzzy relation. This is due to the fact that not every system of fuzzy relation equations is solvable.

6 Approximation of a fuzzy function. Approximate solutions to a system of fuzzy relation equations and their approximation quality

The problem of approximation of a partially given fuzzy function arises when we want to complete partially given mapping, but we do not insist on a precise agreement with the given data. The other reason to consider approximation is that the interpolation problem may not be solvable in the chosen class of interpolating functions. For example, if interpolating fuzzy functions are those which are determined by fuzzy relations then the interpolation problem is equivalent to the existence of a solution to system (7). Because the latter may not be solvable, this implies that there exist fuzzy data (A_i, B_i), $i = 1, \ldots, n$, which cannot be "joined" by any fuzzy relation. In this situation we may weaken the interpolation problem and consider the problem of approximation. We start with a rough formulation of this problem and then, after explanation of details, give a precise formulation.

Given fuzzy data (A_i, B_i) where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$, $i = 1, \ldots, n$, find a fuzzy function, determined by a fuzzy relation which gives an approximate solution to system (7).

By this formulation, we reduce the problem of finding of an approximating fuzzy function to the problem of finding an approximate solution to system (7). The latter will be the core of our investigation in the rest of this paper. However, it requires further specification. Two things have to be specified: an approximating space and a quality of approximation. Below we will introduce three different approximating spaces and different qualities of approximation in them.

1. The widest approximating space consists of all fuzzy relations on $\mathbf{X} \times \mathbf{Y}$

$$\mathcal{R} = \{R \mid R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})\}. \tag{19}$$

   However, we will not deal with this space in this paper, because it is too wide to find an optimal approximation in it.

   We will consider two other, more restrictive approximation spaces which are subspaces of $\mathcal{R}$ (Perfilieva & Gottwald [17]). Unlike $\mathcal{R}$, they are determined by parameters A_i, B_i of system (7).

2. The space of lower approximations

$$\mathcal{R}_l = \{R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y}) \mid A_i \circ R \leq B_i, \quad 1 \leq i \leq n\} \tag{20}$$

   consists of those relations which make compositions lower than the intended right hand sides.

3. The space of upper approximations

$$\mathcal{R}_u = \{R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y}) \mid A_i \circ R \geq B_i, \quad 1 \leq i \leq n\}. \tag{21}$$

   consists of those relations which make compositions greater than the intended right hand sides.

An evaluation of a quality of approximation in $\mathcal{R}$ stems from a comparison of the intended values B_i and those ones determined by the composition $R \in \mathcal{R}$ and A_i, i.e. from a value (Gottwald [6])

$$\delta(R) = \bigwedge_{i=1}^{n} \bigwedge_{y \in \mathbf{Y}} (B_i(y) \leftrightarrow (A_i \circ R)(y)). \tag{22}$$

Being equipped with the evaluation $\delta(R)$ of a quality of approximation we may compare two different approximate solutions, saying that $R' \in \mathcal{R}$ is better than $R'' \in \mathcal{R}$ if and only if its solution degree is higher; formally

$$R' \leq_\delta R'' \quad \text{iff} \quad \delta(R'') \leq \delta(R'). \tag{23}$$

The same index $\delta(R)$ may serve as a quality of approximation in two other spaces $\mathcal{R}_l$ and $\mathcal{R}_u$.

It is easy to see that with help of $\delta(R)$ we have introduced a preorder $\leq_\delta$ (i.e. reflexive and transitive binary relation) on each of the approximation spaces $\mathcal{R}$, $\mathcal{R}_l$ and $\mathcal{R}_u$.

Though a quality of approximation in $\mathcal{R}_l$ and $\mathcal{R}_u$ may be estimated by $\delta(R)$, we will also use another, non-numeric estimation according to the following preorders $\leq_l$ on $\mathcal{R}_l$

$$R' \leq_l R'' \quad \text{iff} \quad R', R'' \in \mathcal{R}_l \quad \text{and} \quad A_i \circ R'' \leq A_i \circ R', \quad 1 \leq i \leq n, \tag{24}$$

and $\leq_u$ on $\mathcal{R}_u$

$$R' \leq_u R'' \quad \text{iff} \quad R', R'' \in \mathcal{R}_u \quad \text{and} \quad A_i \circ R' \leq A_i \circ R'', \quad 1 \leq i \leq n. \tag{25}$$

Let us remark that in the literature on fuzzy relation equations, the preorder $\leq_l$ has been implicitly used in Wu [20] and later on in Klir & Yuan [13] for estimation of approximation quality in $\mathcal{R}_l$.

7 Optimal approximations

In a certain sense, any element from an approximating space can be taken as an approximate solution so that the respective quality of approximation can be computed. However, we would prefer to have an approximate solution with the best possible quality of approximation. This leads us to the following definitions (cf. Perfilieva & Gottwald [17]).

Definition 17. A fuzzy relation R_{opt} is a best approximate solution to system (7) in the approximation space $\mathcal{R}$ ($\mathcal{R}_l$ or $\mathcal{R}_u$) with respect to the quality $\delta(R)$ if

$$\delta(R_{\text{opt}}) = \sup_{R \in \mathcal{R}} \delta(R) \tag{26}$$

$$(\delta(R_{\text{opt}}) = \sup_{R \in \mathcal{R}_l} \delta(R) \quad \text{or} \quad \delta(R_{\text{opt}}) = \sup_{R \in \mathcal{R}_u} \delta(R)). \tag{27}$$

In the approximation spaces $\mathcal{R}_l$ and $\mathcal{R}_u$, we may also define best approximation with respect to preorders $\leq_l$ and $\leq_u$.

Definition 18.
- $R^l_{\text{opt}} \in \mathcal{R}_l$ is a best approximate solution to system (7) w.r.t. $\leq_l$ if there is no fuzzy relation $R \in \mathcal{R}_l$ such that $R \leq_l R^l_{\text{opt}}$ and $A_i \circ R \neq A_i \circ R^l_{\text{opt}}$ for at least one $i \in \{1, \ldots, n\}$.

- $R^u_{\mathrm{opt}} \in \mathcal{R}_u$ is a best approximate solution to system (7) w.r.t. $\leq_u$ if there is no fuzzy relation $R \in \mathcal{R}_u$ such that $R \leq_u R^u_{\mathrm{opt}}$ and $A_i \circ R \neq A_i \circ R^u_{\mathrm{opt}}$ for at least one $i \in \{1, \ldots, n\}$.

As we will see later, a best approximate solution to system (7) in the approximating space $\mathcal{R}_l$ w.r.t. $\leq_l$ maximizes forms $A_i \circ R, i = 1, \ldots, n$ (see Theorem 21), and a best approximate solution to system (7) in the approximating space $\mathcal{R}_u$ w.r.t. $\leq_u$ minimizes forms $A_i \circ R, i = 1, \ldots, n$ (see Theorem 27).

As a consequence of this, a best approximate solution to system (7) in $\mathcal{R}_l$ or $\mathcal{R}_u$ with respect to above introduced approximation qualities if it exists, may not be unique. In Subsection 10 we will see where it may happen. Therefore, in those particular cases we will take into consideration additional characteristics of approximate solutions.

Our next goal is to show that pseudo-solutions $\hat{R}$ and $\check{R}$ are the best approximate solutions to system (7) in the spaces $\mathcal{R}_l$, $\mathcal{R}_u$ with respect to the introduced preorders.

For the pseudo-solution $\hat{R}$, an optimality in the approximation space $\mathcal{R}_l$ and a preorder similar to (24), has been proved in [20, 13]. Below, we will prove a more rigid result.

8 Optimality of pseudo-solution $\hat{R}$

We will show that $\hat{R}$ is a best approximate solution to system (7) in the approximation space $\mathcal{R}_l$ with respect to both preorders $\leq_l$ and $\leq_{\delta(R)}$. Moreover, $\hat{R}$ is the greatest element in this space with respect to the ordinary ordering $\leq$ between fuzzy sets.

Lemma 19. *If the system (7) is unsolvable then the fuzzy relation $\hat{R}$ is the greatest element in the approximation space $\mathcal{R}_l$ w.r.t. the ordinary ordering $\leq$.*

8.1 Optimality of $\hat{R}$ with respect to the preorder $\leq_l$

In the theorem given below, we prove the first of the best approximation results about $\hat{R}$ in $\mathcal{R}_l$ with respect to $\leq_l$.

Theorem 20. *Let the system (7) be unsolvable. Then the fuzzy relation*

$$\hat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \rightarrow B_i(y))$$

is a best approximate solution to system (7) in the space $\mathcal{R}_l$ under the preorder $\leq_l$ (cf. (24)).

The following simple theorem shows even more. If the original system (7) is unsolvable then the first solvable system (when decreasing the right hand sides of (7)) is the system with B_i replaced by $A_i \circ \hat{R}$.

Theorem 21. *Let system (7) be unsolvable and fuzzy sets $C_i \in \mathcal{F}(\mathbf{Y})$ fulfill the inequalities*

$$C_i \leq B_i, \quad i = 1, \ldots, n.$$

Then if the system

$$A_i \circ R = C_i, \quad i = 1, \ldots, n.$$

is solvable then

$$C_i \leq \hat{B}_i, \quad i = 1, \ldots, n,$$

where $\hat{B}_i = A_i \circ \hat{R}$.

8.2 Optimality of $\hat{R}$ with respect to the preorder $\leq_\delta$

The theorem below contains the second of the best approximation results about $\hat{R}$ in $\mathcal{R}_l$ with respect to $\leq_\delta$.

Theorem 22. *Let system (7) be unsolvable. Then the fuzzy relation*

$$\hat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y))$$

is a best approximate solution to system (7) in $\mathcal{R}_l$ with respect to the approximation quality $\delta(R)$ (cf. (22)).

9 Optimality of pseudo-solution $\check{R}$

The relation $\check{R}$ given by (9) is not an optimal approximate solution to system (7) in the space $\mathcal{R}_u$ either with respect to the preorder $\leq_u$ or with respect to the quality $\delta(R)$. This result has been proved in [17]. However, we will obtain the optimality of $\check{R}$ in both cases for special systems of fuzzy relation equations, such that they are solvable if and only if when they are $\check{R}$-solvable.

9.1 Solvability and $\check{R}$-solvability

We put restrictions on fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathbf{X})$ assuming that they are normal and form a semi-partition of $\mathbf{X}$. For this, we recall the definition of a semi-partition (see [3]).

Definition 23. Normal fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathbf{X})$ form a semi-partition of $\mathbf{X}$ if

$$(\forall i)(\forall j) \qquad \left(\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) \leq \bigwedge_{x \in \mathbf{X}} (A_i(x) \leftrightarrow A_j(x)) \right). \tag{28}$$

Throughout this section we will suppose that fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathbf{X})$ in system (7) are normal and form a semi-partition of $\mathbf{X}$.

Definition 24. We say that system (7) of fuzzy relation equations is $\check{R}$-*solvable* if its pseudo-solution $\check{R}$ given by (9) is a solution to this system. We also denote

$$\check{B}_i(y) = (A_i \circ \check{R})(y), \quad 1 \leq i \leq n. \tag{29}$$

Although solvability and $\check{R}$-solvability of system (7) are not in general equivalent, this is true under the accepted assumption about semi-partitioning of $\mathbf{X}$. The theorem given below proves this fact.

Theorem 25. *Let fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathbf{X})$ be normal and form a semi-partition of $\mathbf{X}$. Then system (7) is solvable if and only if it is $\check{R}$-solvable.*

9.2 Optimality of $\check{R}$ with respect to the preorder $\leq_u$

For systems of fuzzy relation equations whose parameters $A_i, 1 \leq i \leq n$, form a semi-partition of $\mathbf{X}$, we will prove the optimality of $\check{R}$ in $\mathcal{R}_u$ with respect to the preorder $\leq_u$, and with respect to $\leq_\delta$ in the next subsection.

Theorem 26. *Let system (7) be unsolvable and fuzzy sets A_i, $1 \leq i \leq n$, be normal and form a semi-partition of $\mathbf{X}$. Then the fuzzy relation*

$$\check{R}(x,y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y))$$

is a best approximate solution to system (7) in the space $\mathcal{R}_u$ with respect to the preorder $\leq_u$ (cf. (25)).

The following theorem shows that if the original system (7) is unsolvable then the first solvable system (when increasing the right hand sides of (7)) is the system with B_i replaced by $\check{B}_i$.

Theorem 27. *Let the conditions of Theorem 26 be fulfilled and fuzzy sets $C_i \in \mathcal{F}(\mathbf{Y})$ be such that*

$$C_i \geq B_i, \quad i = 1, \ldots, n.$$

Then if the system

$$A_i \circ R = C_i, \quad i = 1, \ldots, n.$$

is solvable then

$$C_i \geq \check{B}_i, \quad i = 1, \ldots, n,$$

where $\check{B}_i = A_i \circ \check{R}$.

9.3 Optimality of $\check{R}$ with respect to the preorder $\leq_\delta$

As the last result of this section, we will prove that $\check{R}(x,y)$ is an optimal solution to system (7) with respect to $\leq_\delta$ too.

Theorem 28. *Let the conditions of Theorem 26 be satisfied. Then fuzzy relation $\check{R}(x,y)$ is a best approximate solution to system (7) in $\mathcal{R}_u$ with respect to the approximation quality $\delta(R)$.*

10 Optimality of other pseudo-solutions

Though we introduced various approximation spaces, only two representatives, i.e. $\hat{R}$ and $\check{R}$ have been considered as their members. We have introduced in [8] another candidate for optimal approximation — the iterated relation

$$\check{\hat{R}}(x,y) = \bigvee_{i=1}^{n} (A_i(x) * \hat{B}_i(y)) = \bigvee_{i=1}^{n} (A_i(x) * \bigvee_{x \in \mathbf{X}} (A_i(x) * \bigwedge_{j=1}^{n} (A_j(x) \to B_j(y)))).$$

As before, we use the notation

$$\hat{B}_i(y)) = (A_i \circ \hat{R})(y) = \bigvee_{x \in \mathbf{X}} (A_i(x) * \bigwedge_{j=1}^{n} (A_j(x) \to B_j(y))).$$

The idea lying in the construction of $\check{\check{R}}$ is to replace B_is in the relation $\check{R}$ by $A_i \circ \hat{R}$ (which are smaller) and by this, create a new relation which is smaller than $\check{R}$. Actually,

$$\check{\check{R}} \leq \check{R}$$

and, as shown in [8],

$$A_i \circ \hat{R} \leq A_i \circ \check{\check{R}} \leq A_i \circ \check{R}.$$

Therefore, the optimality of $\check{\check{R}}$ is expected, and this is proved in the theorem below.

Theorem 29. *Let system (7) be not solvable and suppose that the system*

$$A_i \circ R = \hat{B}_i \tag{30}$$

is $\check{R}$-solvable. Then the iterated relation $\check{\check{R}}$ is a best approximate solution to (7) in $\mathcal{R}_l$ with respect to the preorder $\leq_l$ as well as with respect to the quality $\delta(R)$.

Remark 30.

- It follows from Theorem 29 that there are at least two best approximate solutions to (7) in $\mathcal{R}_l$, both with respect to the preorder $\leq_l$ as well as to the quality $\delta(R)$.

  The non-uniqueness of a best approximation is a consequence of the fact that the solvability of (7) is not equivalent to the existence of exactly one solution. Let us explain this claim in more details.

  Our optimality criteria have been chosen in such a way that they measure a deviation from the original right-hand side of system (7). Therefore, if some approximate solution $\tilde{R}$ is optimal then any other fuzzy relation which solves (7) with the same right-hand side as $\tilde{R}$ does, is optimal as well.

- If we want to distinguish various best approximations more subtly, we should specify fuzzy relations (solutions) according to their additional properties. For example, the approximate solution $\hat{R}$ is the greatest element in $\mathcal{R}_l$ (with respect to the ordinary ordering), and this distinguishes it among other (best) approximate solutions.

- We conlucde from Theorems 22, 28, 29 that $\delta(R)$ can be taken as a universal measure of approximation quality in the approximation spaces $\mathcal{R}_l$ and $\mathcal{R}_u$.

11 Optimality under the stronger criterion

Let us summarize the above used methodology for construction of approximate solutions to system (7). We replaced the right-hand sides of equations in (7) by those which guarantee the solvability and took the guaranteed solution as the approximate one. Then we have noticed that the guaranteed solutions composed with the fixed left-hand sides of equations in (7) produced either lower or upper approximations of the given right-hand sides. This observation led us to the introduction of two approximating spaces consisting of those fuzzy relations which, when composed with the fixed left-hand sides, produce various lower or upper approximations of the given right-hand sides. In each approximating space the respective guaranteed solution was among the best approximate solutions to system (7).

In this section, we will extend the approximating space by fuzzy relations which, when composed with fuzzy sets greater than the given left-hand sides of equations in (7), produce smaller right-hand sides than the given ones. We will show that in such extended space the known fuzzy relation $\hat{R}$ is again among the best approximate solutions with respect to the below introduced preorder $\leq_\gamma$.

Suppose as before that system (7) is not solvable and introduce the approximating space

$$\begin{aligned}\mathcal{R}_{lr} = \{R \in \mathcal{F}(\mathbf{X}\times\mathbf{Y}) \mid D_i \circ R = C_i, \quad 1 \leq i \leq n,\\ \text{for some} \quad D_1,\dots,D_n \in \mathcal{F}(\mathbf{X}), C_1,\dots,C_n \in \mathcal{F}(\mathbf{Y}) \quad \text{such that}\\ A_i \leq D_i, C_i \leq B_i\}\end{aligned} \quad (31)$$

and the following quality of approximation

$$\gamma(R) = \bigwedge_{i=1}^{n} \left(\bigwedge_{y\in\mathbf{Y}} (B_i(y) \leftrightarrow (A_i \circ R)(y)) \wedge \wedge \bigwedge_{x\in\mathbf{X}} (A_i(x) \leftrightarrow \bigwedge_{y\in\mathbf{Y}} (R(x,y) \to B_i(y))) \right). \quad (32)$$

The second term in (32) arises from the expression in (**??**) which gives the maximal solution to (7) with respect to unknown A_i.

We can compare different relations saying that $R' \in \mathcal{R}_{lr}$ is better than $R'' \in \mathcal{R}_{lr}$ if and only if its γ-quality $\gamma(R')$ is higher. Formally:

$$R' \leq_\gamma R'' \quad \text{iff} \quad \gamma(R'') \leq \gamma(R'). \quad (33)$$

Moreover, we can define an optimal approximation as follows.

Definition 31. A fuzzy relation R_{opt} is a best approximate solution to system (7) in the approximation space $\mathcal{R}_{lr}$ with respect to the quality $\gamma(R)$ if

$$\gamma(R_{\text{opt}}) = \sup_{R\in\mathcal{R}_{lr}} \gamma(R). \quad (34)$$

The following theorem shows that the relation $\hat{R}$ is again a best one with respect to the quality $\gamma(R)$.

Theorem 32. *Let system (7) be not solvable. Then the set $\mathcal{R}_{lr}$ is non-empty and fuzzy relation $\hat{R}$ is a best approximate solution in the set $\mathcal{R}_{lr}$ with respect to the quality $\gamma(R)$.*

Corollary 33. *The fuzzy relation $\hat{R}$ is the largest approximate solution in $\mathcal{R}_{lr}$ with respect to the ordinal ordering $\leq$.*

12 Concluding remarks

Most of the known results about solvability of systems of fuzzy relation equations have practical importance only in the case when universes of discourse X and Y are finite. In case when these universes are infinite, the complexity of verifying theoretical conditions is comparable with a direct checking of a solvability. Therefore, the problem of discovering easy to check conditions or criteria is still actual. This paper is (among others) a contribution to this topic.

A number of new criteria of the so called Mamdani relation to be a solution to the system is suggested. At the same time, these criteria are sufficient conditions for solvability of the system in general. A new, easy to check criterion of a solvability of the system with special fuzzy parameters is found.

With the notion of a fuzzy function as a mapping between universes of fuzzy sets we threw a new light on the problem of solvability and approximate solvability. In this setting, precise and approximate solutions to a system of fuzzy relation equations are considered as the interpolating and approximating fuzzy functions with respect to the given data. We concentrated on a problem of approximate solvability of a system of fuzzy relation equations. Different approximating spaces and different criteria of approximation have been introduced. We have proved that the widely known fuzzy relations introduced by E. Sanchez and E. H. Mamdani are the best approximations in the respective spaces and under the respective criteria.

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Point-set lattice-theoretic (poslat) topology: a (partly) categorical perspective

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Point-set lattice-theoretic (or poslat) topology refers to that sort of topology for which a space, roughly speaking, is determined from a (carrier) set X, a lattice L of some sort, and an associated topology—either a family of L-valued mappings on X or an operator on the powerset of all L-valued mappings, and for which there are appropriate continuous morphisms. Such topology is also called lattice-valued, many-valued, fuzzy, etc.

For the last 35 years, poslat topology has been intensely developed, aided in significant measure by the *International Seminar on Fuzzy Set Theory*, also known as the *Linz Seminar*. It is our purpose to outline certain aspects of this poslat topology from a (partly) categorical point of view with the general goal of identifying some categories which serve as relevant frameworks for poslat topology, *relevant* in the sense that these categories are topological and contain important examples.

This goal is pursued by doing the following: sampling well-known lattice-theoretic and ground categories and overlying fixed-basis and variable-basis categories for poslat topology; discussing their relationships to point-free categories for topology, Wang's category for lattice-valued topology, and Vicker's category for topological systems arising from domains in computer science; indicating in what sense these categories are topological; sketching functorial relationships between these categories; and inventoring some important examples of objects and morphisms for poslat topology.

1 Preliminaries

1.1 Lattice-theoretic conditions

The most general lattice structure we will consider is that of a **complete quasi-monoidal lattice** (cqml) as defined in [61]: a complete lattice equipped with a binary operation, called a **tensor product**, which is isotone in both arguments and has the top element as an idempotent. See [23] for stronger versions of this definition. Many examples of cqml's are catalogued in [23, 64].

1.2 Lattice-theoretic categories

The category **Cqml** [23, 61] comprises the class of all cqml's, together with the class of all mappings between cqml's which preserve tensor products, arbitrary joins, and top elements. The dual category $\mathbf{Cqml}^{op}$ is denoted **Loqml** and is called the **category of localic quasi-monoidal lattices.** Most of the lattice-theoretic categories of interest in poslat topology are isomorphic to subcategories of **Cqml** or **Loqml**, including [61] **SFrm** (semiframes), its dual **SLoc** (semilocales), **Frm** (frames), **Loc** (locales), **Dmrg** (complete deMorgan algebras), its dual $\mathbf{Dmrg}^{op}$, **Hut** (Hutton algebras), and its dual **FuzLat**, as well as various categories in which the tensor is not the binary meet.

1.3 Ground categories and powerset operators

1.3.1 Fixed-basis grounds and powerset operators

For the case when the cqml L is fixed, the ground category is **Set**, with the associated Zadeh powerset operators $f_L^{\rightarrow}$, $f_L^{\leftarrow}$ between L^X and L^Y for a ground morphism $f : X \rightarrow Y$ [10, 46, 59, 60]. Many properties and characterizations are known for these Zadeh operators, including that $f_L^{\rightarrow} \dashv f_L^{\leftarrow}$.

1.3.2 Variable-basis grounds and powerset operators

For the case when the cqml L may vary, a subcategory **C** of **Loqml**—within which L varies—is fixed and the ground category is $\mathbf{Set} \times \mathbf{C}$, with ground morphisms of the form $(f, \phi) : (X, L) \rightarrow (Y, M)$ with $f : X \rightarrow Y$ in **Set** and $\phi^{op} : L \leftarrow M$ in $\mathbf{C}^{op} \subset \mathbf{Cqml}$, and with the associated powerset operators $(f, \phi)^{\rightarrow}, (f, \phi)^{\leftarrow}$ are between L^X and M^Y [10, 54, 55, 56, 59, 60]

Theorem. $(f, \phi)^{\rightarrow} \dashv (f, \phi)^{\leftarrow}$ if and only if ϕ^{op} preserves arbitrary meets. The consequent holds if: $\phi \in \mathbf{Dmrg}^{op}(L, M)$; ϕ^{op} is a backward Zadeh operator; i.e. $\exists N \in |\mathbf{CQML}|, \exists g \in \mathbf{Set}(W, Z), \phi^{op} = g_N^{\leftarrow}$; ϕ is any of the examples constructed in 7.1.7.2 of [61] or 9.9(2(b), 3) of [62]; or ϕ is an isomorphism in **Loqml**.

1.4 Adjoint Functor Theorem

Let $f : L \rightarrow M$, $g : L \leftarrow M$ be isotone maps between preordered sets. Then $f \dashv g$ provided $[\forall a \in L, a \leq g(f(a))]$ and $[\forall b \in M, f(g(b)) \leq b]$, or equivalently, $[\forall a \in L, b \in M, a \leq g(b) \Leftrightarrow f(a) \leq b]$. If $f \dashv g$, then we write $g = f^{\vdash}$ and $f = g^{\dashv}$.

Theorem (Adjoint Functor Theorem [26]**).** Let $f : L \rightarrow M$ $[g : L \leftarrow M]$ be a function such that L $[M]$ has arbitrary $\bigvee$ $[\bigwedge]$ and f $[g]$ preserves arbitrary $\bigvee$ $[M$, respectively]. Then f $[g]$ is isotone, $\exists!\, f^{\vdash} : L \leftarrow M$ $[g^{\dashv} : L \rightarrow M]$, and $f^{\vdash}$ $[g^{\dashv}]$ preserves all $\bigwedge$ $[\bigvee]$ existing in M $[L]$.

2 Categories For Poslat Topological Structures

2.1 Some Fixed-Basis Categories

Fixed-basis categories are fixed with respect to the underlying cqml L, but varying with respect to the underlying ground object (or set).

Fixing L in **Cqml**, the well-known category L-**Top** [2, 11, 23, 61] has ground category **Set**, with the topology being a crisp subset of the L-powerset closed under (binary) tensor products and arbitrary

$\bigvee$ and containing the top L-subset; and the well-known category L-**FTop** [15, 31, 71, 22, 40, 23] has ground category **Set**, with the topology being an L-subset of the L-powerset which assigns as degree of openness to tensor products the least degree of the tensorands, to arbitrary joins the least degree of the disjuncts, and to the top L-subset the top element of L. See the analysis of important subcategories in [23], often using underlying L with richer structure or with additional conditions on the topology (such as in [42]).

2.2 Some Point-Free Categories

The category **Loc** may be considered to have ground $\mathbf{Set}^{op}$. Each locale may be regarded as the (sober) topology of some L-topological space; and if L turns out to be **2** in that statement, the locale is called **spatial** [55, 56, 57, 58]. More generally, we may replace **Loc** with **Loqml** or $\mathbf{C} \hookrightarrow \mathbf{Loqml}$; restated, each subcategory of **Loqml** can be viewed as a point-free category of topological structures.

Fixing **C** a subcategory of **Loqml**, the category **C-HTop** [25, 55, 61] has ground category **C**, with the topology being a crisp subset of some L in **C** that is closed under the tensor and arbitrary joins and containing the top element; the famous definition originally given in [25] used $\mathbf{C} = \mathbf{FuzLat}$ as the ground category. Further, the category **C-HFTop** has ground category **C**, with the topology being an L-subset of some L in **C** which has properties analogous to those of the topologies in L-**FTop**.

It is our contention that every point-free approach is essentially a variable-basis approach (see below). We have listed these separately from the variable-basis approaches since, with the exception of **C-HFTop**, their origins were independent of, and prior to, variable-basis topology.

2.3 Some Variable-Basis Categories

The underlying set is free to change in fixed-basis topology while the lattice-theoretic base is fixed; and the underlying set is fixed in point-free topology (as a singleton—see below) while the lattice-theoretic base is free to change. In variable-basis topology, both the underlying set and the lattice-theoretic base are free to change.

Fixing **C** a subcategory of **Loqml**, the category **C-Top** [5, 6, 7, 10, 52, 53, 55, 56, 61] has ground category $\mathbf{Set} \times \mathbf{C}$, objects being of the form (X, L, τ), with $(X, \tau) \in |L\text{-}\mathbf{Top}|$, and morphisms being of the form $(f, \phi) : (X, L, \tau) \to (Y, M, \sigma)$, with $\tau \supset ((f, \phi)^{\leftarrow})^{\rightarrow}(\sigma)$. Further, the category **C-FTop** [61] has ground category $\mathbf{Set} \times \mathbf{C}$, objects being of the form $(X, L, \mathcal{T})$, with $(X, \mathcal{T}) \in |L\text{-}\mathbf{FTop}|$, and morphisms being of the form $(f, \phi) : (X, L, \mathcal{T}) \to (Y, M, \mathcal{S})$, with $\mathcal{T} \circ (f, \phi)^{\leftarrow} \geq \phi^{op} \circ \mathcal{S}$ on M^Y.

2.4 Category Of Topological Systems

Topological systems [75] stem from placing domain theory of computer science into a topological setting [68, 69, 70]. The central idea in topological systems is that of a *satisfaction* or *modeling relation*.

We initially view topological systems as categorically having $\mathbf{Set} \times \mathbf{Loc}$ as ground. The category **TopSys** [75] has objects of the form $(X, A, \models)$, with $(X, A) \in |\mathbf{Set} \times \mathbf{Loc}|$ and $\models \subset X \times A$ satisfying:

1. $\forall x \in X, \forall S \subset A,\ x \models \bigvee S \Leftrightarrow \exists a \in A, x \models a$.
2. $\forall x \in X, \forall$ finite $S \subset A,\ x \models \bigwedge S \Leftrightarrow \forall a \in A, x \models a$.

And morphisms are of the form $(f,\phi):(X,A,\models)\to(Y,M,\models)$, where $\forall x\in X, \forall b\in B,\ f(x)\models b \Leftrightarrow x\models\phi^{op}(b)$. (The same symbol is used for both the domain and codomain satisfaction relations.)

Clearly, **TopSys** is a variable-basis approach. But we have separately listed this approach for two reasons: the notion of topological system arose independently of, and subsequent to, variable-basis topology; and the categorical behavior of topological systems is strikingly different than that of variable-basis topology (see below).

2.5 Category Of Wang Topological Spaces

From [76, 77, 78] comes a schemum of categories not having an obvious ground category. Let $\mathbf{C}\hookrightarrow\mathbf{Dmrg}^{op}$ (the original definition requires $\mathbf{C}=\mathbf{FuzLat}$).

Given $L,M\in|\mathbf{C}|$, a set mapping $\phi:L\to M$ is an **order homomorphism** if ϕ preserves arbitrary $\bigvee$ and $\forall b\in M, \left(\phi^{\vdash}(b')\right)=\left(\phi^{\vdash}(b)\right)'$ (i.e. $\phi^{\vdash}\in\mathbf{Dmrg}$). The category **C-WTop** has objects of the form $\left(L^X,\tau\right)$, where $X\in|\mathbf{Set}|$, $L\in|\mathbf{C}|$, and $(X,\tau)\in|L\text{-}\mathbf{Top}|$, and morphisms of the form $\phi:\left(L^X,\tau\right)\to\left(M^Y,\sigma\right)$, where $\phi:L^X\to M^Y$ is an order homomorphism and $\tau\supset\left(\phi^{\vdash}\right)^{\to}(\sigma)$.

As will be seen below, the Wang approach is essentially a point-free approach and is isomorphic to a subcategory of singleton spaces in **C-Top**.

3 Topological Categories For Poslat Topological Structures

3.1 Definition Of Topological Categories

The definition of “**A** is **topological w.r.t. category X and functor** V” comes from [1]; see commentary on this definition in [61]. These variations are also useful:

1. **A** is **small topological w.r.t. category X and functor** V if the indexing class for V-structured sources is always a set.

2. **A** is **quasi-topological w.r.t. category X and functor** V if the unique existence of the lifted morphism in the definition of initiality is replaced by existence.

3. **A** is **c.e.m. topological w.r.t. category X and functor** V if the V-structured source is collection-wise extremally monomorphic in the language of [49] or a mono-extremal source in the language of [1].

4. **A** is **essentially topological [small toplogical, quasi-topological, c.e.m. topological] w.r.t. category X and functor** V if “unique initial V-lift” is replaced by the condition that initial V-lifts of the same V-structured source are isomorphic in the appropriate definitions above.

3.2 Examples Of Topological Categories

In the following statements, the functor V is the forgetful functor, such a functor being obvious once the ground category is specified (using the word “over”).

Theorem [23, 61]. If $L \in |\mathbf{Cqml}|$, then L-**Top** and L-**FTop** are topological over **Set**; if $\mathbf{C} \hookrightarrow \mathbf{Loqml}$, then **C-Top** and **C-FTop** are topological over $\mathbf{Set} \times \mathbf{C}$; and if $\mathbf{C} \hookrightarrow \mathbf{Loqml}$, then **C**, **C-HTop, C-HFTop** are topological over $\mathbf{Set} \times \mathbf{C}$ w.r.t. the forgetful functor of the previous theorem as modified by the embeddings given below of these categories into **C-Top**, **C-Top**, and **C-FTop**, respectively (so that **Loc** is topological in this way over $\mathbf{Set} \times \mathbf{Loc}$).

3.3 Special Case Of Topological Systems

In view of the motivation of topological systems and their relationship to variable-basis spaces given later, the behavior of **TopSys** is rather surprising.

Theorem. **TopSys** is not topological over $\mathbf{Set} \times \mathbf{Loc}$ in any sense or with any modifier as defined above—V-structured sources comprising only one morphism need not even have lifts; **TopSys** is essentially small topological over **Set**—each small V-structured source has a initial lift that is unique up to isomorphism; and **SobTopSys** is essentially c.e.m. topological and essentially quasi-topological over **Loc**.

Conjecture. **TopSys** is neither topological over **Set** nor over **Loc**.

3.4 Special Case Of Wang Topological Spaces

Let $\mathbf{C} \hookrightarrow \mathbf{Dmrg}^{op}$. The problem with **C-WTop** is the lack so far of a well-defined ground category. It is therefore not known in what sense (if any) **C-WTop** is topological.

4 Relationships Between Categories For Poslat Topological Structures

4.1 Adjoint Pairs Between Top And L-Top

The relationships between **Top** and L-**Top** may be classified as concrete or nonconcrete.

4.1.1 Concrete Adjunctions Between Top And L-Top

Many of the concrete adjoint relationships between **Top** and L-**Top** can be unified by the concept of indexed families of mappings between the traditional and L-based fibres [48]. Fix $X \in |\mathbf{Set}|$ and $L \in |\mathbf{SFrm}|$, and let $\mathfrak{T}_{XL}$ be the fibre of all traditional topologies on X and τ_{XL} be the fibre of all L-topologies on X. A pair of isotone maps $F_{XL} : \mathfrak{T}_X \to \tau_{XL}, G_{XL} : \mathfrak{T}_X \leftarrow \tau_{XL}$ is said to be an (L-)**fibre pair (of maps)** and this fibre-pair is **covariant [contravariant]** if $G_{XL} \dashv F_{XL}$ $[F_{XL} \dashv G_{XL}]$. An indexed family $\{F_{XL}, G_{XL}\}_{X \in |\mathbf{Set}|}$ of such maps is said to be an **covariant [contravariant] indexed family of (**L-)**fibre pairs**, and the following conditions can be considered:

1. Such a family **joint-covariantly [joint-contravariantly] generated** if $\forall X \in |\mathbf{Set}|$, $\exists$ a **generator**

$$g_{XL} : \mathbf{2}^{\wp(X)} \leftarrow L^X \quad [f_{XL} : \mathbf{2}^{\wp(X)} \to L^X]$$

such that $\forall \mathfrak{T} \in \mathfrak{T}_X, \forall \tau \in \tau_{XL}$,

$$F_{XL}(\mathfrak{T}) = \langle\langle g_{XL}^{\leftarrow}(\downarrow(\mathfrak{T}))\rangle\rangle, \quad G_{XL}(\tau) = \left\langle\left\langle \bigcup_{u\in\tau} g_{XL}(u) \right\rangle\right\rangle$$

$$\left[F_{XL}(\mathfrak{T}) = \langle\langle g_{XL}^{\leftarrow}(\downarrow(\mathfrak{T}))\rangle\rangle, \quad G_{XL}(\tau) = \left\langle\left\langle\left\{\bigcup \mathfrak{U} : \mathfrak{U} \in f_{XL}^{\leftarrow}(\tau)\right\}\right\rangle\right\rangle\right]$$

2. A jointly-covariantly [jointly-contravariantly] generated family $\{F_{XL}, G_{XL}\}_{X\in|\mathbf{Set}|}$ is **joint-covariantly [joint-contravariantly] natural** if $\forall X, Y \in |\mathbf{Set}|$, the diagram commutes:
$$g_{XL} \circ f_L^{\leftarrow} = (f^{\leftarrow})^{\rightarrow} \circ g_{YL}$$
$$[f_{XL} \circ (f^{\leftarrow})^{\rightarrow} = f_L^{\leftarrow} \circ f_{YL}]$$

3. An indexed family $\{F_{XL}, G_{XL}\}_{X\in|\mathbf{Set}|}$ of fibre-pairs is **separately generated** if $\forall X \in |\mathbf{Set}|$, $\exists$ **generators**
$$f_{XL} : \wp(X) \to L^X, g_{XL} : \wp(X) \leftarrow L^X$$
such that $\forall \mathfrak{T} \in \mathfrak{T}_X, \forall \tau \in \tau_{XL}$,
$$F_{XL}(\mathfrak{T}) = \langle\langle f_{XL}^{\rightarrow}(\mathfrak{T})\rangle\rangle, \quad G_{XL}(\tau) = \langle\langle g_{XL}^{\rightarrow}(\tau)\rangle\rangle$$

4. A separately generated family $\{F_{XL}, G_{XL}\}_{X\in|\mathbf{Set}|}$ of fibre-pairs is **separately natural** if these diagrams commute:
$$f_{XL} \circ f^{\leftarrow} = f_L^{\leftarrow} \circ f_{YL}, \quad g_{XL} \circ f_L^{\leftarrow} = f^{\leftarrow} \circ g_{YL}$$

Examples. The characteristic and Martin G_χ, M_χ fibre maps [47, 61] comprise a joint-contravariantly natural family of fibre-pairs as well as a separately natural, contravariant family of fibre-maps; the Kubiak-Lowen ω_L, ι_L fibre maps [42, 34] comprise a joint-covariantly natural family of fibre-pairs; and the level fibre maps F_α, S_α [43, 51, 55, 61] comprise a joint-covariantly natural family of fibre-pairs (α prime).

Theorem. Let $L \in |\mathbf{SFrm}|$, let $\{F_{XL}, G_{XL}\}_{X\in|\mathbf{Set}|}$ be an indexed family of L-fibre pairs, and let the bi-level mappings $F : \mathbf{Top} \to L\text{-}\mathbf{Top}$, $G : \mathbf{Top} \leftarrow L\text{-}\mathbf{Top}$ be defined as follows:

$$\begin{aligned} F(X, \mathfrak{T}) &= (X, F_{XL}(\mathfrak{T})), \quad F(f) = f \\ G(X, \tau) &= (X, G_{XL}(\tau)), \quad G(f) = f \end{aligned}$$

1. If $\{F_{XL}, G_{XL}\}_{X\in|\mathbf{Set}|}$ is joint-covariantly natural, then F is a concrete functor, G is a concrete functor, and $F \dashv G$.

2. If $\{F_{XL}, G_{XL}\}_{X\in|\mathbf{Set}|}$ is joint-contravariantly natural, then F is a concrete functor, G is a concrete functor, and $G \dashv F$.

3. If $\{F_{XL}, G_{XL}\}_{X\in|\mathbf{Set}|}$ is separately natural covariant [contravariant], then F is a concrete functor, G is a concrete functor, and $F \dashv G$ [$G \dashv F$].

Corollary. The examples and the theorem imply the $M_\chi \dashv G_\chi$, $\omega_L \dashv \iota_L$, $F_\alpha \dashv S_\alpha$ (α prime) adjunctions between **Top** and L-**Top**.

4.1.2 Non-Concrete Adjunctions Between Top And L-Top

Examples of non-concrete adjunctions include the hypergraph functor [67, 43, 51, 9] and the adjunction based on it (assuming L a spatial frame) in [17, 18], as well as the adjunction based on the L-**2** and **2**-L soberification functors [62]. The role of the hypergraph functor in fuzzy addition and fuzzy multiplication can be seen in the references of [63], and the role of the soberification adjunction in building alternative fuzzy real lines and unit intervals can be seen in Sections 2 and 8 of [62]..

4.2 Embedding Of Fixed-Basis And Crisp Variable-Basis Into Fuzzy Variable-Basis

Given $L \in |\mathbf{C}|$, L-**Top** embeds into **C-Top** and L-**FTop** embeds into **C-FTop** by simply choosing $\phi = id_L$. The adjunction between **C-Top** and **C-FTop** (Section 6 of [61]) induces from an "indexed family of fibre pairs" which are an extension of the characteristic-Martin fibre pairs referenced above. In this more general setting, given (X, L, τ) and $(X, L, \mathcal{T})$, $G_\chi(\tau) = \bigwedge_{\mathcal{T} \geq \chi_\tau} \mathcal{T}$ and $M_\chi(\mathcal{T}) = coker(\mathcal{T})$.

4.3 Singleton Embeddings Of Point-Free And Wang Into Variable-Basis

The embeddings of **Loc**, $\mathbf{C} \hookrightarrow$ **Loqml**, **C-HTop** into **C-Top** and the embedding of **C-HTop** into **C-FTop** are given in [54, 55, 61] and are all **singleton functors** making each point-free category isomorphic to a subcategory of singleton spaces.

To illustrate, **Loc** embeds into **Loc-Top** via $A \mapsto (\mathbf{1}, A, A^{\mathbf{1}})$, $[\phi : A \to B] \mapsto [(id, \phi) : (\mathbf{1}, A, A^{\mathbf{1}}) \to (\mathbf{1}, B, B^{\mathbf{1}})]$. Letting $S : \mathbf{Loc} \to \mathbf{Loc\text{-}Top}$ be the embedding just described and $\mathbf{Loc\text{-}Top}_{sk}$ be the full subcategory of stratified singleton spaces, $S \dashv \Omega|_{\mathbf{Loc\text{-}Top}_{sk}} \dashv S$, where $\Omega(X, L, \tau) = \tau$ and $\Omega(f, \phi) = [(f, \phi)^{\leftarrow}]^{op}$. It follows that S is an isomorphism onto $\mathbf{Loc\text{-}Top}_{sk}$ and we should regard **Loc** as a special case of variable-basis point-set lattice-theoretic topology, namely **Loc** is a special case of *singleton variable-basis topology* or *variable-basis topology of singleton spaces*. From this point of view, point-free topology is not a generalization of topology, but rather the special and important case of singleton space topology which focuses on the lattice-theoretics of poslat topology.

The case of **C-WTop** (with $\mathbf{C} \hookrightarrow \mathbf{Dmrg}^{op}$) requires only a slight modification of the singleton functor embedding **C-HTop** into **C-Top**:

$$(L^X, \tau) \mapsto \left(\mathbf{1}, L^X, \tau^{\mathbf{1}}\right),$$

$$\left[\phi : (L^X, \tau) \to (M^Y, \sigma)\right] \mapsto \left[\left(id, \left(\phi^{\vdash}\right)^{op}\right) : \left(\mathbf{1}, L^X, \tau^{\mathbf{1}}\right) \to \left(\mathbf{1}, M^Y, \sigma^{\mathbf{1}}\right)\right]$$

This embedding means that the Wang approach is isomorphic to a subcategory of singleton spaces, *despite the set exponent in Wang objects being non-singleton*.

Essentially, the Wang morphisms do not recognize these non-singleton sets and treats them as if they are singletons. Restated, the Wang approach is essentially point-free. For categories of the form **C-WTop**, it would seem that the mixed syntax, lack of a clearly defined ground category, and seeming lack of being a topological category are issues and questions that need resolution for this popular approach.

4.4 Embeddings Of Fixed-Basis Into Topological Systems [4]

For many $L \in |\mathbf{Frm}|$, there are simple embeddings of L-**Top** into **TopSys**. Fix $L \in |\mathbf{Frm}|$ such that L has a prime element α, and let $(X,\tau) \in |L\text{-}\mathbf{Top}|$. Note $(X,\tau) \in |\mathbf{Set} \times \mathbf{Loc}|$. Define $\models_{\tau,\alpha}$ on (X,τ) by putting $\forall x \in X,\, u \in \tau,\, x \models_{\tau,\alpha} u \Leftrightarrow u(x) > \alpha$. Further, given $f : (X,\tau) \to (Y,\sigma) \in L\text{-}\mathbf{Top}$, define the ground morphism $(f, (f_L^{\leftarrow})^{op}) : (X,\tau) \to (Y,\sigma)$ in $\mathbf{Set} \times \mathbf{Loc}$. Then $F_\alpha(X,\tau) = (X,\tau,\models_{\tau,\alpha})$, $F_\alpha(f) = (f, (f_L^{\leftarrow})^{op})$ defines F_α as a functor from L-**Top** to **TopSys** which is an embedding. This generalizes the spatialization embedding of **Top** into **TopSys** of [75].

4.5 Embedding Of Topological Systems Into Variable-Basis [4]

The relationship between **TopSys** and **Loc-Top** is induced by another variety of maps between posets of structures.

Let $(X,A,\models) \in |\mathbf{TopSys}|$ be given, put

$$\begin{aligned} F(\models) &= \tau_{\models} \\ &\equiv \left\{u \in A^X : (\forall x \in X)(x \models u(x)) \text{ or } (\forall x \in X)(u(x) = \bot)\right\} \end{aligned}$$

$F(X,A,\models) = (X,A,\tau_{\models})$, $F(f,\phi) = (f,\phi)$. Then $F : \mathbf{TopSys} \to \mathbf{Loc}\text{-}\mathbf{Top}$ is a functorial embedding. We note **TopSys** is isomorphic to a proper subcategory of **Loc-Top** since the latter is topological over $\mathbf{Set} \times \mathbf{Loc}$ and **TopSys** is not topological over $\mathbf{Set} \times \mathbf{Loc}$ and the forgetful functor from **TopSys** to $\mathbf{Set} \times \mathbf{Loc}$ factors through F and the forgetful functor from **Loc-Top** to $\mathbf{Set} \times \mathbf{Loc}$.

Given that each of **Loc** and **Top** embed into **TopSys**—the former [latter] by the localification [spatialization] functor of [75], we can now answer a long-standing question whether **Loc-Top** is the smallest supercategory, up to embedding, of **Loc** and **Top**: the answer is *no*, namely, **Loc** and **Top** embed properly into **TopSys** and **TopSys** embeds properly into **Loc-Top**.

Finally, the fact that **TopSys** is not topological means that only in **Loc-Top** can the initial and final lifts of forgetful functor structured sources from **TopSys** be constructed.

5 Examples Of Objects And Morphisms For Poslat Topological Structures

It is not sufficient to have topological categories. Such categories must also exhibit important examples of objects and morphisms justifying the study of such categories and the approaches to topology they represent.

From [4, 8, 17, 18, 19, 23, 24, 30, 33, 34, 35, 36, 37, 38, 39, 44, 45, 50, 51, 55, 58, 59, 61, 62, 63, 64] and their bibliographies an inventory of many important examples can be constructed.

Here is a sample of significant objects in poslat topology:

1. $\mathbb{R}(L)$ and $\mathbb{I}(L)$, for L a deMorgan quasi-monoidal lattice (which includes distributive and non-distributive deMorgan algebras).

2. $\mathbb{R}$ and $\mathbb{I}$ equipped with the dual L-topologies induced from $\mathbb{R}(L)$ and $\mathbb{I}(L)$ (L as above).

3. $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$, the alternative L-fuzzy real line and L-fuzzy unit interval, formed by the L-**2** soberification functor acting on $\mathbb{R}$ and $\mathbb{I}$ for any complete quasi-monoidal lattice (which includes all complete lattices)—and indeed each complete lattice A generates a canonical L-sober space $LPT(A)$.

4. The space of probability measures on the Borel sets of a separable metric space, which gives a stratified, non-generated $\mathbb{I}$-topological space.

5. Traditional limit spaces generate for each complete Heyting algebra a class of L-topological spaces.

6. $\mathbb{I}$-rigid topological spaces constructed using τ-smooth Borel probability measures on ordinary spaces and Radon measures on ordinary compact Hausdorff spaces, constructions allowing Boolean negation to extend continuously to Łukasiewicz negation.

7. Each ordinary T_1 space X with at most finitely many components generates an L-topological space $X(L)$ for $L \in |\mathbf{Hut}|$ with $\perp$ meet-irreducible such that if $L = \mathbf{2}$, $X(L)$ is L-homeomorphic to $G_\chi(X)$, and if $X = \mathbb{R}$ or $\mathbb{I}$, $X(L)$ is L-homeomorphic to $\mathbb{R}(L)$ or $\mathbb{I}(L)$.

8. Variable-basis spaces generated from specific topological systems.

   Here is a sample of significant morphisms in poslat topology:

9. Fuzzy addition and fuzzy multiplication in $\mathbb{R}(L)$.

10. Fuzzy translation and fuzzy scaling (especially in light of the behavior of the inverse mappings of these maps)

11. Fuzzy addition as uniformly continuous

12. Units of adjunctions having universal lifting and extension properties, such as the L-continuous and variable-basis morphisms generated by compactification reflectors from any non-(Chang) compact space such as the canonical $\mathbb{R}(L)$, $\mathbb{R}^*(L)$, $(0,1)(L)$, $(0,1)^*(L)$, etc, catalogued above.

13. Extensions of important continuous maps.

14. The rich inventory of variable-basis morphisms between fuzzy real lines, between induced spaces, between soberifications, all with different underlying bases.

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A bridge between fuzzy set theory and coherent conditional probabilities (I)

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In this talk (strictly linked with that by Giulianella Coletti with the same title) we expound our interpretation of fuzzy set theory (both from a semantic and a syntactic point of view) in terms of conditional events and *coherent* conditional probabilities. During past years, many papers have been devoted to support the negative view maintaining that probability is inadequate to capture what is usually treated by fuzzy theory. In our approach we emphasize the role of *conditioning* (in a proper framework, *i.e.* de Finetti's coherence) to get rid of many controversial aspects. Moreover, we introduce suitable operations between fuzzy subsets, looked on as corresponding operations between conditional events endowed with the relevant conditional probability.

Let us start from the intuitive idea of fuzzy subset: where does it come from and what is its "operational" meaning? We will refer to the state of information (at a given moment) of a real (or fictitious) person (for instance, a "randomly" chosen one) that will be denoted by "You".

If X is a (not necessarily numerical) random quantity with range C_X, let A_x, for any $x \in C_X$, be the event $\{X = x\}$. The family $\{A_x\}_{x \in C_x}$ is obviously a *partition* of the certain event $\Omega = C_X$. Now, let φ be any *property* related to the random quantity X: from a pragmatic point of view, it is natural to think that You have some information about possible values of X, which allows You to refer to a suitable membership function of the fuzzy subset of "elements of C_X with the property φ".

For example, if X is a numerical quantity, for You the membership function may be put equal to 1 for values of X less than a given x_1, while it is put equal to 0 for values greater than x_2; then it is taken as decreasing from 1 to 0 in the interval from x_1 to x_2: this choice of the membership function implies that, for You, elements of C_X less than x_1 have the property φ, while those greater than x_2 do not. So the real problem is that You are uncertain on having or not the property φ those elements of C_X between x_1 and x_2.

Then the interest is in fact directed toward *conditional events* such as $E|A_x$, where x ranges over the interval from x_1 to x_2, with

$$E = \{\text{You claim the property } \varphi\},$$
$$A_x = \{\text{the value of } X \text{ is } x\}.$$

It follows that You may assign a subjective probability $P(E|A_x)$ equal, e.g., to 0.2 without any need to assign a degree of belief of 0.8 to the event E under the assumption A_x^c (i.e., the value of X is not x), since an additivity rule *with respect to the conditioning events* does not hold.

In other words, it seems sensible to identify the values of the membership function with suitable conditional probabilities. In particular, putting

$$H_o = \{\text{the value of } X \text{ is greater than } x_2\},$$
$$H_1 = \{\text{the value of } X \text{ is less than } x_1\},$$
we may assume that E and H_o are incompatible and that H_1 implies E, so that, by the properties of a conditional probability, $P(E|H_o) = 0$ and $P(E|H_1) = 1$.

Notice that the conditional probability $P(E|A_x)$ has been *directly* introduced as a function on the set of conditional events (and without assuming any given algebraic structure). Is that possible? In the usual (Kolmorogovian) approach to conditional probability the answer is NO, since the introduction of $P(E|A_x)$ would require the consideration (and the assessment) of $P(E \wedge A_x)$ and $P(A_x)$ (assuming positivity of the latter). But this is a *not* a simple task: in fact in this context the only sensible procedure is to assign directly $P(E|A_x)$. For example, to assign the (conditional) probability that You claim "Mary is young" knowing her age x, but not that of "the probability that Mary has the age x" (not to mention that, for different choices of the random quantity X, the corresponding probability can be zero).

The probabilistic approach adopted here differs radically from the usual theory based on a measure-theoretic framework, which assumes that a *unique* probability measure is defined on an *algebra* (or σ-algebra) of events constituting the so-called *sample space* Ω. Directing attention to events as subsets of the sample space (and to algebras of events) may be unsuitable for many real world situations, which make instead very significant both giving events a more general meaning and not assuming any specific structure for the set where probability is assessed.

Probability is seen as a *measure of belief in a given proposition*. Notice that a proposition – which can be either *true* or *false* – must not be looked on as an *assertion*: so, even if beliefs may come from various sources, they can be treated in the same way, since the relevant events (including possibly statistical data) need always to be considered (going back to a terminology due to Koopman) as being *contemplated* (or, similarly, *assumed*) and not *asserted* propositions.

This aspect is very crucial, since in our approach an essential role is played by *conditioning*: in fact the very concept of conditional probability is deeper than the usual restrictive view emphasizing $P(E|H)$ only as *a probability for each given H* (looked on as a given *fact*). Regarding instead also the conditioning event H as a "variable", we get something which is *not* just a probability (notice that H also – like E – plays the role of an *uncertain* event whose truth value is not necessarily given and known).

Our probabilistic framework is that based on the concept of *conditional event* and on the ensuing concept of *coherent conditional probability*. Our concept of conditional events differs from those adopted by many others in the relevant literature. Actually, in [1] we showed that, if we do not assign the same "third value" $t(E|H) = u$ (undetermined) to *all* conditional events, but make it suitably depend on $E|H$, it turns out that this function $t(E|H)$ can be taken as a general conditional *uncertainty measure* (and conditional *probability* corresponds to a particular choice of the relevant operations between conditional events).

Then a conditional probability $P(E|H)$ can be – through coherence – *directly* introduced and it is *not* defined as the ratio of the (unconditional) probabilities $P(E \wedge H)$ and $P(H)$, assuming positivity of the latter. This allows to deal with *conditioning events of zero probability*, avoiding to resort, as in the classic approach, to the Radon-Nikodym framework, which (rather than make conditional probability just depend on the given *conditioning* event) requires the *knowledge of the whole conditioning distribution*, a situation which is clearly unsound and contradicts the "inferential" meaning of a conditional event.

Finally, among the peculiarities of the concept of coherent *conditional* probability versus the usual one, we underline the possibility for $P(E|H)$ of assuming the extreme values 0 and 1 also for situations which are different, respectively, from the trivial ones $E \wedge H = \emptyset$ and $H \subseteq E$; moreover, we emphasize the "natural" *looking at the conditional event $E|H$ as "a whole", and not separately at the two events E and H.*

A complete account of probabilistic logic in a coherent setting is in the book [2]. We just mention that a coherent conditional probability can be characterized by suitably representing it by means of a *class* $\{P_\alpha\}$ of unconditional probabilities giving rise to the so-called *zero-layers* (for details, see [2], p.81).

In particular, given a family $\mathcal{C}$ of conditional events $\{E_i|H_i\}_{i\in I}$, where $card(I)$ is arbitrary and the events H_i's are a *partition* of Ω, we recall the following two corollaries of the aforementioned characterization theorem:

(A) *Any* function $f : \mathcal{C} \to [0,1]$ such that $f(E_i|H_i) = 0$ if $E_i \wedge H_i = \emptyset$ and $f(E_i|H_i) = 1$ if $H_i \subseteq E_i$ is a coherent conditional probability.

(B) If $P(\cdot|\cdot)$ is a coherent conditional probability such that $P(E|H_i) \in \{0,1\}$, then the following two statements are equivalent

(i) $P(\cdot|\cdot)$ is the *only* coherent assessment on $\mathcal{C}$;

(ii) it is $H_i \wedge E = \emptyset$ for every $H_i \in \mathcal{H}_0$ and $H_i \subseteq E$ for every $H_i \in \mathcal{H}_1$, where $\mathcal{H}_r = \{H_i : P(E|H_i) = r\}$, $r = 0,1$.

The results that follow are taken from [3]. Let φ be any *property* related to the random quantity X: notice that a *property*, even if expressed by a statement, does not single–out an *event*, since the latter needs to be expressed by a *nonambiguous* proposition that can be either *true* or *false*.

Consider now the **event** E_φ = "You claim φ" and a coherent conditional probability $P(E_\varphi|A_x)$, looked on as a real function $\mu_{E_\varphi}(x) = P(E_\varphi|A_x)$ defined on C_X.

Since the events A_x are incompatible, then – by **(A)** – every $\mu_{E_\varphi}(x)$ with values in $[0,1]$ is a coherent conditional probability. So we can *define* a fuzzy subset in this way:

Given a random quantity X with range C_X and a related property φ, a *fuzzy subset* E^*_φ of C_X is the pair

$$E^*_\varphi = \{E_\varphi\,,\,\mu_{E_\varphi}\},$$

with $\mu_{E_\varphi}(x) = P(E_\varphi|A_x)$ for every $x \in C_X$.

So a coherent conditional probability $P(E_\varphi|A_x)$ is a measure of how much You, given the event $A_x = \{X = x\}$, are willing to *claim* the property φ, and it plays the role of the membership function of the fuzzy subset E^*_φ.

Notice also that (as already remarked above) the significance of the conditional event $E_\varphi|A_x$ is reinforced by looking on it as "a whole", avoiding a separate consideration of the two propositions E_φ and A_x.

Obviously, a fuzzy subset E^*_φ is a *crisp set* when there is *only* a coherent assessment $\mu_{E_\varphi}(x) = P(E_\varphi|A_x)$ with range $\{0,1\}$.

Then, by property **(B)** above, a fuzzy subset E^*_φ is a crisp set when the property φ is such that, for every $x \in C_X$, either $E_\varphi \wedge A_x = \emptyset$ or $A_x \subseteq E_\varphi$.

Given two fuzzy subsets E^*_φ, E^*_ψ, corresponding to the random quantities X and Y (possibly $X = Y$), assume that, for every $x \in C_X$ and $y \in C_Y$, both the following equalities hold

$$(1) \qquad P(E_\varphi | A_x \wedge A_y) = P(E_\varphi | A_x)\,,\ P(E_\psi | A_x \wedge A_y) = P(E_\psi | A_y)\,,$$

with $A_y = \{Y = y\}$. The *definitions* of the binary operations of *union* and *intersection* and that of *complementation* are as follows:

Given two fuzzy subsets (respectively, of C_X and C_Y) E^*_φ and E^*_ψ, put

$$E^*_\varphi \cup E^*_\psi = \{E_{\varphi\vee\psi}\,,\, \mu_{E_{\varphi\vee\psi}}\}\ ,\ E^*_\varphi \cap E^*_\psi = \{E_{\varphi\wedge\psi}\,,\, \mu_{E_{\varphi\wedge\psi}}\}\ ,\ (E^*_\varphi)' = \{E_{\neg\varphi}\,,\, \mu_{E_{\neg\varphi}}\}\,,$$

where (by a fairly improper notation) $\varphi \vee \psi$, $\varphi \wedge \psi$ denote, respectively, the properties "φ or ψ", "φ and ψ", and $E_{\varphi\vee\psi} = E_\varphi \vee E_\psi$, $E_{\varphi\wedge\psi} = E_\varphi \wedge E_\psi$, while $\mu_{E_{\varphi\vee\psi}}$ and $\mu_{E_{\varphi\wedge\psi}}$ are defined on $C_{XY} = C_X \times C_Y$ by putting

$$\mu_{E_{\varphi\vee\psi}}(x,y) = P(E_\varphi \vee E_\psi | A_x \wedge A_y)\ ,\ \mu_{E_{\varphi\wedge\psi}}(x,y) = P(E_\varphi \wedge E_\psi | A_x \wedge A_y)\,.$$

The conditional event $(E_\varphi \vee E_\psi)|(A_x \wedge A_y)$ is true iff $A_x \wedge A_y$ and $E_\varphi \vee E_\psi$ are both true: and the latter event is true, by definition of disjunction, when at least one of the two events is true, that is when "You claim φ" or when "You claim ψ". On the other hand, $E_{\varphi\vee\psi}$ is true when "You claim φ or ψ", and this requires to put $E_{\varphi\vee\psi} = E_\varphi \vee E_\psi$. Similar considerations apply to the events $E_{\varphi\wedge\psi}$ and $E_\varphi \wedge E_\psi$. Notice also the following relation: $E_{\neg\varphi} \neq (E_\varphi)^c$, where $(E_\varphi)^c$ denotes the *contrary* of the event E_φ (while the equality holds only for a crisp set); for example, the propositions "You *claim not young*" and "You *do not claim young*" are logically independent. Then, while $E_\varphi \vee (E_\varphi)^c = C_X$, we have instead $E_\varphi \vee E_{\neg\varphi} \subseteq C_X$. We could also introduce the *tautological* property $T = \varphi \vee \neg\varphi$ (for any φ), which satisfies (trivially) the relation $E_T \subseteq \Omega$, and the *void* property $V = \varphi \wedge \neg\varphi$ (for any φ), which satisfies the relation $E_V \neq \emptyset$. Therefore, if we consider the union of a fuzzy subset and its complement

$$E^*_\varphi \cup (E^*_\varphi)' = \{E_{\varphi\vee\neg\varphi}\,,\, \mu_{E_{\varphi\vee\neg\varphi}}\}$$

we obtain in general a *fuzzy subset* of (the universe) C_X.

On the other hand, it is easy to check that the complement of a crisp set is also a crisp set: in fact, from $E_\varphi \wedge A_x = \emptyset$ it follows $A_x \subseteq (E_\varphi)^c = E_{\neg\varphi}$, and from $A_x \subseteq E_\varphi$ it follows $(E_\varphi)^c \wedge A_x = \emptyset$, that is $E_{\neg\varphi} \wedge A_x = \emptyset$.

Consider now two fuzzy subsets E^*_φ and E^*_ψ: the rules of conditional probability give, taking into account (1),

$$(2) \qquad P(E_\varphi \vee E_\psi | A_x \wedge A_y) = P(E_\varphi | A_x) + P(E_\psi | A_y) - P(E_\varphi \wedge E_\psi | A_x \wedge A_y)\,.$$

Therefore, to evaluate $P(E_\varphi \vee E_\psi | A_x \wedge A_y)$ it is necessary (and sufficient) to know also the value of the conditional probability $p = P(E_\varphi \wedge E_\psi | A_x \wedge A_y)$, and vice versa.

By resorting to the theorem characterizing coherent conditional probability assessments, it is not difficult to prove that the *only* constraint for the value of p is

$$\max\{P(E_\varphi|A_x) + P(E_\psi|A_y) - 1\,,\, 0\} \leq p \leq \min\{P(E_\varphi|A_x), P(E_\psi|A_y)\}\,.$$

Three possible choices for the value of the conditional probability p give rise to different well-known (see, e.g., [4]) t-norms and t-conorms:

(a) give p the *maximum possible value*, that is $p = \min\{P(E_\varphi|A_x), P(E_\psi|A_y)\}$; then in this case we necessarily obtain, by (2), that

$$P(E_\varphi \vee E_\psi | A_x \wedge A_y) = \max\{P(E_\varphi|A_x), P(E_\psi|A_y)\}.$$

This assignment corresponds to the choice of the so-called T_M and S_M as T-norm and T-conorm.

(b) give p the *minimum value*, that is $\max\{P(E_\varphi|A_x) + P(E_\psi|A_y) - 1\,,\, 0\}$, *i.e.* the Łukasiewicz T-norm. In this case we necessarily obtain, again by (2), that

$$P(E_\varphi \vee E_\psi | A_x \wedge A_y) = \min\{P(E_\varphi|A_x) + P(E_\psi|A_y)\,,\, 1\}$$

i.e. the Łukasiewicz T-conorm.

(c) give p the value $P(E_\varphi|A_x)P(E_\psi|A_y)$, that is assume that E_φ is stochastically independent of E_ψ given $A_x \wedge A_y$. In this case we necessarily obtain

$$P(E_\varphi \vee E_\psi | A_x \wedge A_y) = P(E_\varphi|A_x) + P(E_\psi|A_y) - P(E_\varphi|A_x)P(E_\psi|A_y)\,,$$

i.e. the so-called probabilistic sum S_P and product T_P.

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Fuzzy group in a natural interpretation

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We present a natural interpretation of fuzzy groups in a cumulative Heyting valued model for intuitionistic set thoery. With the interpretation we can deduce the essential part of the definitions of fuzzy groups in the literature.

In the natural interpretation fuzzy sets and fuzzy relations are interpreted as sets and relations in the model. Membership functions are related to fuzzy sets by using the canonical embedding from the class of all crisp sets into the model, which assigns each crisp set to its check set. We can deduce most of the standard equations or inequalities of definitions or properties on the basic concepts of fuzzy sets or fuzzy relations ([3]). Fuzzy mappings are interpreted as mappings in the same model, and we can obtain a characterization of fuzzy mappings with membership functions, which is different from all known definitions. The meaning of the extension princilple by Zadeh is made clear with the interpretation of fuzzy mappings ([5]). We can also consider notions such as operations of fuzzy subsets of different universes, fuzzy relations and mappings between fuzzy subsets ([2]). Moreover fuzzy equivalence relations and corresponding fuzzy partitions can be naturally considered with the interpretation ([4]).

Therefore, as far as fuzzy sets, fuzzy relations, etc. are considered as extensions of crisp sets, relations etc., this interpretation seems to be most natural.

In the following we first recall briefly some properties on the canonical embedding and fuzzy mappings, then we consider fuzzy subgroups of a crisp group and present a characterization of fuzzy subgroup with membership functions, which is almost the same as the defining equations in the literature. Our interpretation has its origin from [1], where the interpretation is applied only to elements of a group.

Let H be a complete Heyting algebra and V^H be the cumulative H-valued model. The Heyting value $\|\varphi\|$ is defined for every sentence φ of V^H. For $u, v \in V^H$, u and v are *similar* iff $\|u = v\| = \mathbf{1}$.

For every crisp set x in V, $\check{x} \in V^H$ is defined recursively by:

$$\mathcal{D}(\check{x}) = \{\check{y}; y \in x\}, \quad E\check{x} = \mathbf{1}, \quad \check{x} : \check{y} \longmapsto \mathbf{1}.$$

We call $\check{x}$ the *check set of* x. The check set of a pair (resp. an ordered pair or a cartesian product) of crisp sets is exactly identical with the pair (resp. the ordered pair or the cartesian product) of the check sets of the crisp sets.

Proposition 1. *Suppose* $\varphi(a_1, \cdots, a_n)$ *is a bounded formula of* V^H *and* $x_1, \cdots, x_n \in V$. *Then*

$$\varphi(x_1, \cdots, x_n) \text{ holds} \quad \text{iff} \quad \|\varphi(\check{x}_1, \cdots, \check{x}_n)\| = \mathbf{1}, \text{ and}$$
$$\neg\varphi(x_1, \cdots, x_n) \text{ holds} \quad \text{iff} \quad \|\varphi(\check{x}_1, \cdots, \check{x}_n)\| = \mathbf{0}.$$

Basic operations such as intersection, union, and complement of sets, composition and inverse of relations (and mappings) are naturally defined in the model.

Every set A in V^H is called an *H-fuzzy set*, and for a crisp set X every subset in V^H of the check set $\check{X}$ is called an *H-fuzzy subset of X*. The mapping $\mu_A : X \longrightarrow H; x \longmapsto \|\check{x} \in A\|$ is called the *membership function of A on X*. There is a natural correspondence between H-fuzzy subsets of X and mappings from X to H, which preserves order and basic set operations.

An H-fuzzy subset R of $X \times Y$ is called an *H-fuzzy relation from X to Y*. An *H-fuzzy mapping from X to Y* is a mapping from $\check{X}$ to $\check{Y}$ in V^H.

Lemma 2. *Let $\varphi : X \longrightarrow Y$ be a crisp mapping between crisp sets. Then the check set $\check{\varphi}$ is an H-fuzzy mapping from X to Y, and $\check{\varphi}(\check{x})$ is similar to the check set of $\varphi(x)$ for every $x \in X$.*

In the model various algebras such as groups, rings etc. can be considered. Here a crisp group means a crisp set which is a group with suitable operations. Then the canonical embedding preserves the group structre as following.

Proposition 3. *For every set G, G is a crisp group iff $\check{G}$ is a group in V^H.*

The check sets of the operations (multiplication, inverse, and unit) on G become the corresponding operations on the check set $\check{G}$ by Proposition 1 and Lemma 2. Since the axioms of group are bounded, Proposition 1 is used in the proof.

For a crisp group G, a set K in V^H is called an *H-fuzzy subgroup of G* if $\|K$ is a subgroup of $\check{G}\| = \mathbf{1}$. Obviously an H-fuzzy subgroup of G is an H-fuzzy subset of G.

Theorem 4. *Let G be a crisp group with the unit e, K be an H-fuzzy subset of $\check{G}$, and μ_K be the membership function of K on G. Then K is an H-fuzzy subgroup of G iff it satisfies the following three conditions:*

$$(1)\ \mu_K(x) \wedge \mu_K(y) \leq \mu_K(xy) \quad (\forall x, y \in X),$$
$$(2)\ \mu_K(x) \leq \mu_K(x^{-1}) \quad (\forall x \in X),$$
$$(3)\ \mu_K(e) = \mathbf{1}.$$

In general, a subgroup K of a group G is *normal* iff $xy \in K$ implies $yx \in K$ for every $x, y \in G$. Then in the theorem K is a normal subgroup of $\check{G}$ in V^H iff it additionally satisfies the following condition:

$$(4)\ \mu_K(xy) = \mu_K(yx) \quad (\forall x, y \in X).$$

Theorem 5. *Let G be a crisp group with the unit e and μ be a crisp mapping from G to H. Suppose μ satisfies the following three conditions:*

$$(1)\ \mu(x) \wedge \mu(y) \leq \mu(xy) \quad (\forall x, y \in X),$$
$$(2)\ \mu(x) \leq \mu(x^{-1}) \quad (\forall x \in X),$$
$$(3)\ \mu(e) = \mathbf{1}.$$

Then there is an H-fuzzy subgroup K of G such that $\mu = \mu_K$, where μ_K is the membership function of K on G.

In the theorem if μ also satisfies the following condition:

$$(4)\ \ \mu(xy) = \mu(yx) \quad (\forall x, y \in X),$$

then the H-fuzzy subgroup K becomes normal.

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On many-valued topologies on L-powersets of many-valued sets

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Let $M = (M, \le, \wedge, \vee, *)$ be a GL-monoid with universal upper and lower bounds 1 and 0 resp. and let $\mapsto : E \times E \longrightarrow E$ be the corresponding residuation. Following U. Höhle [1] by a *(global) M-valued equality on a set X* we call a mapping $E : X \times X \longrightarrow M$ such that:

1. $E(x,x) = 1\ \forall x \in X$;
2. $E(x,y) = E(y,x)\ \forall x,y \in X$;
3. $E(x,y) * E(y,z)) \le E(x,z)\ \forall x,y,z \in X$.

An M-valued equality E is called *separated* if $E(x,y) = 1$ implies $x = y$. In case E satisfies at least the first two of these conditions, it will be called an *M-valued similarity relation*.

A many-valued, or an M-valued set is a pair (X,E) where X is a set and E is an M-valued equality on it. Let **SET**(M) denote the category whose objects are M-valued sets and whose morphisms are mappings $f : (X,E_X) \longrightarrow (Y,E_Y)$ s.t. $E_X(x,x') \le E_Y(f(x), f(x'))$ for all $x,x' \in X$ (cf [1]), and let **SET**(M_s) denote its full subcategory consisting of separated M-valued sets. In some cases we restrict the set of values which E can accept by a complete submonoid $K \subset M$. The corresponding full subcategory of **SET**(M) is denoted by **SET**(M,K).

Further, let L be a complete sublattice of M. An L-subset A of (X,E) is called *extensional* if $A(x) * E(x,x') \le A(x')$ for all $x,x' \in X$. Let L^X (resp. $L^{(X,E)}$) denote the family of all (resp. all extensional) L-subsets of X.

Given L-subsets A,B of X we define *the degree of similarity* as follows:

$$\mathcal{E}(A,B) = I(A,B) \wedge I(B,A) \text{ where } I(A,B) := \bigwedge_x \Big(A(x) \mapsto \bigvee_{x'} (E(x,x') * B(x'))\Big).$$

Proposition 1. *The mapping $\mathcal{E} : L^X \times L^X \longrightarrow M$ thus defined is an M-valued similarity relation on L^X and its restriction to $L^{(X,E)}$ is an M-valued equality.*

Note that if E is crisp and $L = M = K$, then $\mathcal{E}$ is the natural equality relation on L^X considered in [3, p. 157]. On the other hand for any M-valued equality E the induced M-valued equality $\mathcal{E}$ when restricted to $L^{(X,E)}$ also coincides with the natural equality.

Given a morphism $f : (X,E_X) \longrightarrow (Y,E_Y)$ in **SET**(M) let $f^{\longrightarrow} : L^X \longrightarrow L^Y$ be the corresponding (forward) L-powerset operator (see e.g. [5]).

Proposition 2. *If $f : (X,E_X) \longrightarrow (Y,E_Y)$ is a morphism in* **SET**(*M*) *and L is completely distributive, then $\mathcal{E}_X(A,B) \le \mathcal{E}_Y\big(f^{\longrightarrow}(A), f^{\longrightarrow}(B)\big)\ \forall A,B \in L^X$.*

Proposition 3. *If $f : (X,E_X) \longrightarrow (Y,E_Y)$ is a morphism in* $\mathbf{SET}(M,K)$ *and $C,D \in L^{(X,E)}$, then $\mathcal{E}_Y(C,D) \leq \mathcal{E}_X(C \circ f, D \circ f)$.*

Proposition 4. *Let $f : (X,E_X) \longrightarrow (Y,E_Y)$ be a morphism in* $\mathbf{SET}(M,K)$ *and L be completely distributive. Then for any extensional L-sets $A,B \in L^{(X,E)}$ it holds $f^{\rightarrow}(A) * \mathcal{E}(A,B) \leq f^{\rightarrow}(B)$.*

Let L-$\mathbf{SET}(M,K)$ denote the category whose objects are quadruples $(X,E,L^X,\mathcal{E})$ where $(X,E) \in \mathcal{O}\lfloor(\mathbf{SET}(M,K))$, L^X is the L-powerset of X and $\mathcal{E}$ is the similarity relation on L^X induced by E and whose morphisms are pairs $(f, f^{\rightarrow})$ where $f : (X,E_X) \longrightarrow (Y,E_Y)$ is a morphism in $\mathbf{SET}(M,K)$ and $f^{\rightarrow} : (L^X, \mathcal{E}_X) \longrightarrow (L^Y, \mathcal{E}_Y)$ is the corresponding powerset operator. Further, let $\mathbb{E}L$-$\mathbf{SET}(M,K)$ be the full subcategory of L-$\mathbf{SET}(M,K)$ whose objects are of the form $(X,E,L^{(X,E)},\mathcal{E})$.

Theorem 5. *By assigning to an M-valued set (X,E) the quadruple $\Phi_L(X,E) := (X,E,L^X,\mathcal{E})$ and assigning to a morphism $f : (X,E_X) \longrightarrow (Y,E_Y)$ the pair $\Phi_L(f) := (f, f^{\rightarrow})$ we define a functor Φ_L :* $\mathbf{SET}(M,K) \longrightarrow L - \mathbf{SET}(M,K)$. *Besides, if $A,B \in L^{(X,E)}$, then $\Phi_L(f)(A) * \mathcal{E}(A,B) \leq \Phi_L(f)(B)$. The forgetful functor $\Psi_L : L - \mathbf{SET}(M,K) \longrightarrow \mathbf{SET}(M,K)$ defined by $\Psi_L(X,E,L^X) = (X,E)$ on objects and $\Psi_L(f, f^{\rightarrow}) = f$ on morphisms is obviously left inverse of Φ_L.*

Recall that an M-valued topology on the L-powerset L^X or an (L,M)-topology on a set X for short is a mapping $\mathcal{T} : L^X \longrightarrow M$ such that

1. $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1$;
2. $\mathcal{T}(U \wedge V) \geq \mathcal{T}(U) \wedge \mathcal{T}(V)\ \forall U,V \in L^X$;
3. $\mathcal{T}(\bigvee_{i\in I}(U_i) \geq \bigwedge_{i\in I} \mathcal{T}(U_i)\ \forall \{U_i \mid i \in I\} \subset L^X$.

A mapping $f : (X,\mathcal{T}_X) \longrightarrow (Y,\mathcal{T}_Y)$ is called continuous if $\mathcal{T}_X(V \circ f) \geq \mathcal{T}_Y(V)\ \forall V \in L^Y$. Theory of M-valued L-topologies in case when E is crisp (and mostly when $M = L$) was developed in [3], [2], [4], and in other works.

Since in our case the ground categories $L - \mathbf{SET}(M,K)$ and $\mathbb{E}L - \mathbf{SET}(M,K)$ are defined on the basis of many-valued sets (X,E), our pricipal interest concerns extensional topologies, that is topologies such that

$$\mathcal{T}(U) * \mathcal{E}(U,V) \leq \mathcal{T}(V)\ \forall U,V \in L^X \text{ (resp.} U,V \in L^{(X,E)}).$$

Sometimes we restrict the codomain of $\mathcal{T}$ by a complete sublattice N of M.

[Lattices of (L,N)-topologies] Let (X,E) be an object of $\mathbf{SET}(M,K)$ and let $\mathfrak{T}^K_M(L,N,X)$ denote the family of all (L,N)-topologies on it. Let

$$\mathcal{T}_1 \preceq \mathcal{T}_2 \text{ iff } \mathcal{T}_1(A) \leq \mathcal{T}_2(A) \text{ for all } A \in L^X.$$

Then $\mathfrak{T}^K_M(L,N,X)$ endowed with relation $\preceq$ becomes a complete lattice, its upper bound and lower bounds are respectively the discrete and indiscrete (L,N)-topologies $\mathcal{T}_{dis}$ and $\mathcal{T}_{ind}$ Further, since intersection of a family of extensional (L,N)-topologies is extensional, the family $\mathbb{E}\mathfrak{T}^K_M(L,N,X)$ of all extensional (L,N)-topologies on (X,E) is a complete sublattice of $\mathfrak{T}^K_M(L,N,X)$.

[Generation of (L,N)-topologies] Given $(X,E) \in \mathcal{O}\lfloor(\mathbf{SET}(M,K))$ and a mapping $\mathcal{S} : L^X \longrightarrow N$ let $\mathfrak{T}_{\mathcal{S}}$ (resp. $\mathbb{E}\mathfrak{T}_{\mathcal{S}}$) denote the family of all (resp. all extensional) (L,N)-topologies on (X,E) such that

$\mathcal{S}(A) \leq \mathcal{T}(A)$ for all $A \in L^X$. The infimum $\mathcal{T}_\mathcal{S}$ of $\mathfrak{T}_\mathcal{S}$ belongs to $\mathfrak{T}_\mathcal{S}$ and hence is the minimal element of this family; $\mathcal{S}$ is called a subbase of the (L,N)-topology $\mathcal{T}_\mathcal{S}$. Respectively, the infimum $\mathcal{T}_{\mathbb{E}\mathcal{S}}$ of $\mathbb{E}\mathfrak{T}_\mathcal{S}$ is the minimal element of this family; in this case $\mathcal{S}$ is called a subbase of the extensional (L,N)-topology $\mathcal{T}_{\mathbb{E}\mathcal{S}}$.

We define categories L-$\mathbf{TOP}^N(M,K)$, L-$\mathbb{E}\mathbf{TOP}^N(M,K)$, $\mathbb{E}L$-$\mathbf{TOP}^N(M,K)$ and $\mathbb{E}L$-$\mathbb{E}\mathbf{TOP}^N(M,K)$ as follows:

1. Objects of L-$\mathbf{TOP}^N(M,K)$ are pairs $(\mathcal{X},\mathcal{T})$ where $\mathcal{X} = (X,E,L^X,\mathcal{E})$ is an object of L-$\mathbf{SET}(M,K)$ and $\mathcal{T} : L^X \longrightarrow N$ is an (L,N)-topology on it.

2. Objects of $\mathbb{E}L$-$\mathbf{TOP}^N(M,K)$ are pairs $(\mathcal{X},\mathcal{T})$ where $\mathcal{X} = (X,E,L^X,\mathcal{E})$ is an object of $\mathbb{E}L$-$\mathbf{SET}(M,K)$ and $\mathcal{T} : L^{(X,E)} \longrightarrow N$ is an (L,N)-topology on it.

3. Objects of L-$\mathbb{E}\mathbf{TOP}^N(M,K)$ are pairs $(\mathcal{X},\mathcal{T})$ where $\mathcal{X} = (X,E,L^X,\mathcal{E})$ is an object of L-$\mathbf{SET}(M,K)$ and $\mathcal{T} : L^X \longrightarrow N$ is an extensional (L,N)-topology on it.

4. Objects of $\mathbb{E}L$-$\mathbb{E}\mathbf{TOP}^N(M,K)$ are pairs $(\mathcal{X},\mathcal{T})$ where $\mathcal{X} = (X,E,L^{(X,E)},\mathcal{E})$ is an object of $\mathbb{E}L$-$\mathbf{SET}(M,K)$ and $\mathcal{T} : L^X \longrightarrow N$ is an extensional (L,N)-topology on it.

As morphisms between $(\mathcal{X},\mathcal{T}_X)$ and $(\mathcal{Y},\mathcal{T}_Y)$ in all these categories we take those morphisms $(f,f^{\rightarrow}) : \mathcal{X} \longrightarrow \mathcal{Y}$ which are continuous with respect to the corresponding (L,N)-topologies.

Example (1) The category 2-$\mathbf{TOP}^2(2_s, 2)$ is the category of ordinary topological spaces.
(2)The category L-$\mathbf{TOP}^L(L_s,2)$ is the category L-**FTOP** studied in [3];
(3) The category $\mathbb{E}L$-$\mathbf{TOP}^L(M_s, 2)$ is isomorphic to the category $\mathbb{E}L$-$\mathbb{E}\mathbf{TOP}^L(M_s,2)$ and in case $M = L$ it is the category $\mathbb{E}L$-**FTOP** introduced in [3].
(4) The category L-$\mathbf{TOP}^2(M_s,2)$ is isomorphic to the category L-**TOP** of Chang-Goguen L-topological spaces.

Proposition 6. *Let $(\mathcal{X}_1,\mathcal{T}_1) := (X_1,E_1,L^{X_1},\mathcal{T}_1)$ and $(\mathcal{X}_2,\mathcal{T}_2) := (X_2,E_2,L^{X_2},\mathcal{T}_2)$ be objects of L-$\mathbf{TOP}^N(M,K)$ and $(f,f^{\rightarrow}) : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ be a morphism in L-$\mathbf{SET}(M,K)$. Further, let $\mathcal{S} : L^{X_2} \longrightarrow N$ be a subbase for $\mathcal{T}_2$. Then the following are equivalent:*

1. *$(f,f^{\rightarrow}) : (\mathcal{X}_1,\mathcal{T}_1) \longrightarrow (\mathcal{X}_2,\mathcal{T}_2)$ is continuous;*

2. *$\mathcal{S}(B) \leq \mathcal{T}_1(B \circ f)$ for every $B \in L^{X_2}$.*

Proposition 7. *Let $(f,f^{\rightarrow}) : \mathcal{X}_1 \longrightarrow \mathcal{X}_2$ be a morphism in L-$\mathbf{SET}(M,K)$ and let $\mathcal{T}_1$ be an extensional (L,N)-topology on $\mathcal{X}_1$. Then the mapping $\mathcal{R} : L^{(X_2,E_2)} \longrightarrow N$ defined by $\mathcal{R}(B) := \mathcal{T}_1(B \circ f)$ for all $B \in L^{(X_2,E_2)}$ is an extensional (L,N)-topology on $\mathcal{X}_2,\mathcal{E}_2,L^{(X_2,E_2)},\mathcal{E}_2)$.*

Theorem 8. *(a) Category L-$\mathbf{TOP}^N(M,K)$ is topological over the ground category L-$\mathbf{SET}(M,K)$ with respect to the forgetful functor $\mathfrak{F} : L$-$\mathbf{TOP}^N(M,K) \longrightarrow L$-$\mathbf{SET}(M,K)$.*

(b) Category $\mathbb{E}L$-$\mathbf{TOP}^N(M,K)$ is topological over the ground category $\mathbb{E}L$-$\mathbf{SET}(M,K)$ with respect to the forgetful functor $\mathfrak{F} : \mathbb{E}L$-$\mathbf{TOP}^N(M,K) \longrightarrow \mathbb{E}L$-$\mathbf{SET}(M,K)$.

Theorem 9. *(a) L-$\mathbb{E}\mathbf{TOP}^N(M,K)$ is a coreflective subcategory of the category L-$\mathbf{TOP}(M,K)$.*

(b) $\mathbb{E}L$-$\mathbb{E}\mathbf{TOP}^N(M,K)$ is a coreflective subcategory of the category $\mathbb{E}L$-$\mathbf{TOP}(M,K)$

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A categorical fabric for fuzzy predicate logic

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Much of the existing work on categorical foundations for Fuzzy sets deals with a single category of fuzzy sets with values in a particular lattice with sufficient a dditional properties to capture the connectives used in fuzzy propositional logic. Goguen's early characterization of fuzzy set categories [1], my work relating fuzzy set categories to topoi and quasitopoi [10, 11, 12, 13] (particularly using the Higgs [2] approach to sheaves on a complete Heyting algebra and the fuzzy powerset of Pultr [9] as starting point2), Höhle's work on structures based on MV algebras, and further consideration of monoidal structures and weak classification of subobjects of various kinds [4, 5, 3] all fix the lattice in which the fuzzy sets are to have their truth values. The categories we have looked at all allow for a certain amount of internalization of the higher order logic of fuzzy sets with values in a particular complete lattice ordered semigroup– including both quantification and powerobject formation paralleling, though somewhat more difficult because of non-uniqueness concerns– paralleling the theory in topoi.

At the Linz seminar in 2000 I presented some preliminary work on properties of the lattice change functors between categories of fuzzy sets using the Goguen definition and the predicate logic structure given by unbalanced subobjects and a second monoidal structure arising from a t-norm as in [12]. Through participation in the Linz seminar I have become aware of Rodabaugh's work in fuzzy topology in which a much larger category is considered in fuzzy topologies with values in many different lattices are all objects in a single category and constructions are allowed to change lattice to solve topological problems. That suggested to me that it might be valuable to look at a single kind of structure incorporating categorical viewpoints on the propositional and predicate logic of fuzzy sets over many different lattices. Bart Jacobs's work on the use of fibrations as a framework for categorical logic [6] suggested to me that looking at a double fibration (over both **Sets** and **Closg** might combine the structures in categories of fuzzy sets into one rich structure. This paper takes a different approach, making a structure out of several closely linked categories rather than putting all of the objects into a single category.

This paper presents an approach to predicate logic in a fuzzy setting using a categorical fabric. This structure has two dimensions woven together: one dimension connects the predicates of different types (where types are taken from the "warp" category, often **Sets** for us) but with a fixed propositional logic given by a complete lattice; the other dimension connects predicates of a single type with variation of the lattice for propositional logic, making a category of lattices of possible truth values into the "weft" of our fabric.

If we restrict our attention to fuzzy predicate logic over **Sets** with values in a particular lattice L for each set S we get a category $\mathcal{P}_L(\mathcal{S})$ (typically a partial order) of predicates about S. These categories of predicates are connected to each other using trios of functors: for any function $f : S \longrightarrow T$ there are functors $f^* : \mathcal{P}_L(T) \longrightarrow \mathcal{P}_L(S)$, $\exists_f : \mathcal{P}_L(S) \longrightarrow \mathcal{P}_L(T)$ and $\forall_f : \mathcal{P}_L(S) \longrightarrow \mathcal{P}_L(T)$ with $\exists_f \dashv f^* \dashv \forall_f$. Furthermore, a pullback square in **Sets**

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ h\downarrow & \text{pull} & \downarrow g \\ U & \xrightarrow{k} & V \end{array}$$

gives rise to the Beck conditions

$$\exists_h f^* = k^* \exists_g \text{ and } \forall_h f^* = k^* \forall_g$$

as in the internal logic of topoi. This representation of predicate logic has its roots in the early work of Lawvere in [7, 8].

The truth functional nature of fuzzy sets shows up in our ability to recapture the lattice of truth values L from the structures on the terminal $\top$ (a one element set) and then use the fact that the terminal is a generator in **Sets** to recover $\mathcal{P}_L(S)$ as a colimit of the diagram consisting of the functors $\ulcorner a \urcorner^* : \mathcal{P}_L(S) \longrightarrow \mathcal{P}_L(\top)$ for all of the functions $\ulcorner a \urcorner : \top \longrightarrow S$.

If we restrict our attention to a particular set S and look at how variation in the propositional logic affects predicates we again get from a suitable function of lattices $\lambda : L \longrightarrow L'$ a trio of functors $\lambda^\uparrow, \lambda^\circ, \lambda^\downarrow$. In the cases of fuzzy sets with values in the lattices these have the following effects:

$$\begin{array}{lll} \lambda^\circ : \mathcal{P}_L(S) \longrightarrow \mathcal{P}_{L'}(S) & \text{takes } \alpha : S \longrightarrow L \text{ to} & \lambda \circ \alpha : S \longrightarrow L' \\ \lambda^\uparrow : \mathcal{P}_{L'}(S) \longrightarrow \mathcal{P}_L(S) & \text{takes } \beta : S \longrightarrow L' \text{ to} & s \mapsto \bigvee \{l \in L | \lambda(l) \leq \beta(s)\} \\ \lambda^\downarrow : \mathcal{P}_{L'}(S) \longrightarrow \mathcal{P}_L(S) & \text{takes } \beta : S \longrightarrow L' \text{ to} & s \mapsto \bigwedge \{l \in L | \lambda(l) \geq \beta(s)\} \end{array}$$

With these definitions $\lambda^\uparrow$ is the smallest left inverse for λ° and $\lambda^\downarrow$ is the largest left inverse. In particular, if λ preserves $\bigvee$ then $\lambda^\uparrow \dashv \lambda^\circ$; if λ preserves $\bigwedge$ then $\lambda^\circ \dashv \lambda^\downarrow$.

If we think of all of the categories $\mathcal{P}_L(S)$ as objects in a category where the arrows are functors between them, then the assignment of $\mathcal{P}_L(S)$ to a set S with functors f^* assigned to functions $f : S \longrightarrow T$ gives a contravariant functor for each lattice L. All of the functors $\lambda^\downarrow$, λ°, and $\lambda^\uparrow$ then give natural transformations. Naturality of any of these with the covariant functors using $\exists_f$ or $\forall_f$ will require that λ have further preservation properties or that the relevant lattices be completely distributive.

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Residuum-based approximate reasoning with distance-based uninorms

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In fuzzy control system the system state is described by a fuzzy rule base system, and the relationship between fuzzy rule base system, system output and system input is modeled by compositional rule of inference. The first successful practical applications of fuzzy sets were realized by means of the Mamdani inference [12], but the Mamdani's approach is not fully coherent with the paradigm of approximate reasoning [1, 11]. In the fuzzy rule based control theory and usually in the approximate reasoning the covering over of fuzzy rule base input and rule premise of a rule determine the importance of that fuzzy rule and the rule output, too. The practical realization of that notion usually depends on the application. A very thorough overview of mathematical background of that principle can be found in [4, 7]. The Mamdani type controller is based on Generalized Modus Ponens (GMP) inference rule, and the rule output is given with a fuzzy set, which is derived from rule consequence, as a cut of them. This cut is the generalized degree of firing level of the rule, considering actual rule base input, and usually it is the supremum of the minimum of the rule premise and rule input (calculating with their membership functions, of course). In fact the uninorms [5] offer new possibilities in fuzzy approximate reasoning, because the low level of covering over of rule premise and rule input has measurable influence on rule output as well. In some applications the meaning of that novel approach, has practical importance. The modified Mamdani's approach , with similarity measures between rule premises and rule input, does not rely on the compositional rule inference any more, but still satisfies the basic conditions supposed for the approximate reasoning for a fuzzy rule base system [14]. The using of distance based operators in fuzzy logic control theory (FLC) was described in [13]. From mathematical point of view, and having results from [3], we can introduce residuum-based inference mechanism ([9]) using distance-based uninorms.

The distance-based operators can be expressed by means of the min and max operators. The modification of the distance based operators from [10] is related to the boundary condition for the neutral element e. The maximum distance minimum operator with respect to $e \in]0,1]$ is defined by

$$\max_e^{\min} = \begin{cases} \max(x,y) & \text{if } y > 2e - x, \\ \min(x,y) & \text{if } y < 2e - x, \\ \min(x,y) & \text{if } y = 2e - x. \end{cases}$$

The minimum distance minimum operator with respect to $e \in [0,1[$ is defined by

$$\min_e^{\min} = \begin{cases} \min(x,y) & \text{if } y > 2e - x, \\ \max(x,y) & \text{if } y < 2e - x, \\ \min(x,y) & \text{if } y = 2e - x. \end{cases}$$

The maximum distance maximum operator with respect to $e \in]0,1]$ is defined by

$$\max_e^{\max} = \begin{cases} \max(x,y) & \text{if } y > 2e - x, \\ \min(x,y) & \text{if } y < 2e - x, \\ \max(x,y) & \text{if } y = 2e - x. \end{cases}$$

The minimum distance maximum operator with respect to $e \in [0,1[$ is defined by

$$\min_e^{\min} = \begin{cases} \min(x,y) & \text{if } y > 2e - x, \\ \max(x,y) & \text{if } y < 2e - x, \\ \max(x,y) & \text{if } y = 2e - x. \end{cases}$$

The distance-based operators have the following properties

- $\max_e^{\min}$ and $\max_e^{\max}$ are uninorms,
- the dual operator of the uninorm $\max_e^{\min}$ is $\max_{1-e}^{\max}$, and
- the dual operator of the uninorm $\max_e^{\max}$ is $\max_{1-e}^{\min}$.

In [3] and [2] there were studied two important classes of uninorms: the class of left-continuous and the class of right-continuos ones. We can find there also the properties of the conjunctive left-continuous idempotent uninorm with neutral element $e \in]0,1]$, and of the disjunctive right-continuous idempotent uninorm with neutral element $e \in [0,1[$ with a super-involutive decreasing unary operator g. Based on [3] and [2], we conclude: Operator $\max_{0.5}^{\min}$ is a conjunctive left-continuous idempotent uninorm with neutral element $e \in]0,1]$ with the super-involutive decreasing unary operator $g(x) = 2e - x = 1 - x$. Operator $\min_{0.5}^{\max}$ is a disjunctive right-continuous idempotent uninorm with neutral element $e \in]0,1]$ with the sub-involutive decreasing unary operator $g(x) = 2e - x = 1 - x$.

The paper [3] contain general theoretical results related the residual implicators of uninorms, based on residual implicators of t-norms and t-conorms. Residual operator R_U, considering a uninorm U can be represented in the following form for all $(x,y) \in [0,1]^2$

$$R_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}.$$

Uninorms with neutral elements $e = 0$ and $e = 1$ are t-norms and t-conorms, respectively, and related residual operators are investigated in [3, 5, 6, 8, 9]. If we consider a uninorm U with neutral element $e \in]0,1[$, then the binary operator R_U is an implicator if and only if $(\forall z \in]e,1[)(U(0,z) = 0)$. The residual implicator R_U of uninorm U is denoted by Imp_U. According to Theorem 8. in [3] we introduce implicator of the distance based operator $\max_{0.5}^{\min}$. Operator $\max_{0.5}^{\min}$ is a conjunctive left-continuous idempotent uninorm with the unary operator $g(x) = 1 - x$, and its residual implicator $Imp_{\max_{0.5}^{\min}}$ is given by

$$Imp_{\max_{0.5}^{\min}} = \begin{cases} \max(1-x,y) & \text{if } x \leq y, \\ \min(1-x,y) & \text{elsewhere.} \end{cases} \quad (1)$$

In the theory of approximate reasoning introduced by Zadeh in 1979, the knowledge of system behavior and system control can be stated in the form of if-then rules. In Mamdani-based sources it was suggested to represent an

if x is A then y is B

rule simply as a connection (for example as a t-norm, $T(A,B)$, specially min) between the so called rule premise: x is A and rule consequence: y is B. Let x be from universe X, y from universe Y, and let x and y be linguistic variables. Normal fuzzy set A on $X \subset \mathbb{R}$ finite universe is characterized by its membership function $\mu_A : X \longrightarrow [0,1]$, and normal fuzzy set B on universe $Y \subset \mathbb{R}$ is characterized by its membership function $\mu_B : Y \longrightarrow [0,1]$. The Generalized Modus Ponens reflects the real influences of the implication or connection choice on the inference mechanisms in fuzzy systems. Usually the general rule consequence $B_i'(y)$ for i^{th} rule from a rule system, for rule base input $A'(x)$ is obtained by

$$B_i^{'}(y) = \sup_{x\in X}(T(A'(x), Imp(A_i(x), B_i(y))). \tag{2}$$

The FLC rule base output is constructed as a crisp value calculated with a defuzzification model, from rule base output

$$B_{out}^{'}(y) = S(B_n^{'}, S(B_{n-1}^{'}, S(\ldots, S(B_1^{'}, B_2^{'}, B_1^{'})))).$$

Rule base output is an aggregation of all rule consequences $B_i^{'}(y)$ from the rule base ($i = 1,2,\ldots,n$). As aggregation operator, t-conorms are usually used.

Although the minimum plays an exceptional role in fuzzy control theory, there are situations requiring new models. In system control one would intuitively expect: to make the powerful coincidence between fuzzy sets stronger, and the weak coincidence even weaker. The distance-based operators group satisfy these properties. The papers [13, 14] contain the basics of approximate reasoning with distance-based operators using Degree of Coincidence (Doc) in the inference mechanism.

Let we consider residuum-based approximate reasoning and inference mechanism for special class of distance based operators. Hence, and because of the results from (2), we can consider the general rule consequence for i^{th} rule from a rule system as

$$B_i^{'}(y) = \sup_{x\in X}(\max_{0.5}^{\min}(A'(x), Imp_{\max_{0.5}^{\min}}(A_i(x), B_i(y))),$$

or using (1)

$$B_i^{'}(y) = \sup_{x\in X}\begin{cases} \max_{0.5}^{\min}(A'(x), \max(1-A_i(x), B_i(y)) & \text{if } A_i(x) \leq B_i(y), \\ \max_{0.5}^{\min}(A'(x), \min(1-A_i(x), B_i(y)) & \text{elsewhere.} \end{cases} \tag{3}$$

The crisp rule base output is constructed also with a defuzzyfication model, from rule base output $B_{out}^{'}$ by (3). As aggregation operator for rule consequences in this case, dual operator $\max_{0.5}^{\max}$ of the $\max_{0.5}^{\min}$ can be used.

Taken into account Proposition 13 from [3], it can be conclude, that conjunctive left-continuous idempotent uninorm $\max_{0.5}^{\min}$ and its implicator $Imp_{\max_{0.5}^{\min}}$ satisfy the inequality

$$\max_{0.5}^{\min}(A'(x), Imp_{\max_{0.5}^{\min}}(A_i(x), B_i(y)) \leq B_i(y)$$

if $A'(x) = A_i(x)$ for all $x \in X$.

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Fuzzy deductive and inductive systems with similarity based unification

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We will present mathematical results on deductive and inductive aspects of different rule based systems of fuzzy logic motivated by computer science applications and related to fuzzy logic programming (FLP), fuzzy databases, fuzzy inductive logic programming (FILP) and fuzzy similarity based unification. We refer on results obtained with several coauthors. Our results are mainly generalizations of older results of many other researchers in the direction of extending them to a wider class of operators. In the talk we will try to put them into a suitable historical perspective (which we cannot list completely here in this extended abstract).

We split our presentation to results on rule based systems and to results on fuzzy similarity based unification.

In the classical logic the implication $H \leftarrow B$ is equivalent to the clause $H \vee \neg B$. This is no more true in fuzzy logic in general. So, it is natural to study two types of rule systems – those where rules are implications and those where rules are clauses ([2]).

Implication rule systems without negation. We will study an FLP system based on the fuzzy modus ponens for weighted formulas

$$\frac{(B,b),(H \leftarrow B,r)}{(H,C_I(b,r))}$$

where I is the truth function of the implication $\leftarrow$, and C_I is the residual conjunctor (not necessary a t-norm). The FLP computation can be based on the backward use of this rule, namely, starting with query $?-H$, having the rule $(H \leftarrow B,r)$ we proceed with query $?-C_I(B,r)$, and having the fact (B,b) we finish with the computed answer $C_I(b,r)$. The notion of a correct answer is based on satisfaction of truth functional fuzzy logic in narrow sense ([4]). To model the aggregation of partial results, bodies of our rules have the form $@(B_1,\dots,...B_n)$.

We prove ([10]) a Pavelka-like completeness results for implication rule based FLP systems without negation under condition that all C_I's and aggregations @ in body are left continuous.

We show ([6]) that FLP are equivalent to a variant of generalized annotated programs GAP under following transformations:

FLP $(H \leftarrow @(B_1,\dots,B_n),r)$ transform to GAP $H : C_I(@(x_1,\dots,x_n),r) \leftarrow B_1 : x_1 \& \dots \& B_n : x_n$

where $C_I(@(x_1,\dots,x_n),r)$ is here considered as an head annotation term and

GAP $H : f(x_1,\dots,x_n) \leftarrow B_1 : x_1 \& \dots \& B_n : x_n$ transform to FLP $(H \leftarrow @_f(B_1,\dots,B_n),1)$

where $@_f$ is a body aggregation induced by the head annotation term f.

We study a fuzzy relational algebra based on this FLP and discuss join evaluation strategies for finding best, top-k, threshold and ε-best answer based on an upper residuation operator.

A model of fuzzy inductive logic programming ([11]) is based on a multiple use of classical ILP system learning the annotation term of the transformed GAP program for a graded classification example. A comparison with classification trees on a small example will be given. A problem of learning with qualitative condition will be formulated.

Our acquaintance is that FLP systems are more suitable for deductive (database) applications and GAP systems are more suitable for inductive tasks. Equivalence between FLP and GAP yields a system with unified deductive and inductive part. Informally, we can say, that what is in FLP hidden in the aggregation operator of the body, this is in the GAP represented by the annotation term of the head of the rule.

We will discuss connections of FLP, and more directly of GAP, to Bayesian networks, where the probability production operator corresponds to the head annotation term.

Fuzzy resolution for clausal rule systems. We study operators $f_\vee$ for which the fuzzy resolution rule with weighted clauses

$$\frac{(\gamma \vee \alpha, x), (\beta \vee \neg\alpha, y)}{(\gamma \vee \beta, f_\vee(x,y))}$$

is sound and compare it to results in the deMorgan logic with involutive negation ([1,5]), approximate reasoning, possibilistic logic and different forms of residuation and fuzzy operators. Having D the truth function of the disjunction and R_D the corresponding residual, for the operator

$$f_\vee(x,y) = \inf_{a \in [0,1]} (D(R_D(a,x), R_D(1-a,y)))$$

we prove the soundness result ([9]).

Similarity based unification. Based on the presented model of FLP ([7]), a similarity based unification approach is constructed by adding axioms of fuzzy equality to a fuzzy logic program. Connections to several max-min similarity based systems ([3], [8]]) are discussed. Several models of generating fuzzy similarities are presented (e.g. from the geometry of the sample space, from fuzzy sets of linguistic expressions, ...).

From a point of view of flexible querying systems, we consider the object-attribute model. We distinguish, whether the data type of the attribute value is an element of the attribute domain from the case when the data type is a subset of the domain. In the case when a fuzzy set acts as an attribute value of data type being an element of the domain, we discuss several possibilities of defining the degree of unification (e.g. degree of fuzzy equality of fuzzy sets, measure theoretic and metric space approach, generalization of the probability of the equality of two random variables, ...).

We will illustrate our approach on several small illustrative examples. Several problems will be formulated.

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