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Abstracts

Non-Classical Measures and Integrals

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Abstracts

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Editors

LINZ 2013

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NON-CLASSICAL MEASURES AND INTEGRALS

Dedicated to the memory of Lawrence Neff Stout

ABSTRACTS

Radko Mesiar, Endre Pap, Erich Peter Klement
Editors

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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2013 will be the 34th seminar carrying on this tradition and is devoted to the theme “Non-Classical Measures and Integrals”. The goal of the seminar is to present and to discuss recent advances in non-classical measure theory and corresponding integrals and their various applications in pure and applied mathematics.

A large number of highly interesting contributions were submitted for possible presentation at LINZ 2013. In order to maintain the traditional spirit of the Linz Seminars — no parallel sessions and enough room for discussions — we selected those thirty-three submissions which, in our opinion, fitted best to the focus of this seminar. This volume contains the abstracts of this impressive selection. These regular contributions are complemented by six invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

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Contents

On the interpretation of ambiguity

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In his axiomatic treatment of the Subjective Expected Utility (SEU) theory for behavior in the face of uncertainty, Savage (1954) takes as primitive notions a state space Ω, a set of outcomes X and preferences $\succsim_\Omega$ over Savage acts, defined as functions mapping states onto outcomes and whose set is denoted $\mathcal{A}_\Omega$. Savage designs a set of axioms on $\succsim_\Omega$ under which there exists a unique probability measure $\mathbb{P}$ over Ω and an (almost) unique utility function $u : X \to \mathbb{R}$ such that a SEU representation is ensured:

$$\forall F, G \in \mathcal{A}_\Omega, F \succsim_\Omega G \Longleftrightarrow \mathbb{E}_\mathbb{P} u(F) \geq \mathbb{E}_\mathbb{P} u(G)$$

The Savage axioms include the Sure Thing Principle (STP), which proves to be equivalent to the fundamental property of dynamic consistency (Hammond, 1988; Ghirardato, 2002) and requires the existence of a family $(\succsim_E)_{E\subseteq\Omega}$ of preferences over Savage acts such that:

(i) $\forall E \subseteq \Omega$, $\forall f, g \in \mathcal{A}_\Omega$, if $f = g$ over E, then $f \sim_E g$
(ii) $\forall E \subseteq \Omega$, $\forall f, g \in \mathcal{A}_\Omega$, if $f \succsim_E g$ and $f \succsim_{E^c} g$, then $f \succsim_\Omega g$

At an intuitive level, dynamic consistency requires consistency between strategies that are optimal *ex ante* and strategies actually implemented *ex post*. It is therefore usually seen as a normative requirement for behavior under uncertainty. But it also serves a behavioral justification for the optimality of backward induction solutions and for the Bayesian updating rule for probability measures (Al-Najjar and Weinstein, 2009).

The Ellsberg paradox challenges the Savage SEU theory and is usually understood as a violation of STP (Ellsberg, 1961). The idea is that people do not always know the 'true' probabilities of relevant events and are reluctant to assign any precise value to them. As a result, they do not behave as if they knew these values. When this happens, they are said to perceive *ambiguity* and their behavior is inconsistent with STP. For instance, in the Schmeidler (1989) model of ambiguity, an individual is characterized by a non-additive probability measure, or capacity, and compares Savage acts through the criterion of Choquet Expected utility (CEU). In the Gilboa and Schmeidler (1989) model of ambiguity, an individual is rather characterized by a set of additive probability measures and uses the criterion of Minimal Expected Utility with respect to that set to compare Savage acts.

Since these models of ambiguity typically weaken STP, they typically imply dynamic inconsistencies. Concretely, it is not clear whether an individual who perceives ambiguity ends up implementing *ex post* one of his optimal strategies, backward induction can not be applied and there is no consensus on how to update such non-additive beliefs, conditional upon receiving new information (Al-Najjar and Weinstein, 2009).

In this paper, we develop a new approach to ambiguity, meant to make it possible to rationalize the Ellsberg pattern of behavior, while maintaining a certain form of dynamic consistency. Our primitive notions are, as in the Savage framework, a state space Ω, a set of outcomes X and preferences $\succsim_\Omega$ over Savage acts. The framework is enriched by two more primitive objects: a set of observations O and a mapping Φ transforming each feasible act (defined as a function mapping observations onto outcomes, their set being denoted $\mathcal{A}_O$) into a Savage act. In addition, the Savage preferences $\succsim_\Omega$ and the mapping Φ assumed to conform to certain appropriate axioms, which include STP for $\succsim_\Omega$.

This framework makes it first possible to derive *observable preferences* $\succsim_O$ over $\mathcal{A}_O$ in the following way:

$$\forall f, g \in \mathcal{A}_O, f \succsim_O g \Longleftrightarrow \Phi(f) \succsim_\Omega \Phi(g)$$

Depending on the properties of the mapping Φ, such observable preferences $\succsim_O$ may rationalize the Ellsberg pattern of behavior, even when Savage preferences $\succsim_\Omega$ conform to STP. For instance, it is simple to give examples of mappings Φ that imply observable preferences following Schmeidler's or Gilboa and Schmeidler's models of ambiguity. The fact that STP is no longer incompatible with the Ellsberg choices in this framework is precisely what makes it always possible to derive a certain form of dynamic consistency for observable preferences.

More precisely, it is always possible to construct a family $(\succsim_\omega)_{\omega\in\Omega}$ of preferences over $\mathcal{A}_O$, which in turn leads to two epistemic operators $K : 2^O \to 2^\Omega$ and $B : 2^O \to 2^\Omega$. For each $A \subseteq O$, $K(A)$ can be seen as the set of states in which the individual thinks that A is necessarily realized, while $B(A)$ can be seen as the set of states in which the individual thinks that A is not impossible. Then, it is possible to define an algebra $\mathcal{M} = \{A \subseteq O, K(A) = B(A)\}$ of subsets of O, whose elements are called measurable. A subset $A \subseteq O$ is measurable, if, in any state, it is either necessarily realized or impossible. At last, it is possible to construct two families of *ex post* preferences $(\succsim_A^K)_{A\subseteq O}$ and $(\succsim_A^B)_{A\subseteq O}$ such that the conditions hereafter hold:

(i) $\forall A \subseteq O, \forall f, g \in \mathcal{A}_O$, if $A \in \mathcal{M}$, then $f \succsim_A^K g \Longleftrightarrow f \succsim_A^B g$
(ii) $\forall A \subseteq O, \forall f, g \in \mathcal{A}_O, f \succsim_A^K g$ and $f \succsim_{A^c}^B g \Longrightarrow f \succsim_O g$
(iii) $\forall A \subseteq O, \forall f, g \in \mathcal{A}_O, \forall \omega \in K(A), f \sim_\omega g \Longrightarrow f \sim_A^K g$
(iv) $\forall A \subseteq O, \forall f, g \in \mathcal{A}_O, \forall \omega \in B(A), f \sim_\omega g \Longrightarrow f \sim_A^B g$

The appropriate interpretation of $\succsim_A^K$ (resp. $\succsim_A^B$) is that of the preferences that represent *ex post* behavior when A is known to be necessarily realized (resp. A is known not to be impossible). The usual form of dynamic consistency presented above can be obtained as a particular case, when all subsets $A \subseteq O$ are measurable. The main difference with this more usual form lies in that *ex post* behavior now depends on information and constraints, but also on one of the two modalities, Knowledge or Belief, through which information is processed. The potential of this new form of dynamic consistency is illustrated on the issue of updating capacities as new information is acquired.

At last, we also discuss what seems to be the main limitation of our approach: the fact that observable preferences $\succsim_O$ do not make it possible to identify uniquely Savage preferences. In contrast, in a Savage framework and under the assumption that states of nature are nothing but observations, observable preferences and Savage preferences simply coincide and the former fully determine the latter. This problem might be overcome, we suggest, by accepting the idea that Ω, X and $\succsim_\Omega$ stand for the state space, the set of outcomes and the preferences of some observer facing the uncertainty generated by the individual's behavior.

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Overlap indices and fuzzy measures

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In many applications it is important to recognize and analyze the relations between different pieces of information. In classification problems, for instance, when instances need to be classified in different classes, it may happen that this classes are not disjoint [2]. Or in image processing, given an image, it may be necessary to distinguish between different objects in the same image, even when these objects are not completely different and may have non-empty intersections [1]. So in this kind problems, two different steps can be considered in order to get a final solution to a given problem (i.e., a valid classification or a useful segmentation, for instance).

1. Identify correctly all the information that is provided, including the possible relations between the different parts.
2. Put together all this information so that a feasible solution of the proposed problem is obtained.

For the second step, one of the most commonly used tools is that of aggregation functions [3]. Aggregation functions provide a meaningful, mathematically rich way of putting together pieces of (numerical) information in an analytical way. A particular instance of these aggregation functions are those obtained by means of the so called fuzzy measures, including (but not limited to) ordered weighted aggregation operators or, more generally, Choquet integrals.

Regarding the first step, many different approaches can be found in the literature. Of particular interest for us is the use of the so-called overlap functions and overlap indices [1], which allow to measure in an analytical way up to what extent different pieces of information share common features.

In this work we propose a link between steps one and two. More specifically, our objective is to build fuzzy measures from overlap functions, so that the way the identification is carried out for step 1 is taken into account in the aggregation for a final result in step 2.

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On comonotonically modular functions

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1 Preliminaries

The discrete Choquet and the discrete Sugeno integrals are well-known aggregation functions that have been widely investigated due to their many applications in decision making (see the edited book [9]). A convenient way to introduce the discrete Choquet integral is via the concept of Lovász extension. An n-place Lovász extension is a continuous function $L\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each of the $n!$ subdomains

$$\mathbb{R}^n_\sigma = \{\mathbf{x} = (x_1,\ldots,x_n) \in \mathbb{R}^n : x_{\sigma(1)} \leqslant \cdots \leqslant x_{\sigma(n)}\}, \qquad \sigma \in S_n,$$

is an affine function, where S_n denotes the set of permutations on $[n] = \{1,\ldots,n\}$. Equivalently, Lovász extensions can be defined via the notion of pseudo-Boolean function, i.e., a mapping $\psi\colon \mathbb{B}^n \to \mathbb{R}$; its corresponding set function $v_\psi\colon 2^{[n]} \to \mathbb{R}$ is defined by $v_\psi(A) = \psi(\mathbf{1}_A)$ for every $A \subseteq [n]$, where $\mathbf{1}_A$ denotes the n-tuple whose i-th component is 1 if $i \in A$, and is 0 otherwise. The *Lovász extension* of a pseudo-Boolean function $\psi\colon \mathbb{B}^n \to \mathbb{R}$ is the function $L_\psi\colon \mathbb{R}^n \to \mathbb{R}$ whose restriction to each subdomain $\mathbb{R}^n_\sigma$ ($\sigma \in S_n$) is the unique affine function which agrees with ψ at the $n+1$ vertices of the n-simplex $[0,1]^n \cap \mathbb{R}^n_\sigma$ (see [11, 12]). We then have $L_\psi|_{\mathbb{B}^n} = \psi$.

It can be shown (see [8, §5.4.2]) that the Lovász extension of a pseudo-Boolean function $\psi\colon \mathbb{B}^n \to \mathbb{R}$ is the continuous function

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + \sum_{i\in[n]} x_{\sigma(i)}\left(L_\psi(\mathbf{1}_{A^\uparrow_\sigma(i)}) - L_\psi(\mathbf{1}_{A^\uparrow_\sigma(i+1)})\right), \qquad \mathbf{x} \in \mathbb{R}^n_\sigma, \tag{1}$$

where $A^\uparrow_\sigma(i) = \{\sigma(i),\ldots,\sigma(n)\}$, with the convention that $A^\uparrow_\sigma(n+1) = \varnothing$. Indeed, for any $k \in [n+1]$, both sides of (1) agree at $\mathbf{x} = \mathbf{1}_{A^\uparrow_\sigma(k)}$. Let ψ^d denote the *dual* of ψ, that is the function $\psi^d\colon \mathbb{B}^n \to \mathbb{R}$ defined by $\psi^d(\mathbf{x}) = \psi(\mathbf{0}) + \psi(\mathbf{1}) - \psi(\mathbf{1}-\mathbf{x})$. Then

$$L_\psi(\mathbf{x}) = \psi(\mathbf{0}) + L_\psi(\mathbf{x}^+) - L_{\psi^d}(\mathbf{x}^-), \tag{2}$$

where $\mathbf{x}^+ = \mathbf{x} \vee \mathbf{0}$ and $\mathbf{x}^- = (-\mathbf{x})^+$. An n-place *Choquet integral* is a nondecreasing Lovász extension $L_\psi\colon \mathbb{R}^n \to \mathbb{R}$ such that $L_\psi(\mathbf{0}) = 0$. It is easy to see that a Lovász extension $L\colon \mathbb{R}^n \to \mathbb{R}$ is an n-place Choquet integral if and only if its underlying pseudo-Boolean function $\psi = L|_{\mathbb{B}^n}$ is nondecreasing and vanishes at the origin (see [8, §5.4]).

Similarly, a convenient way to introduce the discrete Sugeno integral is via the concept of (lattice) polynomial functions, i.e., functions which can be expressed as combinations of variables and constants using the lattice operations $\wedge$ and $\vee$. It can be shown that polynomial functions are exactly those representable by expressions of the form

$$\bigvee_{A\subseteq[n]} c_A \wedge \bigwedge_{i\in A} x_i, \quad \text{see, e.g., [4, 7].}$$

Over real intervals $I \subseteq \mathbb{R}$, the discrete Sugeno integrals are exactly those polynomial functions $p : I^n \to I$ that are idempotent, i.e., satisfying $p(x,\ldots,x) = x$.

Natural generalizations of Lovász extensions and polynomial functions are the quasi-Lovász extensions and quasi-polynomial functions, which are best described by

$$f(x_1,\ldots,x_n) = L(\varphi(x_1),\ldots,\varphi(x_n)) \quad \text{and} \quad f(x_1,\ldots,x_n) = p(\varphi(x_1),\ldots,\varphi(x_n)), \text{ resp.,}$$

where L is a Lovász extension, p is a polynomial function, and φ a nondecreasing function such that $\varphi(0) = 0$. Such aggregation functions are used in decision under uncertainty, where φ is a utility function and f an overall preference functional. It is also used in multi-criteria decision making where the criteria are commensurate (i.e., expressed in a common scale). For a recent reference, see Bouyssou et al. [1]. In this paper we show that all of these classes of functions can be axiomatized in terms of so-called comonotonic modularity by introducing variants of homogeneity. To simplify our exposition when dealing with these different objects simultaneously in a unified framework, we will assume hereinafter that $I = [-1,1] \subseteq \mathbb{R}$, and we set $I_+ = [0,1]$, $I_- = [-1,0]$ and $I^n_\sigma = I^n \cap \mathbb{R}^n_\sigma$.

2 Comonotonic Modularity

A function $f\colon I^n \to \mathbb{R}$ is said to be *modular* (or a *valuation*) if

$$f(\mathbf{x}) + f(\mathbf{x}') = f(\mathbf{x}\wedge\mathbf{x}') + f(\mathbf{x}\vee\mathbf{x}') \tag{3}$$

for every $\mathbf{x},\mathbf{x}' \in I^n$. It was proved (see Topkis [13, Thm 3.3]) that a function $f\colon I^n \to \mathbb{R}$ is modular if and only if it is *separable*, that is, there exist n functions $f_i\colon I \to \mathbb{R}$, $i \in [n]$, such that $f = \sum_{i\in[n]} f_i$. In particular, any 1-place function $f\colon I \to \mathbb{R}$ is modular.

Two n-tuples $\mathbf{x},\mathbf{x}' \in I^n$ are said to be *comonotonic* if $\mathbf{x},\mathbf{x}' \in I^n_\sigma$ for some $\sigma \in S_n$. A function $f\colon I^n \to \mathbb{R}$ is said to be *comonotonically modular* (or, shortly, *comodular*) if (3) holds for every comonotonic n-tuples $\mathbf{x},\mathbf{x}' \in I^n$. Note that for any function $f\colon I^n \to \mathbb{R}$, condition (3) holds for tuples $\mathbf{x} = x\mathbf{1}_A$ and $\mathbf{x}' = x'\mathbf{1}_A$, where $x,x' \in I$ and $A \subseteq [n]$. Note that if $f\colon I^n \to \mathbb{R}$ is comodular, then by setting $\mathbf{x}' = \mathbf{0}$ in (3) we have

$$f_0(\mathbf{x}) = f_0(\mathbf{x}^+) + f_0(-\mathbf{x}^-) \quad (\text{where } f_0 = f - f(\mathbf{0}).)$$

Theorem 1. ([6]) *For any function $f\colon I^n \to \mathbb{R}$, the following are equivalent.*

(i) f is comodular.
(ii) There are $g\colon I^n_+ \to \mathbb{R}$ and $h\colon I^n_- \to \mathbb{R}$ comodular s.t. $f_0(\mathbf{x}) = g_0(\mathbf{x}^+) + h_0(-\mathbf{x}^-)$ for every $\mathbf{x} \in I^n$. In this case, we can choose $g = f|_{I^n_+}$ and $h = f|_{I^n_-}$.

(iii) *There are* $g\colon I_+^n \to \mathbb{R}$ *and* $h\colon I_-^n \to \mathbb{R}$ *s.t. for every* $\sigma \in S_n$ *and* $\mathbf{x} \in I_\sigma^n$,

$$\begin{aligned} f_0(\mathbf{x}) \;=\; & \sum_{1\leqslant i\leqslant p} \big(h(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_{\sigma}(i)}) - h(x_{\sigma(i)}\mathbf{1}_{A^{\downarrow}_{\sigma}(i-1)})\big) \\ & + \sum_{p+1\leqslant i\leqslant n} \big(g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i)}) - g(x_{\sigma(i)}\mathbf{1}_{A^{\uparrow}_{\sigma}(i+1)})\big), \end{aligned}$$

where $x_{\sigma(p)} < 0 \leqslant x_{\sigma(p+1)}$. *In this case, we can choose* $g = f|_{I_+^n}$ *and* $h = f|_{I_-^n}$.

In the next section we will propose variants of homogeneity, which will show that the class of comodular functions subsumes important aggregation functions (such as Sugeno and Choquet integrals) as well as several extensions that are pertaining to decision making under uncertainty. We finish this section with a noteworthy consequence of Theorem 1 that provides a "comonotonic" analogue of Topkis' characterization [13] of modular functions as separable functions, and which provides an alternative description of comodular functions.

Corollary 1. *A function* $f\colon I^n \to \mathbb{R}$ *is comodular if and only if it is comonotonically separable, that is, for every* $\sigma \in S_n$, *there exist functions* $f_i^\sigma\colon I \to \mathbb{R}$, $i \in [n]$, *such that*

$$f(\mathbf{x}) = \sum_{i=1}^{n} f_i^\sigma(x_{\sigma(i)}) = \sum_{i=1}^{n} f^\sigma_{\sigma^{-1}(i)}(x_i), \qquad \mathbf{x} \in I^n \cap \mathbb{R}^n_\sigma.$$

Remark 1. (i) Quasi-polynomial functions were axiomatized in [2] in terms of two well-known conditions in aggregation theory, namely, comonotonic maxitivity and comonotonic minitivity. It is not difficult to verify that both properties imply comonotonic modularity, and hence quasi-polynomial functions are comodular or, equiv., comonotonically separable.

(ii) The discrete Shilkret integral can be seen as an aggregation function $f\colon I^n \to \mathbb{R}$, $I \subseteq \mathbb{R}$, that can be represented by an expression of the form

$$f(\mathbf{x}) = \bigvee_{A\subseteq[n]} s_A \cdot \bigwedge_{i\in A} x_i\,, \qquad \mathbf{x} \in \mathbb{R}^n.$$

Essentially the Shilkret integral differs from the Choquet integral in the fact that meet-terms are aggregated by join rather than by sum, and from the Sugeno integral in the fact that each meet-term is transformed by scalar multiplication rather than by scalar meet.
Surprisingly and despite these similarities, unlike the Choquet and Sugeno integrals, the Shilkret integral is not comodular, and hence not comonotonically separable: Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the Shilkret integral $f(x_1,x_2) = 0.2\cdot x_1 \vee 0.4\cdot x_2$, and let $\mathbf{x} = (0.1, 0.1)$ and $\mathbf{x}' = (0.2, 0)$.

3 Homogeneity variants

Despite the negative result concerning the Shilkret integral, the class of comodular functions subsumes a wide variety of integral-like functions, such as quasi-polynomial functions (see Remark 1). The next theorem introduces different variants of homogeneity which, together with comonotonic modularity, provide axiomatizations for the various classes of (extended) integrals we consider in this paper.

Theorem 2. *A function* $f\colon I^n \to \mathbb{R}$ *is a*

1. *quasi-Lovász extension* **iff** *f is comodular and there is $\varphi\colon I \to \mathbb{R}$ nondec. s.t.*

$$f(x\mathbf{1}_A) = sign(x)\varphi(x)\, f(sign(x)\,\mathbf{1}_A) \tag{4}$$

2. *Lovász extension* **iff** *f is comodular and there is $\varphi\colon I \to \mathbb{R}$ nondecreasing s.t.*

$$f(x\mathbf{1}_A) = sign(x)\,x\, f(sign(x)\,\mathbf{1}_A) \tag{5}$$

3. *quasi-polynomial function* **iff** *it is comodular and there is $\varphi\colon I \to \mathbb{R}$ nondec. s.t.*

$$f(x \wedge \mathbf{1}_A) = \varphi(x) \wedge f(\mathbf{1}_A) \quad \textit{and} \quad f(x \vee \mathbf{1}_A) = \varphi(x) \vee f(\mathbf{1}_A) \tag{6}$$

4. *polynomial function* **iff** *it is comodular and for every x in the range of f*

$$f(x \wedge \mathbf{1}_A) = x \wedge f(\mathbf{1}_A) \quad \textit{and} \quad f(x \vee \mathbf{1}_A) = x \vee f(\mathbf{1}_A)$$

Proof. The first two assertions follow immediatly from (2) and Theorem 1. Necessity in the last two assertions follows from Remark 1 and the fact that quasi-polynomials and polynomial functions are quasi-min and quasi-max homogeneous, and range-min and range-max homogeneous, resp. (see [2, 3]). For sufficiency in the third, note that from (6) and Theorem 1 (by applying the left identity on I_+ and the right on I_-), it follows that f is nondecreasing and quasi-min and quasi-max homogeneous, and thus it is a quasi-polynomial function (see Theorem 17 in [2]). Sufficiency in the fourth assertion follows similarly but using results from [3]. □

Remark 2. (i) For the symmetric variants of quasi-Lovász extensions and Lovász extensions replace (4) and (5) by $f(x\mathbf{1}_A) = \varphi(x)f(\mathbf{1}_A)$ (φ odd) and $f(x\mathbf{1}_A) = xf(\mathbf{1}_A)$, resp. (see [6]).
(ii) For Choquet integrals add nondecreasing monotonicity, and for Sugeno integrals replace "for every x in the range of f" by "for every $x \in I$".

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Convexity and concavity of copulas and co-copulas

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Abstract. Convexity and concavity are properties of general interest in the context of aggregation functions. In this contribution, we collect and introduce some weakened or related versions of these properties and study their interrelationships. We focus in particular on copulas and co-copulas.

1 Segment-based notions of convexity and concavity

The most classical definition of convexity and concavity, expressed for aggregation functions, is given next.

Definition 1. *A (binary) aggregation function A is called:*

(i) convex if for any $\mathbf{x},\mathbf{y} \in [0,1]^2$ *and* $\lambda \in [0,1]$*, it holds that*

$$A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda A(\mathbf{x}) + (1-\lambda)A(\mathbf{y})\,;$$

(ii) concave if for any $\mathbf{x},\mathbf{y} \in [0,1]^2$ *and* $\lambda \in [0,1]$*, it holds that*

$$A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda A(\mathbf{x}) + (1-\lambda)A(\mathbf{y})\,.$$

The above definition can be rephrased in a more compact form as follows: an aggregation function A is convex (resp. concave) if for any $\omega \in \mathbb{R}$ and any $\alpha \in \mathbb{R}$, the functions $A(x,\omega x+\alpha)$ and $A(\omega y+\alpha, y)$ are convex (resp. concave).

Unfortunately, these properties are extremely restrictive for copulas and co-copulas:

(i) $T_{\mathbf{L}}$ is the only convex copula;
(ii) $T_{\mathbf{M}}$ is the only concave copula;
(iii) $S_{\mathbf{L}}$ is the only concave co-copula;
(iv) $S_{\mathbf{M}}$ is the only convex co-copula.

We introduce the following weakened versions of convexity and concavity.

Definition 2. *A (binary) aggregation function A is called:*

(i) pos-convex (resp. pos-concave) if for any $\omega \in \mathbb{R}^+$ *and* $\alpha \in \mathbb{R}$ *the functions* $A(x,\omega x+\alpha)$ *and* $A(\omega y+\alpha, y)$ *are convex (resp. concave) on* $[0,1]$*;*

(ii) neg-convex (resp. neg-concave) if for any $\omega \in \mathbb{R}^-$ and $\alpha \in \mathbb{R}$ the functions $A(x, \omega x + \alpha)$ and $A(\omega y + \alpha, y)$ are convex (resp. concave) on $[0,1]$.

Obviously, one can also consider coordinate-wise convexity and concavity.

Definition 3. *A (binary) aggregation function is called:*

(i) coordinate-wisely convex if its partial mappings are convex;
(ii) coordinate-wisely concave if its partial mappings are concave.

Clearly, convexity implies both pos-convexity and neg-convexity, while each of the latter implies coordinate-wise convexity. The same holds for concavity. The above definitions could be further relaxed by considering a fixed ω only. For instance, setting $\omega = -1$, then for a symmetric aggregation function A, convexity (resp. concavity) on $[0,1]$ of the functions $A(x, -x + \alpha)$, for any $\alpha \in \mathbb{R}$, is known as Schur-convexity (resp. Schur-concavity) [1, 3].

2 Directional convexity and concavity

Following a different line of reasoning, the notion of directional convexity was introduced in [9]. In general, convexity neither implies nor is implied by directional convexity. Note that directional convexity is also called ultramodularity [10] (see also [7]).

Definition 4. *A (binary) aggregation function A is called:*

(i) directionally convex if for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0,1]^2$, with $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} + \mathbf{z} \leq \mathbf{1}$, it holds that

$$A(\mathbf{x} + \mathbf{z}) - A(\mathbf{x}) \leq A(\mathbf{y} + \mathbf{z}) - A(\mathbf{y});$$

(ii) directionally concave if for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in [0,1]^2$, with $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} + \mathbf{z} \leq \mathbf{1}$, it holds that

$$A(\mathbf{x} + \mathbf{z}) - A(\mathbf{x}) \geq A(\mathbf{y} + \mathbf{z}) - A(\mathbf{y}).$$

Proposition 1. *A copula C is directionally convex if and only if its co-copula C^*, defined by $C^*(x,y) = 1 - C(1-x, 1-y)$, is directionally concave.*

It suffices to set $\mathbf{x} = \mathbf{0}$ in the definition of directional convexity, to see that directional convexity of an aggregation function implies super-additivity. Similarly, directional concavity implies sub-additivity.

Proposition 2.

(i) Any directionally convex aggregation function is super-additive.
(ii) Any directionally concave aggregation function is sub-additive.

Furthermore, directional convexity (resp. concavity) is a more stringent version of 2-increasingness (resp. 2-decreasingness), considering parallelograms instead of rectangles. However, in the presence of a neutral element 1 (resp. 0), this convexity (resp. concavity) property turns out to be of interest to copulas (resp. co-copulas) only.

Proposition 3. *The following implications hold.*

(i) If a semi-copula is directionally convex, then it is a copula.
(ii) If a semi-co-copula is directionally concave, then it is a co-copula.

The results in [7] immediately lead to the following proposition.

Proposition 4. *If an aggregation function is directionally convex, then it is also coordinate-wisely convex and pos-convex.*

Moreover, for copulas and co-copulas, many of the notions introduced above turn out to be equivalent (see also [8]).

Proposition 5. *For a copula, the following properties are equivalent:*

(i) directional convexity;
(ii) coordinate-wise convexity;
(iii) pos-convexity.

Proposition 6. *For a co-copula, the following properties are equivalent:*

(i) directional concavity;
(ii) coordinate-wise concavity;
(iii) pos-concavity.

Example 1. Recall that the family of Frank copulas $(F_\lambda)_{\lambda\in\bar{\mathbb{R}}}$ consists of the copulas [6]

$$F_\lambda(x,y) = -\frac{1}{\lambda}\log\left[1+\frac{(e^{-\lambda x}-1)(e^{-\lambda y}-1)}{e^{-\lambda}-1}\right], \quad \lambda \neq 0, \tag{1}$$

and its limits

$$F_{-\infty} = T_{\mathbf{L}}, \quad F_0 = T_{\mathbf{P}}, \quad F_{+\infty} = T_{\mathbf{M}}.$$

The following observations hold:

(i) any Frank copula F_λ with $\lambda \leq 0$ is directionally convex;
(ii) any Frank copula F_λ with $\lambda \geq 0$ is coordinate-wisely concave.

3 Point-based convexity and concavity

A different approach to weakening the convexity and concavity properties is presented next.

Definition 5. *A (binary) aggregation function A is called:*

(i) pos-ray-convex (resp. pos-ray-concave) at $\mathbf{x} = (x_0,y_0)$ if for any $\omega \in \mathbb{R}^+$ the functions $A(x,\omega(x-x_0)+y_0)$ and $A(\omega(y-y_0)+x_0,y)$ are convex (resp. concave) on $[0,1]$;

(ii) *neg-ray-convex (resp. pos-ray-concave) at* $\mathbf{x} = (x_0, y_0)$ *if for any* $\omega \in \mathbb{R}^-$ *the functions* $A(x, \omega(x - x_0) + y_0)$ *and* $A(\omega(y - y_0) + x_0, y)$ *are convex (resp. concave) on* $[0, 1]$.

Proposition 7. *The following equivalences hold.*

(i) *An aggregation function is pos-convex (resp. pos-concave) if and only if it is pos-ray-convex (resp. pos-ray-concave) at every point.*
(ii) *An aggregation function is neg-convex (resp. neg-concave) if and only if it is neg-ray-convex (resp. neg-ray-concave) at every point.*

Proposition 8. *The following implications hold.*

(i) *If an aggregation function is directionally convex, then it is pos-ray-convex at every point.*
(ii) *If an aggregation function is directionally concave, then it is pos-ray-concave at every point.*

In particular, if a copula is coordinate-wisely convex, then it is pos-ray-convex at every point.

Example 2. For the Frank copula family, the following observations hold:

(i) any Frank copula F_λ with $\lambda \leq 0$ is pos-ray-convex at every point;
(ii) any Frank copula F_λ with $\lambda \geq 0$ is pos-ray-convex at $(0, 0)$ and $(1, 1)$.

Example 3. The copula of the uniform circular distribution C_1 and the average C_2 of $T_\mathbf{M}$ and $T_\mathbf{L}$ (both orthogonal grid constructions [2] with $T_\mathbf{P}$ as background copula) satisfy:

(i) C_1 is pos-ray-convex and neg-ray-concave at $(1/2, 1/2)$;
(ii) C_2 is pos-ray-concave and neg-ray-convex at $(1/2, 1/2)$.

However, these copulas are neither directionally convex, pos-convex, neg-convex, pos-concave nor neg-concave.

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Modular aggregation functions and copulas

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Singular copulas act on connected domain subareas disjoint with their support as 1-Lipschitz modular aggregation functions. Inspired by this idea, we study the functions $C_A : [0,1]^n \to [0,1]$ given by

$$C_A(x_1,...,x_n) = min\{x_1,...x_n, A(x_1,...,x_n)\},$$

where $A : [0,1]^n \to [0,1]$ is a 1-Lipschitz modular aggregation function (for more details on copulas and aggregation functions we recommend [2], [4]). For $n = 2$, we show that C_A is a copula. Note that for $n > 2$, C_A is, in general, only a quasi-copula. In particular, if $n = 2$ and if A is symmetric, $C_A = C_d$ is the diagonal copula introduced in [1], $C_d(x,y) = min\left\{x, y, \frac{d(x)+d(y)}{2}\right\}$, where $d : [0,1] \to [0,1]$ is given by $d(x) = min\{x, A(x,x)\}$. Obviously, if d is diagonal of some 2-copula, then $C_d = C_A$, putting $A(x,y) = \frac{d(x)+d(y)}{2}$ (and then A is 1-Lipschitz modular aggregation function). Our approach results into several new parametric classes of binary copulas. For example, for $\lambda \in [1,\infty[$, define $A_\lambda : [0,1]^2 \to [0,1]$ by $A_\lambda = \frac{\lambda x + y^\lambda}{\lambda+1}$. Then the corresponding parametric class $\left(C_{A_\lambda}\right)_{\lambda\in[1,\infty[}$ of copulas is given by

$$C_{A_\lambda}(x,y) = min\left\{x, y, \frac{\lambda x + y^\lambda}{\lambda+1}\right\} = \begin{cases} x & \text{if} \quad x \le y^\lambda, \\ y & \text{if} \quad \frac{(\lambda+1)y - y^\lambda}{\lambda} \le x, \\ \frac{\lambda x + y^\lambda}{\lambda+1} & \text{else.} \end{cases}$$

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Lattice-valued preordered sets as lattice-valued topological systems

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Motivated directly by questions from programming semantics and pattern matching in data mining concerning how to mathematically define and model the compatibility of bitstring comparisons with degrees of bitstring-predicate satisfactions – cf. [7, 9, 16], the authors of [3, 5] applied notions of enriched categories [11] and a variable-basis generalization of enriched functors to motivate the notions of sets as enriched categories over meet-semilattices, together with morphisms between them which respect the enrichments of these sets as well as facilitating the changes in underlying meet-semilattices. It was shown in [3, 5], in parallel with [17], that such sets as enriched categories are precisely many-valued preordered sets as studied extensively in [1–3, 8, 14, 17, 20, 23, 24]; and it was further shown that the generalized enriched functors proposed in [3, 5] are precisely many-valued, variable-basis isotone morphisms between such preordered sets. Such generalization of enriched functors leads to the variable-basis category **EnrSet** in [3], renamed **SLat**$(\wedge)$-**PreSet** in [5] and again in this abstract by **Loc-PreSet** (for reasons given below); and for compatibility with topological systems in the sense of [3, 22] also introduces the important, variable-basis subcategory $\mathbf{EnrSet_{Frm}}$, renamed **Frm-PreSet** in [5] and again in this abstract as **Loc-PreSet** (for reasons given below). Extending the fixed-basis, classical theorem that **PreSet** is topological over **Set**, it is shown in [5] that $\mathbf{SLat}(\wedge)^{op}$-**PreSet** is topological over $\mathbf{Set} \times \mathbf{SLat}(\wedge)^{op}$; and, letting $\mathbf{Loc}(\bigwedge)$ denote that subcategory of **Loc** for which the opposite of each morphism preserves arbitrary $\bigwedge$, it is also shown in [5] that $\mathbf{Loc}(\bigwedge)$-**PreSet** is topological over $\mathbf{Set} \times \mathbf{Loc}(\bigwedge)$.

Replacing the usual carrier sets underlying many-valued topological systems by frame-preordered sets, i.e. by sets as enriched categories, and then insisting that the system's frame-valued satisfaction relation satisfy a "compatibility axiom" with respect to the frame-valued preorders, lead to the category **EnrTopSys** of enriched topological systems with ground category (**Loc**-**PreSet**) $\times$ **Loc**. This category provides the mathematical framework needed to answer the programming questions cited above from [3, 5]. Associated extant spaces of such systems, which ride on the same enriched carrier sets, turn out to have topologies for which the membership functions of the open sets satisfy an analogous compatibility axiom with respect to the frame-valued preorders, and therefore lead to the category **EnrTop** of enriched topological spaces with ground category **Loc**-**PreSet**. The modifier "enriched" is consistent with the language of saturation operations for lattice-valued topologies; and we note that in traditional topology, "(pre-)ordered topological space" refers to a space in which the order builds the topology; but this need not be the case with enriched topological spaces and enriched topological systems. It is shown in [5] that **EnrTop** is topological over **Loc**-**PreSet**, and that **EnrTopSys** is neither essentially algebraic nor topological over (**Loc**-**PreSet**) $\times$ **Loc**.

To summarize so far, enriched topological systems were initially created to model the compatibility of bitstring comparisons with degrees of bitstring-predicate satisfactions. But, in the frame-valued case, the notion of satisfaction relations and topologies compatible with a given preorder appears to have unanticipated consequences. For example, discussions at the recently concluded *Fuzzy Symposium 2012* identified an intimate connection between enriched topologies and "lattice-indexed" specialization and lattice-valued specialization orders – some of this discussion is recorded in [5] and is closely related to [14, 23]. In particular, [5] constructs an antisymmetry condition for frame preorders and shows that each of the "crisp" L-specialization and L-valued specialization preorders generated by an L-topology is logically equivalent to the L-T_0 axiom – see [18] – and hence linked to the Stone representation machinery for lattice-valued topology.

It is the purpose of this abstract to outline another, far-reaching consequence of many-valued systems and topologies compatible (or enriched) with preorders, a consequence which gives a deep categorical motivation and justification for such notions and is related to the notion of specialization preorders referred to above. To begin this part of the story, recall from [10] the existence of a full concrete embedding of the category **PreSet** of preordered sets (sets equipped with a reflexive and transitive binary relation) into the category **Top** of topological spaces, which assigns to a preordered set $(X, \leqslant)$ a topological space $(X, \mathfrak{T}_{\leqslant})$, where $\mathfrak{T}_{\leqslant} = \{U \in \mathcal{P}(X) \mid U = \downarrow U\}$ with $\downarrow U = \{x \in X \mid x \leqslant y$ for some $y \in U\}$. This embedding has a right-adjoint, left-inverse **Top** $\xrightarrow{Spec}$ **PreSet**, which assigns to a topological space $(X, \mathfrak{T})$ a preordered set $(X, \leqslant_{\mathfrak{T}})$, where $\leqslant_{\mathfrak{T}}$ is the (dual of the) *specialization preorder* given in [10], i.e., $x \leqslant_{\mathfrak{T}} y$ iff $y \in cl\{x\}$. Additionally, there exists a full non-concrete embedding of **Top** into the category **TopSys** of topological systems, which has a right-adjoint, left-inverse **TopSys** $\xrightarrow{Spat}$ **Top**, where *Spat* is the *spatialization functor* – see [22] for both functors. These adjunctions show that the category **PreSet** is isomorphic to a full coreflective subcategory of **TopSys**, providing an opportunity, to develop the theory of *domains* [6] inside that of topological systems [22].

To continue the story, we take note of the existence of lattice-valued analogues of topological spaces [19], topological systems [4, 21], and lattice-valued specialization preorders [14] – including both the crisp "lattice-indexed" specialization and fuzzy lattice-valued specialization preorders referred to above. However, a variable-basis, lattice-valued analogue of the above [**PreSet**;**TopSys**] correspondence must be carefully constructed: the classical correspondence [**PreSet**;**TopSys**] is fixed-basis; the lattice-valued topological spaces and systems we wish to use are variable-basis (in the sense of [19]), as are the categories $\mathbf{SLat}(\wedge)^{op}$-**PreSet** and **Loc**-**PreSet** introduced in [3] and commented on above; and variable-basis settings are more sophisticated than their fixed-basis counterparts and can hold striking surprises. One of these surprises is that in traditional and fixed-basis lattice-valued topology, continuous mappings respectively preserve the traditional specialization preorder and both the "crisp" L-specialization and L-valued specialization preorders; but the carrier map of continuous morphisms in **Loc-Top** need not preserve either of the "crisp" L-specialization and L-valued specialization preorders. Thus, in case of variable-basis approach over, e.g., the category **Loc** of locales [10], one gets a full coreflective embedding of a particular *subcategory* of **Loc-PreSet** into a particular *subcategory* of **Loc**-**TopSys**, i.e., the classical domain-system relationships can go astray. This is, however, the point where enriched topological systems come to help. It is possible to coreflectively embed the whole category **Loc**-**PreSet** into the category **EnrTopSys** of enriched topological systems of [3, 5], getting thereby an analogue of the classical correspondence. We mention from [5] that **Loc**-**TopSys** embeds into **EnrTopSys**; indeed the forgetful functor from **EnrTopSys** onto **Loc**-**TopSys** has a left adjoint which is an embedding.

As noted above, **EnrTop** is topological over its ground **Loc**-**PreSet** and $\mathbf{Loc}(\bigwedge)$-**PreSet** is topological over its ground $\mathbf{Set} \times \mathbf{Loc}(\bigwedge)$; but it is not the case that **Loc-PreSet** is topological over its ground $\mathbf{Set} \times \mathbf{Loc}$. So one does not get that **EnrTop** is topological over the ground of its ground, namely, $\mathbf{Set} \times \mathbf{Loc}$. One way to address this, in essence to get a "deeper" topologicity, is by taking the dual of a lattice-valued preorder, which generalizes *quasi-pseudo-metric spaces* [13], also known as *hemimetric spaces* [12]. It is the case that the counterpart to **EnrTop** using quasi-pseudo-metrics yields a category topological over $\mathbf{Set} \times \mathbf{Loc}$. Further, the shift from lattice-valued preorders to lattice-valued quasi-pseudo-metrics paves the way to "enrichment" in, e.g., the category of measurable spaces, posing the problem concerning a full coreflective embedding of the latter into a category of "enriched" topological systems. Ultimately, one should arrive at a schema of [**PreSet**;**TopSys**] correspondences for a whole class of categories and their respective "enriched" systems, providing the possibility of doing various mathematical theories inside suitable categories of systems.

We now formally write out some elements of the above-mentioned theory obtained so far, listing needed definitions and results of [3, 5] without individual citation.

Definition 1. **Loc**-**PreSet** *is the category, concrete over* $\mathbf{Set} \times \mathbf{Loc}$*, whose objects (*lattice-valued preordered sets*) are triples* (X,L,P)*, where* (X,L) *is a* $\mathbf{Set} \times \mathbf{Loc}$*-object, and* $X \times X \xrightarrow{P} L$ *is a map (*lattice-valued preorder *on* X*) such that* $P(x,x) = \top_L$ *for every* $x \in X$*; and* $P(x,y) \wedge P(y,z) \leqslant P(x,z)$ *for every* $x,y,z \in X$*. Morphisms (*frame-valued monotone maps*)* $(X_1,L_1,P_1) \xrightarrow{(f,\varphi)} (X_2,L_2,P_2)$ *are* $\mathbf{Set} \times \mathbf{Loc}$*-morphisms* $(X_1,L_1) \xrightarrow{(f,\varphi)} (X_2,L_2)$ *such that* $P_1(x,y) \leqslant \varphi^{op} \circ P_2(f(x),f(y))$ *for every* $x,y \in X_1$.

The shift from the notation **Frm**-**PreSet** in [5] to **Loc**-**PreSet** is more consistent with the ground category being $\mathbf{Set} \times \mathbf{Loc}$.

Definition 2. **Loc**-**Top** *is the category, concrete over* $\mathbf{Set} \times \mathbf{Loc}$*, whose objects (*lattice-valued topological spaces*) are triples* (X, L, τ)*, where* (X, L) *is a* $\mathbf{Set} \times \mathbf{Loc}$*-object, and* τ *(*lattice-valued topology *on* X*) is a subframe of the product frame* L^X*. Morphisms (*lattice-valued continuous maps*)* $(X_1, L_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, L_2, \tau_2)$ *are* $\mathbf{Set} \times \mathbf{Loc}$*-morphisms* $(X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2)$ *such that* $(f, \varphi)^{\leftarrow}(\alpha) = \varphi^{op} \circ \alpha \circ f \in \tau_1$ *for every* $\alpha \in \tau_2$*.*

Theorem 1. **Loc**-**Top** *is topological over* $\mathbf{Set} \times \mathbf{Loc}$*; and* **Loc**-**PreSet** *is topological over* $\mathbf{Set} \times \mathbf{Loc}(\bigwedge)$*, where* $\mathbf{Loc}(\bigwedge)$ *is that subcategory of* **Loc** *for which the opposite* φ^{op} *of each morphism* φ *preserves arbitrary* $\bigwedge$*.*

Theorem 2. *There is a full concrete embedding* $\mathbf{Loc}\text{-}\mathbf{PreSet} \overset{E}{\hookrightarrow} \mathbf{Loc}\text{-}\mathbf{Top}$ *given by* $E((X_1, L_1, P_1) \xrightarrow{(f,\varphi)} (X_2, L_2, P_2)) = (X_1, L_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, L_2, \tau_2)$*, where* $\tau_i = \{\alpha \in L_i^{X_i} \mid P_i(x, y) \wedge \alpha(y) \leqslant \alpha(x)$ *for every* $x, y \in X_i\}$*.*

It is shown in Lemma 5.4 of [5] that the lattice-valued topologies constructed in Theorem 2 are the largest topologies compatible with the preorder on the underlying carrier set. In particular, given an L-preordered set (X, L, P) and an L-topology on X, then τ is *compatible* with P if for every $x, y \in X$ and every $\alpha \in \tau$, it is the case that

$$P(x, y) \wedge \alpha(y) \leqslant \alpha(x).$$

Given a locale L and $a, b \in L$, recall that $a \to b = \bigvee\{c \in L \mid a \wedge c \leqslant b\}$. $\mathbf{Loc}^*$ is a subcategory of **Loc**, with the same objects, and with morphisms φ such that φ^{op} preserves arbitrary $\bigwedge$ and $\to$.

Theorem 3. *There exists a concrete functor* $\mathbf{Loc}^*\text{-}\mathbf{Top} \xrightarrow{Spec} \mathbf{Loc}^*\text{-}\mathbf{PreSet}$ *defined by* $Spec((X_1, L_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, L_2, \tau_2)) = (X_1, L_1, P_1) \xrightarrow{(f,\varphi)} (X_2, L_2, P_2)$*, where* $P_i(x, y) = \bigwedge_{\alpha \in \tau_i}(\alpha(y) \to \alpha(x))$*, which is a right-adjoint-left-inverse to its respective restriction* E^* *of* E*.*

It should be noted that the P_i's are precisely the lattice-valued specialization preorders referenced in preceding paragraphs. The restriction to $\mathbf{Loc}^*$ is not needed using the crisp lattice-indexed specialization preorders.

The following provides a possible way to avoid the restriction to the category $\mathbf{Loc}^*$ with lattice-valued specialization preorders.

Definition 3. **EnrTop** *is the category, concrete over* **Loc**-**PreSet***, whose objects (*lattice-valued enriched topological spaces*) are tuples* (X, L, P, τ) *viewed as objects* (X, L, P) *of* **Loc**-**PreSet** *equipped with an enriched* L*-topology* τ*, i.e.,* (X, L, τ) *is an object of* **Loc**-**Top** *and* τ *is* L*-topology on* X *compatible with* P*. Morphisms (*lattice-valued monotone continuous maps*)* $(X_1, L_1, P_1, \tau_1) \xrightarrow{(f,\varphi)} (X_2, L_2, P_2, \tau_2)$ *are* **Loc**-**PreSet** *morphisms* $(X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2)$ *which are also* **Loc**-**Top** *morphisms.*

Clearly, since **Loc**-**PreSet** has ground $\mathbf{Set} \times \mathbf{Loc}$, then **EnrTop** can be defined as having ground $\mathbf{Set} \times \mathbf{Loc}$ as well.

Theorem 4.

1. *There exists a full concrete embedding* **Loc**-**PreSet** $\overset{E}{\hookrightarrow}$ **EnrTop** *defined by* $E((X_1,L_1,P_1) \xrightarrow{(f,\varphi)} (X_2,L_2,P_2)) = (X_1,L_1,P_1,\tau_1) \xrightarrow{(f,\varphi)} (X_2,L_2,P_2,\tau_2)$*, where*
$$\tau_i = \{\alpha \in L_i^{X_i} \,|\, P_i(x,y) \wedge \alpha(y) \leqslant \alpha(x) \textit{ for every } x,y \in X_i\}.$$
2. *The embedding E is left-adjoint, right-inverse to the forgetful functor* **EnrTop** $\xrightarrow{V}$ **Loc**-**PreSet***.*
3. **EnrTop** *is topological over* **Loc**-**PreSet** *with respect to V.*

Again, the topologies constructed in Theorem 4 (1) are the largest topologies compatible with the given preorders, and they play a significant role in the proof of the topologicity of **EnrTop** over **Loc**-**PreSet** w.r.t. V.

Definition 4. **EnrTopSys** *is the category, concrete over* $(\textbf{Loc}\text{-}\textbf{PreSet}) \times \textbf{Loc}$*, whose objects (*lattice-valued enriched topological systems*) are tuples $(X,L,P,A,\models)$ viewed as objects (X,L,P) of* **Loc**-**PreSet** *equipped with a locale A and an L-valued satisfaction relation $X \times A \xrightarrow{\models} L$ such that $(X,L,A,\models)$ is an object in* **Loc**-**TopSys** *and $\models$ is compatible with P, i.e.,*
$$P(x,y) \wedge \models(y,a) \leqslant \models(x,a) \textit{ for every } x,y \in X \textit{ and every } a \in A.$$

Theorem 5.

1. **EnrTop** *embeds into* **EnrTopSys** *as a full, coreflective subcategory. Hence* **Loc**-**PreSet** *embeds into* **EnrTopSys***.*
2. **Loc**-**TopSys** *embeds into* **EnrTopSys** *as a full, coreflective subcategory.*

Per our earlier discussion, **EnrTopSys** as the larger category has the "wiggle room" to accommodate **Loc**-**PreSet**, which **Loc**-**TopSys** does not; and, as also mentioned earlier, this accommodation gives a powerful categorical argument for **EnrTopSys** and the approaches and ideas it represents.

Turning our attention now to the dual of preorders, the following notions give partial insights into lattice-valued quasi-pseudo-metric spaces.

Definition 5. **Loc**-**QPMet** *is the category, concrete over* $\textbf{Set} \times \textbf{Loc}$*, whose objects (*lattice-valued quasi-pseudo-metric spaces*) are triples (X,L,ρ), where (X,L) is a* $\textbf{Set} \times \textbf{Loc}$*-object, and $X \times X \xrightarrow{\rho} L$ is a map (*lattice-valued quasi-pseudo-metric *on X) such that $\rho(x,x) = \bot_L$ for every $x \in X$; and $\rho(x,y) \leqslant \rho(x,z) \vee \rho(z,y)$ for every $x,y,z \in X$. Morphisms (*lattice-valued non-expansive maps*) $(X_1,L_1,\rho_1) \xrightarrow{(f,\varphi)} (X_2,L_2,\rho_2)$ are* $\textbf{Set} \times \textbf{Loc}$*-morphisms $(X_1,L_1) \xrightarrow{(f,\varphi)} (X_2,L_2)$ with $\varphi^{op} \circ \rho_2(f(x),f(y)) \leqslant \rho_1(x,y)$ for every $x,y \in X_1$.*

Theorem 6. $(\textbf{Loc}\text{-}\textbf{QPMet}, |-|)$ *is topological over* $\textbf{Set} \times \textbf{Loc}$*.*

Proof. Given a $|-|$-structured source $\mathcal{L} = ((X,L) \xrightarrow{(f_i,\varphi_i)} |(X_i,L_i,\rho_i)|)_{i \in I}$, the $|-|$-initial structure on (X,L) w.r.t. $\mathcal{L}$ is given by $\rho(x,y) = \bigvee_{i \in I} \varphi_i^{op} \circ \rho_i(f_i(x),f_i(y))$. □

Definition 6. **Loc**-**QPMTop** *is the category, concrete over* $\mathbf{Set} \times \mathbf{Loc}$*, whose objects (*lattice-valued quasi-pseudo-metric topological spaces*) are tuples* (X, L, τ, ρ)*, where* (X, L, τ) *(resp.* (X, L, ρ)*) is a lattice-valued topological (resp. quasi-pseudo-metric) space, and* $\alpha(x) \leqslant \rho(x,y) \vee \alpha(y)$ *for every* $\alpha \in \tau$*,* $x, y \in X$*. Morphisms (*lattice-valued non-expansive continuous maps*)* $(X_1, L_1, \tau_1, \rho_1) \xrightarrow{(f,\varphi)} (X_2, L_2, \tau_2, \rho_2)$ *are* $\mathbf{Set} \times \mathbf{Loc}$*-morphisms* $(X_1, L_1) \xrightarrow{(f,\varphi)} (X_2, L_2)$*, which are lattice-valued non-expansive and continuous.*

It should be pointed out that topologies of quasi-pseudo-metric topological spaces are essentially enriched, but not generated, by quasi-pseudo-metrics, and this enrichment is in the form of a compatibility axiom dual to that for preorders.

Theorem 7. $(\mathbf{Loc}\text{-}\mathbf{QPMTop}, |-|)$ *is topological over* **Loc**-**QPMet***.*

Proof. Given a $|-|$-structured source $\mathcal{L} = ((X, L, \rho) \xrightarrow{(f_i,\varphi_i)} |(X_i, L_i, \tau_i, \rho_i)|)_{i \in I}$, the $|-|$-initial structure τ on (X, L, ρ) w.r.t. $\mathcal{L}$ is given by $\tau = \{\alpha \in \overline{\tau} \mid \alpha(x) \leqslant \rho(x,y) \vee \alpha(y)$ for every $x, y \in X\}$, where $\overline{\tau}$ is the subframe of L^X generated by $\bigcup_{i \in I} ((f_i, \varphi_i)^{\leftarrow})^{\rightarrow}(\tau_i)$. $\square$

Corollary 1. $(\mathbf{Loc}\text{-}\mathbf{QPMTop}, |-|)$ *is topological over* $\mathbf{Set} \times \mathbf{Loc}$*.*

The theory as developed so far suggests the following two open problems.

Problem 1. Is it possible to extend the above machinery and its results, replacing the category **PreSet** or **QPMet** with an arbitrary category of structured sets, e.g., with the category of measurable spaces or probabilistic metric spaces?

Problem 2. As explained in the opening paragraphs, standard notions of enriched categories [11] guided development of enriched topological systems and enriched topological spaces in which the carrier sets are enriched categories, initially over meet semilattices and later over locales; and extensions of standard notions of enriched functors guided formulation of the variable-basis morphisms of these new categories of systems and spaces. Can any of these new categories in fact be itself viewed as an enriched category over some monoidal category?

Regarding Problem 2, note that there is a functor $\mathbf{PreSet}^{op} \times \mathbf{PreSet} \xrightarrow{hom} \mathbf{Set}$, whose co-domain can be **PreSet** (and the functor itself Hom). Every locale L has a functor $(L\text{-}\mathbf{PreSet})^{op} \times L\text{-}\mathbf{PreSet} \xrightarrow{Hom} L\text{-}\mathbf{PreSet}$, the L-preorder P on $hom((X_1, P_1), (X_2, P_2))$ being defined through $P(f,g) = \bigwedge_{x \in X_1} P_2(f(x), g(x))$. The extension of Hom to **Loc-PreSet** is unclear to us.

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Recent results about qualitative monotonic set-functions on finite sets

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This talk deals with monotonic set-functions taking values on a finite totally ordered scale. While numerical capacities have been extensively studied, it seems that this kind of non-numerical set-functions, we call qualitative, have not received proper attention in the literature. Yet several authors, such as Grabisch and Mesiar have shown that several well-known notions in the numerical setting possess qualitative counterparts, for instance Moebius transforms. Possibility measures (maxitive set-functions) play the same role in the qualitative setting as probability measures in the quantitative setting. In particular their qualitative Moebius transform coincide with possibility distributions. Interestingly, qualitative necessity measures have nested qualitative Moebius transforms that are the cuts of possibility distributions of their conjugate possibility measures. Likewise Sugeno integrals are counterparts of Choquet integrals, prioritized minimum and maximum are qualitative counterparts of weighted average.

Our work in the last two years has tried to push the analogy between qualitative and quantitative capacities further. Research results obtained so far deal with the following issues:

1. The use of qualitative Moebius transforms highlights an analogy between general qualitative capacities and belief functions. For instance a qualitative counterpart of Dempster rule of combination can be defined. Using Dempster's construction for belief function, we can generate upper and lower possibilities. While an upper possibility measure is a possibility measure (induced by the counterpart to the contour function of a capacity), a lower possibility measure is a general capacity.
2. Conversely, a capacity can always be viewed as a lower possibility measure. The set of possibility measures dominating a capacity is never empty and forms a directed-complete partial order (a sup-semi lattice). So a capacity can always be represented as the eventwise minimum of a set of possibility measures. Alternatively, a capacity can always be represented as the eventwise maximum of a set of necessity measures.
3. A possibility measure is simply characterized by its possibility distribution which is linear in terms of the cardinality of the underlying set. The complexity of a qualitative capacity can be described by the number of possibility measures that may generate it.They are the minimal elements of the set of possibility measures that

dominate it. It leads to a generalisation of the maxitive axiom of possibility measures (resp: of the minitive axiom of necessity measures) whereby classes of qualitative capacities can be ordered according to their representational complexity. More precisely consider the following property of a qualitative capacity γ

$$n\text{-adjunction: } \forall A_i, i = 1, \dots n+1, \min_{i=1}^{n+1} \gamma(A_i) \leq \max_{1 \leq i < j \leq n+1} \gamma(A_i \cap A_j)$$

When $n = 1$, this is the adjunction property $\min(\gamma(A), \gamma(B)) \leq \gamma(A \cap B)$, equivalent to the minitivity axiom of necessity measures : $N(A \cap B) = \min(N(A), N(B))$ so that $\gamma = N$ is then a necessity measure. In the general case, it can be proved that γ is n-adjunctive if and only if there exist n necessity measures N_i such that $\gamma(A) = \max_{i=1}^{n} N_i(A)$. The dual notion of n-maxitivity can be likewise defined for the representation of capacities in terms of possibility measures.
4. Sugeno integral with respect to an n-maxitive capacity is a lower prioritized maximum, i.e., the lower bound of the set of prioritized maxima computed from the n possibility distributions that generate it. This is the counterpart to the fact that for 2-monotone capacities, the Choquet integral is a lower expectation.
5. Alternative aggregation operations to Sugeno integral can be obtained if we equip the range of the capacity with a residuated operation. From a multifactorial point of view, it comes down to alternative ways of using the criteria importance degrees modeled by the capacity. Yet other related aggregation schemes (we call "desintegrals") can be obtained if the local evaluations have a negative flavor (measuring bad properties). An interesting issue is the study of formal properties of such variants of Sugeno integral, and whether they can still be viewed as upper and lower prioritized minimum and maximum.

These results also bear some connections with modal logic. Indeed a possibility (resp. necessity) measure can be viewed as a graded extension of a KD possibility (resp. necessity) modality. Namely, if p is a propositional formula with set of models $[p]$ then interpreting $\Box p$ as $N([p]) \geq \alpha$ yields KD modalities, and the set $\{p : N([p]) \geq \alpha\}$ for a fixed threshold α is deductively closed (it is a filter). The above results concerning n-adjunction indicate that given a qualitative capacity γ, the set $\{p : \gamma([p]) \geq \alpha\}$ is a union of filters (a neighborhood structure in the sense of modal logic), and corresponds to a disjunction of KD necessity modalities, an idea that is closely related to non-regular modal logics.

Some of the above results appear in the papers listed below.

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The multivariate probability integral transform

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It is well–known from elementary probability theory that, if X is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and if its distribution function F is continuous, then the random variable $F \circ X = F(X)$ is uniformly distributed on $[0,1]$. This is called the *probability integral transform* (shortly, PIT) of X. In higher dimensions, however, the concept of PIT is a much richer tool that is also far less understood. We start with its very definition.

Definition 1. *Let* $\mathbf{X}$ *be a continuous random vector on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *whose distribution function is equal to* H*. Then the PIT of* $\mathbf{X}$ *is the random variable* $V = H(\mathbf{X})$.

In contrast to the univariate case, it is not generally true that the distribution function K of $V = H(\mathbf{X})$ is uniform on $[0,1]$ nor is it possible to characterize H or reconstruct it from the knowledge of K alone. In fact the calculation of K depends only on the copula C of $\mathbf{X}$ (see, e.g., [2]) and does not involve the knowledge the marginal distributions. Specifically, for every $t \in [0,1]$, we have

$$K(t) = \mu_C(\{\mathbf{u} \in [0,1]^d : C(\mathbf{u}) \leq t\})$$

where μ_C is the measure induced by the copula C on $[0,1]^d$.

The distribution function K of the PIT has some interesting applications:

– It is related to the population value of Kendall's τ rank correlation coefficient for a pair of random variables (X_1, X_2) via the formula ([5]):

$$\tau(X_1, X_2) = 4E(H(X_1, X_2)) - 1.$$

For this reason, the distribution function of a PIT is also called *Kendall's distribution function*.

– It is connected (under suitable conditions) with a suitable Archimedean copula, i.e. a continuous Archimedean triangular norm that is a copula (see, e.g., [6, 7]). A fact that encouraged various authors (see, e.g., [3, 4]) to develop estimation and goodness-of-fit procedures for Archimedean copula models using an empirical version of K.

Here we aim at discussing a recent application of the PIT and Kendall's distribution function to the determination of a suitable notion of quantile for multivariate random vectors. In fact, while in the univariate case the notion of quantile is unambiguous, in higher dimensions the notion allows different definition, mainly due to the lack of a natural total order in multi-dimensional Euclidean spaces (see [10] and the references therein). Following the approach outlined in [8], we will show how Kendall's distribution function can be used for this purpose. In particular, a notion of multivariate return period will be developed and its use as measure of risk for multivariate events in hydrology will be underlined (see, for instance, [1, 9]).

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On a convergence of $\otimes$-fuzzy integrals over MV-algebras

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1 Introduction

In [5], we proposed a new type of fuzzy integral defined over a complete residuated lattice. It is called $\odot$-fuzzy integral, where $\odot$ is either the operation $\wedge$ or $\otimes$ from the residuated lattice. The primary motivation for a fuzzy integral defined over a lattice structure came from fuzzy logic via the need to model quantifiers of natural language like *most* or *many* by logical tools (see [6, 4, 3]). And, as will be seen from the definition, the formula defining fuzzy integral contains the universal and existential quantifier, where the existential quantifier is applied on fuzzy measurable sets from a given algebra of fuzzy sets and the universal quantifier on the elements that belong to the support of these fuzzy sets, which enable to introduce quantifiers using fuzzy integrals in the framework of a higher-order fuzzy logic. On the other hand, this new type of fuzzy integral is interesting by itself without any link to quantifiers. For example, we proved in [5] that our integral coincides with the Sugeno integral supposing the divisibility of the complete residuated lattice. Furthermore, our integral satisfies a basic type of convergence if we restrict ourselves to globally (strongly) convergent sequences of mappings.[1]

In this contribution, we would like to focus on the pointwise convergence of this type of fuzzy integrals and show several results about their convergence in parallel to results proved for other types of fuzzy integrals.

2 $\otimes$-fuzzy integrals

In this paper, we suppose that the structure of truth values is a complete linearly ordered MV-algebra $\mathbf{L} = (L, \wedge, \vee, \otimes, \rightarrow, \bot, \top)$, which is dense, i.e., for any $a < b$ there is $c \in L$ such that $a < c < b$. For details about MV-algebras, we refer to [1] or [2]. Fuzzy sets are defined as mappings from a given non-empty universe M to L. The set of all fuzzy sets over M is denoted by $\mathcal{F}(M)$. The set of all fuzzy subsets of a fuzzy set A is denoted by $\mathcal{F}(A)$. By $1_\emptyset$ we denote the empty fuzzy set. To refer to the universe of discourse M of A, we will sometimes write $\mathrm{Dom}(A)$ instead of M. The intersection and the union of fuzzy sets is defined standardly, moreover, we define the difference of two fuzzy sets by $(A \setminus B)(m) = A(m) \otimes (B(m) \rightarrow \bot)$. Motivated by our research on fuzzy quantifiers, we provide the following definition of σ-algebra over a fuzzy set (the scope of a quantifier).

[1] Let $\{f_n\} \subset \mathcal{F}(M)$ be a sequence of mappings and $f, X \in \mathcal{F}(M)$. A sequence of mapping $f_1, f_2, \ldots$ *globally (strongly) converges to* f on $\mathrm{Dom}(X)$, if for any $a \in L$, $a < \top$, there exists a natural number n_0 such that $(f_n(m) \leftrightarrow f(m)) > a$ for any $m \in X$ and $n > n_0$.

Definition 1 **(σ-algebra over a fuzzy set).** *Let A be a non-empty fuzzy set on M. A subset $\mathcal{F}$ of $\mathcal{F}(A)$ is a* σ-algebra of fuzzy sets on A, *if the following conditions are satisfied*

1. $1_\emptyset, A \in \mathcal{F}$,
2. *if $X \in \mathcal{F}$, then $A \setminus X \in \mathcal{F}$,*
3. *if $X_i \in \mathcal{F}$, $i = 1, 2, \ldots$, then $\bigcup_{i=1}^{n} X_i \in \mathcal{F}$.*

A pair $(A, \mathcal{F})$ is called a *fuzzy measurable space* (on A), if $\mathcal{F}$ is a σ-algebra of fuzzy sets on A and we say that X is *$\mathcal{F}$-measurable* if $X \in \mathcal{F}$.

Theorem 1. *If $(A, \mathcal{F})$ is a fuzzy measurable space, then $\mathcal{F}$ is closed under countable intersections.*

Definition 2 **(Continuous fuzzy measure).** *Let $(A, \mathcal{F})$ be a fuzzy measurable space. A non-decreasing mapping $\mu : \mathcal{F} \to L$ is a continuous fuzzy measure on $(A, \mathcal{F})$ if*

1. *$\mu(1_\emptyset) = \bot$ and $\mu(A) = \top$;*
2. *$\{Y_n\} \subseteq \mathcal{F}$, $Y_1 \subset Y_2 \subset \cdots$, $Y = \bigcup_{i=n}^{\infty} Y_n \in \mathcal{F}$, then $\lim_{n\to\infty} \mu(Y_n) = \mu(Y)$;*
3. *$\{Y_n\} \subseteq \mathcal{F}$, $Y_1 \supset Y_2 \supset \cdots$, $Y = \bigcap_{i=n}^{\infty} Y_n \in \mathcal{F}$ and there exists n_0 such that $\mu(Y_{n_0}) < \top$, then $\lim_{n\to\infty} \mu(Y_n) = \mu(Y)$.*

A triplet $(A, \mathcal{F}, \mu)$, where $(A, \mathcal{F})$ is a fuzzy measurable space and μ is a continuous fuzzy measure on $(A, \mathcal{F})$, is called a fuzzy measure space.

Definition 3 **($\otimes$-fuzzy integral).** *Let $(A, \mathcal{F}, \mu)$ be a fuzzy measure space with $M =$ $\mathrm{Dom}(A)$, $f : M \to L$ be a mapping and X be an $\mathcal{F}$-measurable fuzzy set. Then $\otimes$-fuzzy integral of f on X is given by*

$$\int_X^{\otimes} f \, d\mu = \bigvee_{Y \in \mathcal{F}_X^-} \bigwedge_{m \in \mathrm{Supp}(Y)} (f(m) \otimes \mu(Y)),$$

where $\mathcal{F}_X^- = \{Y \mid Y \in \mathcal{F}$ and $1_\emptyset \neq Y \subseteq X\}$. If $X = A$, then we denote this integral by $\int^{\otimes} f \, d\mu$.

Remark 1. As we noted in Introduction, one can see that the supremum in the formula defining $\otimes$-fuzzy integral is computed over a set of $\mathcal{F}$-measurable fuzzy sets and this could be interpreted as the existential quantifier applied on a set of fuzzy sets (in a higher-order fuzzy logic). The infimum could be interpreted as the universal quantifier.

3 Convergence theorems for sequences of $\otimes$-fuzzy integrals

Let $\{a_n\} \subset L$ be a sequence of elements and $b \in L$. We say that $a_1, a_2, \ldots$ *converges to* b, if for any $a \in L$, $a < \top$, there exists a natural number n_0 such that

$$(a_n \leftrightarrow b) > a$$

for any $n > n_0$. One can see that the concept of a distance which is commonly used in the definition of convergence is replaced here by a concept of similarity expressed by the biresiduum which can be introduced in each MV-algebra using $a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a)$.

We write $a_n \rightarrow b$ if the sequence $a_1, a_2, \dots$ converges to b. If $a_1, a_2, \dots$ is a non-increasing (non-decreasing) sequence converging to b, then we write $a_n \searrow b$ ($a_n \nearrow b$). Let $f_1, f_2, \dots$ be a sequence of mappings from M to L and X be a fuzzy set. We say that $f_1, f_2, \dots$ converges (pointwise) to f on X if $f_n(m) \rightarrow f(m)$ for any $m \in \mathrm{Dom}(X)$. We write $f_n \rightarrow f$ if the sequence $f_1, f_2, \dots$ converges to f, and also $f_n \searrow f$ ($f_n \nearrow f$), whenever $f_1, f_2, \dots$ is a non-increasing (non-decreasing) convergent sequence of mappings.

Theorem 2. *Let $X \in \mathcal{F}$ and $f_n \searrow f$ on X. If there exists n_0 such that*

$$\mu(X \upharpoonright \{m \mid f_{n_0}(m) > \int_X^{\otimes} f \, d\mu\}) < \top, \tag{1}$$

then $\int_X^{\otimes} f_n \, d\mu \searrow \int_X^{\otimes} f \, d\mu$.

Theorem 3. *Let $X \in \mathcal{F}$ and $f_n \nearrow f$ on X. Then $\int_X^{\otimes} f_n \, d\mu \nearrow \int_X^{\otimes} f \, d\mu$.*

Theorem 4. *Let $X \in \mathcal{F}$ and $f_n \rightarrow f$ on X. If there exists n_0 satisfying the condition (1), then $\int_X^{\otimes} f_n \, d\mu \rightarrow \int_X^{\otimes} f \, d\mu$.*

Note that the proof of the convergence theorem for $\otimes$-fuzzy integral with the idempotent $\otimes$ (i.e., the corresponding MV-algebra is a Boolean algebra) can be done analogously to the proof of Theorem 7.5 in [7]. The proof for the general operation $\otimes$ turned out to be much more complicated, since the idempotency of the operation $\otimes$ cannot be used. A similar question has been investigated in [8], where $\otimes$ is a generalized t-norm defined on $[0, \infty]$.

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On states on quantum and algebraic structures

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The notion of a state is an analogue of the notion of a probability measure in the classical probability theory. When a measurement is preformed for example in quantum mechanics, it was observed that if we measure momentum x and position y of an elementary particle, then

$$\sigma_s(x)\sigma_s(y) \geq \hbar > 0,$$

where $\hbar$ is a strictly positive constant. Therefore, there is no experiment which allows us to measure both observables with prescribed preciseness. The models which study this problem are said to be *quantum structures*, and nowadays, we have a whole hierarchy of different structures like Boolean algebras orthomodular lattices, orthomodular posets, orthoalgebras. In the last two decades, we are studying very intensively D-posets by Kôpka and Chovanec [14] or equivalently, effect algebras by Foulis and Bennett [12]. These structures, E have a partial binary operation, $+$, which model the conjunction of two mutually excluding events, and two constant elements 0 and 1. The state is simply a mapping $s : E \to [0,1]$ such that (i) $s(a+b) = s(a) + s(b)$ whenever $a+b$ is defined in E, and (ii) $s(1) = 1$. Every Boolean algebra has a lot of states, but there is a Boolean σ-algebra which has no σ-additive state. In addition, there are even finite orthomodular posets which are stateless . However, if E is an interval effect algebra, i.e., it is an interval $[0,u]$ in an Abelian po-group, E has at least one state. Therefore, every MV-algebra possesses at least one state. We recall that a state on an MV-algebra M is any mapping $s : M \to [0,1]$ such that $s(a \oplus b) = s(a) + s(b)$ whenever $a \leq b^*$.

The main problem in quantum structures is a problem of commensurability of two events a and b, that is an existence of three events a_1, b_1, c such that $a = a_1 + c$, $b = b_1 + c$ and $a_1 + b_1 + c$ is defined in E. A maximal set of commensurable elements is said to be a *block*, and if an effect algebra is a lattice, every block is an MV-algebra. We note that every effect algebra can be covered by blocks.

Twelve years ago there appeared noncommutative structures like pseudo MV-algebras, [13], or equivalently, generalized MV-algebras [15]. Latter, pseudo effect algebras were introduced in [9, 10]. Every pseudo MV-algebra is an interval in an ℓ-group, [3], and also some pseudo effect algebras are intervals in po-groups not necessarily Abelian. However, these structures can be stateless, [2].

The representability of effect algebras and pseudo effect algebras by intervals in po-groups is guaranteed by the so-called the Riesz Decomposition Property, i.e. if $a_1 + a_2 = b_1 + b_2$, there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $a_1 = c_{11} + c_{12}$, $a_2 = c_{21} + c_{22}$, $b_1 = c_{11} + c_{21}$, and $b_2 = c_{21} + c_{22}$, and their noncommutative generalizations.

If pseudo effect algebra satisfies RDP, then the state space is always either an empty set or a non-empty Choquet simplex. For such algebras every state can be represented as a standard integral over classical probability measure, [4].

The notion of a state for BL-algebras is not straightforward, but there are two notions: a Riečan state and a Bosbach state. This was generalized also for pseudo BL-algebras, [5], and in some cases they coincide, [8].

Recently, the notion of a state was algebraized and the notion of a state MV-algebra was introduced in [11] adding a so-called state operator to the language of MV-algebras. The state operator resembles the properties of MV-algebras. A representation of subdirectly irreducible state-morphism MV-algebras, where the internal state is an idempotent endomorphism, was given in [1] and generators of the variety of state-morphism MV-algebras are presented in [6].

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Decomposition-integral: unifying Choquet and the concave integrals

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In economics, and particularly in decision theory under uncertainty, a rational decision maker is often described as an expected utility maximizer. The expected utility is calculated with respect to (w.r.t.) some prior probability over the state space. Although expected utility theory is useful and convenient to work with, different experiments, among which the Ellsberg's paradox [3], show that decision makers often violate this theory.

Schmeidler [12] proposed a theory of decision making, where the belief of the decision maker is represented by a non-additive probability (henceforth referred to as *capacity*). The representation of the belief by a capacity might reflect an incomplete or imprecise information that the decision maker has about the uncertain aspects of the decision problem under consideration. Schmeidler [12] proposed a model where the expected value of a random variable is calculated according to Choquet integral [1]. According to this model, among all alternatives (in this literature they are called *acts*) the decision maker chooses the one that maximizes Choquet expected utility.

As an integration scheme, Choquet integral posses two essential properties and lacks one. On one hand, it is monotonic w.r.t. first order stochastic dominance and it is translation-invariant. That is, Choquet expected value of a portfolio with an added constant is equal to the expected value of the original portfolio plus the constant. On the other hand, Choquet integral does not respect diversification. In other words, the expected value of two portfolios mixed together is not necessarily greater than, or equal to, the mixture of the expected values of the two portfolios calculated separately.

Lehrer [8] introduced the concave integral with respect to capacities, which differs from Choquet integral. It hinges on the idea underlying the Lebesgue integral and thus respects risk aversion. The concave integral in based on decomposition of random variables to simple ingredients. A *decomposition* is a representation of a random variable as a positive linear combination of indicators.[3] When an indicator is replaced by the value of its corresponding event, the decomposition is transformed to a linear combination of numbers. In other words, a capacity assigns to each decomposition a value: the value corresponding to the linear combination of indicators. This value helps the decision maker to evaluate any portfolio, even when the information available is incomplete or

[3] An indicator of event A, denoted $\mathbb{I}_A$, is the random variable that attains the value 1 on A and the value 0, otherwise.

imprecise. The expected value of a random variable, according to the concave integral, is defined as the maximum value obtained among all its decompositions.

Not only the concave integral can be expressed in terms of decompositions, Choquet integral can also be described in these terms. While the concave integral does not impose any restriction on the decompositions allowed, Choquet integral does. A chain of events is a sequence of decreasing events w.r.t. inclusion. A Choquet decomposition is a decomposition that uses only chains. Like the concave integral, Choquet integral of a random variable is defined as the maximum value obtained among its decompositions, but in this case only among its Choquet decompositions.

Based on the decomposition method, this paper develops a new notion of integral w.r.t. capacities: the *decomposition-integral*. This integral scheme is determined by a vocabulary that dictates which decompositions are allowed and which are not. For instance, when all possible decompositions are allowed, the decomposition-integral coincides with the concave integral, and when only Choquet-decompositions are allowed, the decomposition-integral coincides with Choquet integral.

It turns out that the decomposition approach to integration unifies many other integral schemes. A decomposition of a random variable is partitional if any two of its indicators are disjoint (i.e., obtain the value 1 on disjoint events). Riemann integral coincides with the decomposition-integral when the vocabulary allows only partitional decompositions[4]. Another well known integral that can be expressed in terms of decompositions is Shilkret integral (see Shilkret [13]). Suppose that a vocabulary allows to use only one indicator at a time. In this case the linear combination consists of merely one indicator. Obviously, there is no way to obtain any random variable as an indicator of an event multiplied by a positive scalar. This is the reason why the integral scheme uses also sub-decompositions. A sub-decomposition of a random variable is a linear combination of indicators, but unlike a decomposition, it does not necessarily coincide with the random variable (it may be smaller). Using the language of decomposition-integrals, Shilkret integral of a random variable is the maximum among all its sub-decompositions that employs only one indicator.

A decision maker who holds a non-additive belief would like to use it in order to choose the best act. However, different integration methods might result in different evaluate different evaluations, and ultimately to different decisions. One of the advantages of the decomposition method is that it clarifies the trade-off between different essential properties. Once this trade-off is well formulated, the decision maker can compare between the various available integration methods and it is left for her to choose the integration method that owns the properties she desires.

Few essential properties are maintained by all decomposition-integrals, regardless of the particular vocabulary used. It is said that one random variable is greater than another if the former obtains a higher value than the latter in every possible state. It turns out that when one random variable is higher than another, its decomposition-integral is greater than that of the other. A similar property remains valid when comparing two capacities. A capacity is greater than another if it assigns every event a higher value than the other. Regardless of the vocabulary used, the decomposition-integral of the same random variable w.r.t two capacities maintains the order among the capacities. Further-

[4] No reference to the Riemann integral w.r.t. capacities was found in the literature.

more, decomposition-integral is homogeneous[5] and is independent of irrelevant events [6]. However, there are essential properties that are respected by some decomposition-integrals but not by other, depending on the vocabularies used.

We study in depth three essential properties: concavity (uncertainty-aversion), monotonicity w.r.t. first order stochastic dominance, and translation-invariance. It turns out, for instance, that uncertainty-aversion and monotonicity w.r.t. first order stochastic dominance cannot live together. Roughly speaking, the concave integral is the only plausible scheme that respects risk-aversion, while Choquet integral is the only plausible scheme that respects monotonicity w.r.t. first order stochastic dominance, as well as translation-invariance.

The paper also points to another advantage of the decomposition method. In various contexts Choquet integral is extended to domains that lie beyond classical capacities. For instance, Grabisch and Labreuche [4] introduced the notion of bicapacity which is consonant with the prospect theory of Kahneman and Tversky [6]. Bicapacities reflect different attitudes of decision makers toward gains and losses. Grabisch and Labreuche [5] define an integral that extends Choquet integral to the domain of bicapacities. As it turns out, the decomposition method provides a convenient manner to express this definition and to display its similarity with the classical definition.

Another non-classical domain is that of fuzzy capacities (see Lehrer [8]). It is shown that the decomposition approach allows for a natural way to expand Choquet integral to this terrain as well.

Other integral schemes and unifying approaches : Another well known concept for integration w.r.t. capacities is Sugeno integral [15], also known as the Fuzzy integral. When the capacity takes only the values zero and one (a simple game, in the terminology of cooperative games), Sugeno integral coincides with Choquet integral [11], but it does not coincide with the expected value when the capacity is additive. Sugeno integral is not generalized by the decomposition approach. That is, there is no vocabulary that induces a decomposition-integral which coincides with Sugeno integral.

Other unifying approaches were also proposed in the literature. One approach (see de Campos et al. [2]) unifies Choquet and Sugeno integrals through four essential properties. Another approach (see Klement et al. [7]), which builds on Choquet, Sugeno and Shilkret integrals, defines a universal integral. Both methods use different binary operations instead of the regular addition and multiplication, and both do not generalize the concave integral. It is worth noting also that these unifying approaches do not necessarily coincide with the Lebesgue integral (i.e., the expectation) when the underlying capacity is a probability distribution.

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On the family of marginal copulas: an analysis motivated by multi-attribute target-based utility

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1 Introduction

Let $n \in \mathbb{N}$ and a n-dimensional probability distribution function $F(x_1, ..., x_n)$ be fixed. For $A \subseteq \{1, ..., n\}$ we denote by $F_A(x_A)$ the marginal distribution functions

$$F_A(x_A) := \lim_{x_{\widetilde{A}} \to +\infty} F(x_A, x_{\widetilde{A}}) \quad ,$$

and consider functions $u : \mathbb{R}^n \to \mathbb{R}$ of the form

$$u(x_1, ..., x_n) = \sum_{A \subseteq \{1, ..., n\}} \widehat{u}_A F_A(x_A) \quad , \tag{1}$$

where, for $A \subseteq \{1, ..., n\}$, $\widehat{u}_A \in \mathbb{R}$ are given coefficients such that

$$\sum_{A \subseteq \{1, ..., n\}} \widehat{u}_A = 1 \quad .$$

In the theory of decisions under risk, functions of the form in (1) emerge as the (*multi-attribute*) utility functions suggested by the *Target-Based* approach. As very well-known, the *expected utility principle* prescribes to assign a utility function u and, then, to evaluate a risky *prospect* X (with its probability distribution F_X) in terms of the expected-utility

$$\mathbb{E}(u(X)) = \int u(x) F_X(dx) \quad .$$

In one-attribute decision problems, where X is a scalar random variable, the *Target-Based* approach suggests looking at a bounded utility function (once it has been normalized) as the probability distribution function F_T of a *target* T, a random variable independent of X:

$$u(x) = P(T \leq x) \tag{2}$$

and, under this position, the expected utility $\mathbb{E}(u(X))$ is given the interpretation

$$\mathbb{E}(u(X)) = \mathbb{E}(F_T(X)) = \int P(T \leq x) F_X(dx) = P(T \leq X) \quad . \tag{3}$$

In other words, the decision criterion amounts to choosing a target T and to evaluating the risky prospect X in terms of the probability $P(T \leq X)$. Following the first formalizations given by Castagnoli and Li Calzi and Bordley and Li Calzi (see [3], [1]), more and more papers have been devoted to analyzing different aspects of this approach. See in particular the papers by Bordley and Kirkwood [2] and Tsetlin and Winkler ([4] and [5]) and references cited therein. As first noticed by [2] for the multi-attribute case, a natural extension of (2) leads to selecting a multi-variate random target $\mathbf{T} = (T_1, ..., T_n)$, with joint distribution $F_{\mathbf{T}}$, and considering utility functions $u : \mathbb{R}^n \to \mathbb{R}$ of the form

$$u(x) = u(x_1, ..., x_n) = \sum_{A \subseteq \{1,...,n\}} u_A P\{T_A < x_A, T_{\widetilde{A}} \geq x_{\widetilde{A}}\} \tag{4}$$

with coefficients $u_A \in \mathbb{R}_+ \cup \{0\}$, such that

$$u_\emptyset = 0, \quad u_{\{1,...,n\}} = 1 \ ,$$

and satisfying the natural monotonicity condition

$$A' \subseteq A'' \Rightarrow u_{A'} \leq u_{A''} \ .$$

Notice that, at a first glance, the special choice

$$u(x_1, ..., x_n) = F_{\mathbf{T}}(x_1, ..., x_n)$$

(namely $u_A = 0$ for all $A \subset \{1, ..., n\}$) may appear as the most direct generalization of (2) to the multi-attribute case. However, the formula 1 it is actually too restrictive and this makes the position (4) much more natural (see, in particular the discussion presented in [2]).

On the other hand, (4) can be equivalently reduced to the form (1). Thus, the expected utility corresponding to the choice of a prospect $\mathbf{X} = (X_1, ..., X_n)$ can be written

$$\mathbb{E}\left(u(\mathbf{X})\right) = \sum_{A \subseteq \{1,...,n\}} \widehat{u}_A P\{T_A \leq X_A\} \ . \tag{5}$$

Such a conclusion shows that, when evaluating $\mathbf{X}$, the random vector of interest is $\mathbf{D} = \mathbf{T} - \mathbf{X}$.

Let $G_i(\xi)$ denote the marginal distribution function of D_i, for $i = 1, ..., n$, and put $\gamma = (\gamma_1, ..., \gamma_n)$ with

$$\gamma_i = G_i(0) \ .$$

Let furthermore C_A denote the connecting copula of the marginal distribution of $\mathbf{D}_A$, for $A \subset \{1, ..., n\}$. Then (5) becomes

$$\mathbb{E}\left(u(\mathbf{X})\right) = \sum_{A \subseteq \{1,...,n\}} \widehat{u}_A C_A\left(\gamma_A\right) \ . \tag{6}$$

This formula highlights that, concerning the joint distribution of $\mathbf{D}$, we only need to specify the vector γ and $C = C_{\{1,...,n\}}$, the connecting copula of $\mathbf{D}$. From C, we can derive the family of all marginal copulas C_A.

The present talk will be devoted to analyzing different aspects of the quantity in (6). The form of stochastic dependence of the different copulas C_A clearly has an important role in the analysis of $\mathbb{E}\left(u(\mathbf{X})\right)$.

We in particular aim to analyze comparisons between two different prospects $\mathbf{X}', \mathbf{X}''$ in terms of (6) and of the marginal copulas C'_A, C''_A ($A \subseteq \{1, ..., n\}$) of $\mathbf{X}', \mathbf{X}''$.

Interesting results in this direction have already been pointed out in [5], for the case $n = 2$ (see Proposition 3 and Proposition 4).

Their analysis can be extended to the case $n > 2$, by comparing dependence of C'_A and C''_A. On this purpose a detailed study of the coefficients $\widehat{u}_A$ is needed. We notice, in this respect, that only the coefficients u_A are given exogenously and $\widehat{u}_A$ are to be derived from the knowledge of them. Generally, $\widehat{u}_A$ will be negative for some A.

Such a study can turn out as useful, furthermore, in the analysis of risk-attitude and "risk-aversion"-type properties for the utilities in (1).

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Towards a betting interpretation for belief functions on MV-algebras

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The well known de Finetti's coherence criterion [1], that founds probability theory in terms of betting games between a gambler and a swappable bookmaker, has been generalized to the case of states on MV-algebras by Mundici in [8] and by Kühr and Mundici in [6].

Classical Dempster-Shafer belief functions are a generalization of both probability measures and possibility and necessity measures. In this setting, an interpretation of belief functions in terms of a betting scheme on (Boolean) events have been provided by Jaffray [4] and Paris [9]. On the other hand, in [5, 3, 2] a generalization of belief function theory in the frame of MV-algebras has been proposed. The main idea of this approach is to define a belief function $\mathbf{b}$ over an MV-algebra of fuzzy sets $M = [0,1]^X$ (where X is a finite set of cardinality k that represents the set of possible worlds we will take in consideration) as a state over a separable MV-subalgebra $\mathcal{R}$ of $[0,1]^M$, that strictly contains the free MV-algebra over k generators $Free(k)$ (see [2]). More precisely, we call a mapping $\mathbf{b} : M \to [0,1]$ a *generalized belief function* if there is an state $\mathbf{s} : \mathcal{R} \to [0,1]$ such that, for every $f \in M$,

$$\mathbf{b}(f) = \mathbf{s}(\rho_f),$$

where $\rho_f : M \to [0,1]$ is defined as

$$\rho_f(g) = \inf_{x\in X} g(x) \Rightarrow f(x),$$

with $\Rightarrow$ being Łukasiewicz implication function in the standard MV-algebra $[0,1]_{MV}$. It is worth noticing that, in case the state $\mathbf{s}$ has a countable support, then the belief function $\mathbf{b}$ on an event f can be expressed as $\mathbf{b}(f) = \sum_{g\in M} \rho_f(g) \cdot m(g)$, where the mass $m(g)$ is nothing but $\mathbf{s}(\{g\})$.

In this work we will apply the generalized coherence criterion provided in [6] to provide a betting scheme interpretation for belief functions on MV-algebras.

The necessary modification for that criterion to apply to this general case regards the special behavior on how events are evaluated on possible worlds describing a partial information state. As a matter of fact we will allow possible worlds to partially or totally overlap, so to generate *superpositions* of (many-valued) evaluations. Each of these resulting imprecise worlds will be uniquely generated by a normalized possibility distribution $\pi : X \to [0,1]$ on the set of completely informed worlds X, and the evaluations of events in these imprecise worlds (defined by generalized necessity measures) will be henceforth denoted by N_π. Therefore, the events we will take into consideration will be now realized in each of these imprecise worlds and evaluated by the N_π's functions and not by truth-functional valuations. In what follows we will explain the basic construction and the main result of our contribution.

Let X be a finite set of possible worlds, and let $M = [0,1]^X$ be the MV-algebra of all functions from X into $[0,1]$. We will interpret each function $f \in M$ as an event where, for every $x \in X$, $f(x)$ denotes the truth value of the event f in x.

Fix a finite set of events $\{f_1, \dots, f_n\}$ and an assignment $\alpha : f_i \to \alpha_i \in [0,1]$. Then for each $x \in X$, let

$$\mathbf{x} = f(x) = (f_1(x), \dots, f_n(x)) \in [0,1]^n.$$

The class of $\mathbf{X} = \{\mathbf{x} = (f_1(x), \dots, f_n(x)) \in [0,1]^n : x \in X\}$ is hence a finite subset (of cardinality $|X| = k$) of points in the n-cube $[0,1]^n$.

Let $\Pi = \{\pi : X \to [0,1] \mid \max\{\pi(x) : x \in X\} = 1\}$ be the class of all *normalized* possibility distributions over X. For every $\mathbf{x} \in \mathbf{X}$ and $\pi \in \Pi$, let

$$\mathbf{x}^\pi = (f_1(x) + (1 - \pi(x)), \dots, f_n(x) + (1 - \pi(x))).$$

and for every $\pi \in \Pi$, consider the point $\mathbf{n}^\pi \in [0,1]^n$ such that, for every $j = 1, \dots, n$, its j-th projection is:

$$\mathbf{n}^\pi(j) = N_\pi(f_i) = \bigwedge_{x \in X} \mathbf{x}^\pi(j). \tag{1}$$

that is,

$$\mathbf{n}^\pi = (N_\pi(f_1), \dots, N_\pi(f_n)) \tag{2}$$

Let us denote by $\mathbf{N}^\Pi$ the set of all the points $\mathbf{n}^\pi$ defined trough the above equation (1).

Let now $\alpha : \{f_1, \dots, f_n\} \to [0,1]$ be an assignment and write $\alpha_i = \alpha(f_i)$. Then we say that α is *B-coherent* iff for all $\sigma_1, \dots, \sigma_n$ there exists a $\mathbf{n}^\pi \in \mathbf{N}^\Pi$ such that

$$\begin{aligned}\textstyle\sum_{i=1}^n \sigma_i(\alpha_i - \mathbf{n}^\pi(i)) &= \textstyle\sum_{i=1}^n \sigma_i(\alpha_i - N_\pi(f_i)) \\ &= \textstyle\sum_{i=1}^n \sigma_i(\alpha_i - (\bigwedge_{x \in X} f_i(x) + (1 - \pi(x)))) \\ &\geq 0.\end{aligned}$$

Notice that, comparing this expression with the one for states in [6], the role truth-evaluations play there is played here by the *necessity-evaluations* $\mathbf{N}^\Pi$.

Theorem 1. *Let $f_1, \dots, f_n$ be events, and $\alpha : f_i \mapsto \alpha_i$ be an assignment. Then α is B-coherent iff there exists a belief function $\mathbf{b} : [0,1]^X \to [0,1]$ extending α.*

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Some new results concerning local and relativized local finiteness in t-norm-based structures

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An algebraic structure is *locally finite* iff each of its finite subset generates a finite subalgebra only.

In this short note we discuss local finiteness properties for t-norm based structures and extend some former results presented in this seminar in 2011.

For the case of left-continuous t-norms there are some structural properties which guarantee local finiteness of such structures, particularly for t-norm-bimonoids.

And also for the case of the relativization of the local finiteness property to suitable subsets of the carrier of the t-norm based structures more general results are available.

The core of games with restricted cooperation

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In cooperative game theory, for a given set of players N, TU-games are functions $v : 2^N \to \mathbb{R}$ which express for each nonempty coalition $S \subseteq N$ of players the best they can achieve by cooperation. Capacities, widely used in decision making, are monotone TU-games.

In the classical setting, every coalition may form without any restriction, i.e., the domain of v is indeed 2^N. In practice, this assumption is often unrealistic, since some coalitions may not be feasible for various reasons, e.g., players are political parties with divergent opinions, or have restricted communication abilities, or a hierarchy exists among players, and the formation of coalitions must respect the hierarchy, etc.

Many studies have been done on games defined on specific subdomains of 2^N, e.g., antimatroids [1], convex geometries [3,4], distributive lattices [6], or others [2,5]. In this paper, we focus on the case of distributive lattices. To this end, we assume that there exists some partial order $\preceq$ on N describing some hierarchy or precedence constraint among players, as in [6]. We say that a coalition S is feasible if the coalition contains all its subordinates, i.e., $i \in S$ implies that any $j \preceq i$ belongs to S as well. Then by Birkhoff's theorem, feasible coalitions form a distributive lattice. From now on, we denote by $\mathcal{F}$ the set of feasible coalitions, assuming that $\emptyset, N \in \mathcal{F}$.

The main problem in cooperative game theory is to define a rational solution of the game, that is, supposing that the grand coalition N will form, how to share among its members the total worth $v(N)$. The core is the most popular solution concept, since it ensures stability of the game, in the sense that no coalition has an incentive to deviate from the grand coalition. In the field of decision making, the core of capacities is also a well-known concept, as it is the set of probability measures compatible with the capacity. For a game v on a family $\mathcal{F}$ of feasible coalitions, the core is defined by

$$\mathcal{C}(v) = \{x \in \mathbb{R}^n \mid x(S) \geq v(S), \forall S \in \mathcal{F}, x(N) = v(N)\}$$

where $x(S)$ is a shorthand for $\sum_{i \in S} x_i$. When $\mathcal{F} = 2^N$, the core is either empty or a convex bounded polyhedron. However, for games whose cooperation is restricted, the study of the core becomes much more complex, since it may be unbounded or even contain no vertices (see a survey in [7]). For the case of games with precedence constraints, it is known that the core is *always* unbounded or empty, but contains no line (i.e., it has vertices). The problem arises then, to select a significant bounded part of the core as a reasonable concept of solution, since unbounded payments make no sense.

A simple remedy to this problem is to select a bounded face of the core, by imposing additional equality constraints $x(S) = v(S)$ for all S in some collection $\mathcal{N}$, so that to exclude any extremal ray in the core [11,8]. We call $\mathcal{N}$ a normal collection, making the convention that $N \notin \mathcal{N}$, and we call *restricted core* w.r.t. $\mathcal{N}$ the resulting polytope, denoted by $\mathcal{C}_{\mathcal{N}}(v)$. Taking the union of all possible restricted cores (i.e., all possible bounded faces) gives the so-called *bounded core*, denoted by $\mathcal{C}^b(v)$ [10].

Within the set of normal collections, those which are *nested*, i.e., which form a chain in $\mathcal{F}$ are of particular importance. Also, we consider *minimal* normal collections, i.e., for which no subcollection is normal. We denote by $\mathcal{MNNCF}(\mathcal{F}$ the set of minimal nested normal collections. There exist remarkable normal collections, obtained by simple algorithms operating on the minimal or maximal elements of $(N, \preceq)$, the partial order on the players.

The case of convex games is of particular interest, since the bounded core can be expressed in an irredundant way. Our main result is the following [9].

Theorem 1. *1. For any convex game v and any nested normal collection $\mathcal{N}$ of $\mathcal{F}$, $\mathcal{C}_{\mathcal{N}}(v) \neq \emptyset$. Moreover, if v is strictly convex, then* $\dim \mathcal{C}_{\mathcal{N}}(v) = n - |\mathcal{N}| - 1$.

2. *For any convex game v,*

$$\mathcal{C}^b(v) = \bigcup_{\mathcal{N} \in \mathcal{MNNC}(\mathcal{F})} \mathcal{C}_{\mathcal{N}}(v).$$

   Moreover, no term in the union is redundant if v is strictly convex.

3. *Let $\mathcal{N}$ be a normal collection of $\mathcal{F}$. If v is strictly convex, then $\mathcal{C}_{\mathcal{N}}(v) \neq \emptyset$ if and only if $\mathcal{N}$ is nested.*

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Modeling influence by aggregation functions

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Influence and opinion formation are broadly studied in several scientific fields. Important contributions to the study of these issues can be found in the literature on dynamic aspects of influence; see, e.g., [3] for an overview of dynamic models of imitation and social influence. One of the leading models of opinion and consensus formation is due to [1]. In his model, every individual in a society has an initial opinion on a subject, represented by a number in $[0,1]$, and he aggregates the opinions of other individuals through a weighted arithmetic mean. The interaction patterns are described by a stochastic matrix whose entries represent weights that an agent places on the current opinions of other agents in forming his own opinion for the next period. The opinions are updated over time. Results in Markov chain theory are easily adapted to the model. Several works have been devoted to the DeGroot framework and its different variations have been proposed. However, although the literature on influence and opinion formation is quite vast, most of the related works assume a convex combination as the way of aggregating opinions.

In this paper we investigate a new approach to influence based on *aggregation functions*. The point of departure is a one-step model ([2]) in which agents make a yes-no decision on a certain issue. While each agent has his preliminary opinion (inclination), he may decide differently from that inclination, due to influence between agents. The present paper extends our previous research on influence in several aspects. While influence functions considered so far were deterministic and the framework was a decision process after a single step of influence, we consider now a dynamic influence mechanism which is assumed to be stochastic and to follow a Markov chain.

There are three main contributions of the present paper to the study of influence, and basically three advantages of our framework over other existing models. First of all, we introduce and analyze a new framework of influence based on arbitrary aggregation functions, which to the best of our knowledge has not been proposed before. Each agent modifies his opinion by aggregating the current opinion of all agents (possibly including himself) according to his aggregation function. The framework covers numerous existing models of opinion formation, since we allow for arbitrary aggregation functions. We provide a general analysis of convergence in the aggregation model and find all terminal classes and states. First, we show that possible terminal classes to which the process of influence may converge are terminal states (the consensus states and non trivial states), cyclic terminal classes, and unions of Boolean lattices (called regular terminal classes). Next, we use the concepts of *influential agent* and *graph of influence*. Roughly speaking, an agent j is yes- (or no-) influential for agent i if the opinion of j matters for i. The graph of yes-influence (no-influence) is a directed graph whose nodes are the agents and there is an arc from j to i if j is yes-influential (no-influential) for i. A direct generalization of these notions leads to *influential coalitions* and *hypergraphs of* yes- (or no-) *influence*. It appears that the qualitative description of the convergence is entirely described by the hypergraphs of influence. Based on properties of the hypergraphs and influential coalitions we determine conditions for the existence of the different types of terminal classes. Furthermore, we study a specific family of aggregation models – the family of symmetric decomposable models, in which all influential coalitions are singletons and the graphs of yes- and no-influence coincide. Terminal classes in such models are analyzed.

The second advantage of the present model concerns the reduction of complexity. We assume that the influence mechanism is a Markovian process. Consequently, for the analysis of the qualitative convergence in a model with n agents we need the information on all entries of the $2^n \times 2^n$ (reduced) transition matrix. While the Markovian model of influence is exponentially complex, the subfamily based on aggregation functions is of polynomial complexity. Indeed, in order to determine all terminal classes in the aggregation model we only need to know the hypergraphs of yes- and no-influence, whose maximal size is $2n\binom{n}{\lfloor n/2 \rfloor}$. Note that the size difference between the Markovian model of influence and the aggregation model drastically increases with n.

The third advantage of the present model is related to practical considerations and applicability of the model. When we know exactly how each agent aggregates the current opinions of others when modifying his own opinion for the next step and how they are correlated, we can provide the full analysis of convergence. However, in practice, we frequently do not know how the aggregation is done by the agents. In our model, for the analysis of the qualitative convergence we do not need the full information on the agents' aggregation functions. What we only need to know are all influential coalitions, but this information can usually be obtained by observing the influence process.

In order to show the advantages of the aggregation framework over other existing models we study an empirical example based on the advice network of [4]. He collected data from managers of a small manufacturing firm in the US about who sought advice from whom. Based on these data, [3] developed a social influence matrix as defined in the context of the DeGroot model. We apply our approach to the same advice network of [4] and provide the convergence analysis of the example. Moreover, for simplicity of the illustration and discussion of our results, we additionally analyze a smaller example of the advice network with 3 managers who have to decide whether to introduce a new technology in the company. The discussion on that issue may take many rounds and every manager may seek advice from the others before each round. Apart from the classical approach of weighted averaging aggregation, one can easily imagine different ways of aggregating the opinions by the managers.

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Axiomatic foundations of the universal integral in terms of aggregation functions and preference relations

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Abstract. The concept of universal integral has been recently proposed in order to generalize the Choquet, Shilkret and Sugeno integrals. We present two axiomatic foundations of the universal integral. The first axiomatization is expressed in terms of aggregation functions, while the second is expressed in terms of preference relations.

1 Basic concepts

For the sake of simplicity in this note we present the result in a Multiple Criteria Decision Making (MCDM) setting (for a state of art on MCDM see [1]). Let $N = \{1,\ldots,n\}$ be the set of criteria and let us identify the set of possible alternatives with $[0,1]^n$. For all $\boldsymbol{x} = (x_1 \ldots, x_n) \in [0,1]^n$, the set $\{i \in N \mid x_i \geq t\}, t \in [0,1]$, is briefly indicated with $\{\boldsymbol{x} \geq t\}$. For all $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ we say that $\boldsymbol{x}$ dominates $\boldsymbol{y}$ and we write $\boldsymbol{x} \succeq \boldsymbol{y}$ if $x_i \geq y_i,\ i = 1,\ldots,n$. An aggregation function $f : [0,1]^n \to \mathbb{R}$ is a function such that, $\inf_{\boldsymbol{x}\in[0,1]^n} f = 0$, $\sup_{\boldsymbol{x}\in[0,1]^n} f = 1$, and $f(\boldsymbol{x}) \geq f(\boldsymbol{y})$ whenever $\boldsymbol{x} \succeq \boldsymbol{y}$ [2]. Let $\mathcal{A}_{[0,1]^n}$ be the set of aggregation functions on $[0,1]^n$.
Let M denotes the set of all capacities m on N, i.e. for all $m \in M$ we have $m : 2^N \to [0,1]$ satisfying the following conditions:

– *boundary conditions*: $m(\emptyset) = 0, m(N) = 1$;
– *monotonicity*: $m(A) \leq m(B)$ for all $\emptyset \subseteq A \subseteq B \subseteq N$.

A *universal integral* [3] is a function $I : M \times [0,1]^n \to [0,1]$ satisfying the following properties:

(UI1) I is non-decreasing in each coordinate,
(UI2) there exists a pseudo-multiplication $\otimes$ (i.e. $\otimes : [0,1]^2 \to [0,1]$ is nondecreasing in its two coordinates and $\otimes(c,1) = \otimes(1,c) = c$) such that for all $m \in M$, $c \in [0,1]$ and $A \subseteq N$,

$$I(m, c\mathbf{1}_A) = \otimes(c, m(A)),$$

(UI3) for all $m_1, m_2 \in M$ and $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$, if $m_1(\{\boldsymbol{x} \geq t\}) = m_2(\{\boldsymbol{y} \geq t\})$ for all $t \in]0,1]$, then $I(m_1, \boldsymbol{x}) = I(m_2, \boldsymbol{y})$.

Given a universal integral I with respect to the pseudomultiplication $\otimes$, we shall write

$$I(m, \boldsymbol{x}) = \int_{univ,\otimes} \boldsymbol{x}\, dm$$

for all $m \in M, \boldsymbol{x} \in [0,1]^n$.

2 Axiomatic foundation in terms of aggregation functions

Consider a family $\mathcal{F} \subseteq \mathcal{A}$ with $\mathcal{F} \neq \emptyset$ and consider the following axioms on $\mathcal{F}$:

(A_1) For all $f_1, f_2 \in \mathcal{F}$ and $\boldsymbol{x}, \boldsymbol{y} \in [0,1]^n$ such that for all $t \in [0,1]$

$$f_1\left(\mathbf{1}_{\{\boldsymbol{x} \geq t\}}\right) \geq f_2\left(\mathbf{1}_{\{\boldsymbol{y} \geq t\}}\right),$$

then $f_1(\boldsymbol{x}) \geq f_2(\boldsymbol{y})$;

(A_2) Every $f \in \mathcal{F}$ is idempotent, i.e. for all $c \in [0,1]$ and $f \in \mathcal{F}$,

$$f(c \cdot \mathbf{1}_N) = c;$$

(A_3) For all $m \in M$ there exists $f \in \mathcal{F}$ such that $f(1_A) = m(A)$ for all $A \subseteq N$.

Proposition 1. *Axioms (A_1), (A_2) and (A_3) hold if and only if there exists a universal integral I with a pseudo-multiplication $\otimes_{\mathcal{F}}$ such that, for all $f \in \mathcal{F}$ there exists an $m_f \in M$ for which*

$$f(\boldsymbol{x}) = \int_{univ,\otimes_{\mathcal{F}}} \boldsymbol{x}\, dm_f \qquad \text{for all} \quad \boldsymbol{x} \in [0,1]^n,\ f \in \mathcal{F}.$$

More precisely, for all $f \in \mathcal{F}$ and for all $A \subseteq N$, $m_f(A) = f(\boldsymbol{1}_A)$ and for all $a, b \in [0,1], \otimes_{\mathcal{F}}(a,b) = f(a\boldsymbol{1}_B)$ if $f(\boldsymbol{1}_B) = b$, with $B \subseteq N$.

Remark 1. One can weaken axiom (A_3) as follow.

(A_4) For all $c \in [0,1]$ there exist $A \subseteq N$ and $f \in \mathcal{F}$ such that $f(\mathbf{1}_A) = c$.

In this case above Proposition 1 holds provided that the universal integral is no more defined as a function $I : M \times [0,1]^n \to [0,1]$, but as a function $I : M_{\mathcal{F}} \times [0,1]^n \to [0,1]$ with $M_{\mathcal{F}} \subseteq M$. More precisely, we have $M_{\mathcal{F}} = \left\{m_f | f \in \mathcal{F}\right\}$.

3 Axiomatic foundation in terms of preference relations

We consider the following primitives:

- a set of outcomes X,
- a set of binary preference relations $\mathcal{R} = \{\succsim_t, t \in \mathbf{T}\}$ on $X^n, n \in \mathbf{N}$.

In the following

- we shall denote by α the constant vector $[\alpha, \alpha, \ldots, \alpha] \in X^n$, with $\alpha \in X$;
- we shall denote by $\succ_t$ and $\sim_t$ the asymmetric and the symmetric part of $\succsim_t \in \mathcal{R}$, respectively;
- we shall denote by (α_A, β_{N-A}), $\alpha, \beta \in X, A \subset N$, $\boldsymbol{x} \in X^n$ such that $x_i = \alpha$ if $i \in A$ and $x_i = \beta$ if $i \notin A$.

We consider the following axioms:

A1) $\succsim_t$ is a complete preorder on X^n for all $\succsim_t \in \mathcal{R}$.
A2) For all $\alpha, \beta \in X$ and for all $\succsim_t, \succsim_r \in \mathcal{R}$, $\alpha \succsim_t \beta \Rightarrow \alpha \succsim_r \beta$.
A3) X is infinite and there exists a countable subset $A \subseteq X$ such that for all $\succsim_t \in \mathcal{R}$, for all $\alpha, \beta \in X$ for which $\alpha \succ_t \beta$ there is $\gamma \in A$ such that $\alpha \succsim_t \gamma \succsim_t \beta$.
A4) There are $1, 0 \in X$ such that for all $\succsim_t \in \mathcal{R}$ $\mathbf{1} \succ_t \mathbf{0}$ and for all $\boldsymbol{x} \in X^n$,

$$\mathbf{1} \succsim_t \boldsymbol{x} \succsim_t \mathbf{0}.$$

A5) For each $\boldsymbol{x} \in X^n$ and for each $\succsim_t \in \mathcal{R}$, there exists $\alpha \in X$ such that $\boldsymbol{x} \sim_t \alpha$.
A6) For all $\boldsymbol{x}, \boldsymbol{y} \in X^n$, $\succsim_t, \succsim_r, \succsim_s \in \mathcal{R}$,

$$[(1_{\{i \in N: x_i \succsim_t \alpha\}}, 0_{N-\{i \in N: x_i \succsim_t \alpha\}}) \succsim_r \beta \Rightarrow (1_{\{i \in N: y_i \succsim_t \alpha\}}, 0_{N-\{i \in N: y_i \succsim_t \alpha\}}) \succsim_s \beta, \forall \alpha, \beta \in X]$$
$$\Rightarrow$$
$$[\boldsymbol{x} \succsim_r \gamma \Rightarrow \boldsymbol{y} \succsim_s \gamma, \forall \gamma \in X].$$

A7) For all $\mathcal{A} = \{\alpha_1, \ldots, \alpha_{2^n-2}\} \subset X$ there exists $\succsim_t \in \mathcal{R}$ such that for all $\alpha \in \mathcal{A}$ there is A, $\emptyset \subset A \subset N$, for which $\alpha \sim_t 1_A$.

Theorem. Conditions $A1) - A7)$ hold if and only there exist

- a function $u : X \to [0, 1]$,
- a bijection between $\mathcal{R}$ and M for which each $\succsim_t \in \mathcal{R}$ corresponds to one capacity $\mu_t \in M$,
- a pseudo-multiplication $\otimes$,

such that, for all $\boldsymbol{x}, \boldsymbol{y} \in X^n$ and for all $\succsim_t \in \mathcal{R}$

$$\boldsymbol{x} \succsim_t \boldsymbol{y} \Leftrightarrow \int_{univ,\otimes} \mathbf{u}(\boldsymbol{x}) d\mu_t \geq \int_{univ,\otimes} \mathbf{u}(\boldsymbol{y}) d\mu_t,$$

where $\mathbf{u}(\boldsymbol{x}) = [u(x_1), \ldots, u(x_n)]$ and $\mathbf{u}(\boldsymbol{y}) = [u(y_1), \ldots, u(y_n)]$.

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Bipolar semicopulas

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Abstract. The concept of semicopula plays a fundamental role in the definition of a universal integral. We present an extension of semicopula to the case of symmetric interval $[-1,1]$. We call this extension bipolar semicopula. The last definition can be used to obtain a simplified definition of the bipolar universal integral. Moreover bipolar semicopulas allow for extension of theory of copulas to the interval $[-1,1]$.

1 Bipolar semicopulas

Definition 1. *A semicopula is a function* $\otimes : [0,1]\times[0,1] \to [0,1]$, *which is nondecreasing and has 1 as neutral element, i.e.*

- *if* $a_1 \le a_2$ *and* $b_1 \le b_2$, *then* $a_1 \otimes b_1 \le a_2 \otimes b_2$; *and*
- $1 \otimes a = a \otimes 1 = a$.

Note that a semicopula has 0 as annihilator. Indeed $0 \le a \otimes 0 \le 1 \otimes 0 = 0$ and $0 \le 0 \otimes a \le 0 \otimes 1 = 0$.

Definition 2. *A bipolar semicopula is a function*

$$\otimes_b : [-1,1]^2 \to [-1,1]$$

that is "absolute-nondecreasing", has 1 as neutral element and -1 *as opposite-neutral element, and preserves the sign rule, i.e*

(A1) *if* $|a_1| \le |a_2|$ *and* $|b_1| \le |b_2|$ *then* $|a_1 \otimes_b b_1| \le |a_2 \otimes_b b_2|$;
(A2) $a \otimes_b \pm 1 = \pm 1 \otimes_b a = \pm a$; *and*
(A3) $sign\,(a \otimes_b b) = sign(a) \otimes_b sign(b)$.

Let us note that a bipolar semicopula also satisfies the following additional properties

(A4) $a \otimes_b 0 = 0 \otimes_b a = 0$;
(A5) $sign(a) \otimes_b sign(b) = sign(a \cdot b)$; and
(A6) $|a \otimes_b b| = |a| \otimes_b |b|$.

Indeed, $0 \le |a \otimes_b 0| \le |\pm 1 \otimes_b 0| = |\pm 0| = 0$ and $0 \le |0 \otimes_b a| \le |0 \otimes_b \pm 1| = |\pm 0| = 0$. $(A5)$ is true by $(A4)$ if $a = sign(a) = 0$ or $b = sign(b) = 0$, while is true by $(A2)$ and $(A3)$ if $a = sign(a), b = sign(b) \in \{-1, 1\}$. Regarding $(A6)$, it is sufficient to note that for all $a \in [0,1]$, $|-a| \le |a| \le |-a|$, then $|\pm a \otimes_b (\pm b)| = |a \otimes_b b|$.

Let us consider the binary operation $*$ on $[-1,1]$ given by

$$a * b = \begin{cases} -ab \ if \ (a,b) \ \in]-1,1[^2 \\ ab \quad else. \end{cases}$$

This satisfies axioms $(A1)$ and$(A2)$, but not $(A3)$ (think to $a = -1/3 = b$), then the additional axiom $(A3)$ is necessary in order to consider bipolar semicopulas as symmetric extensions of standard semicopulas in the sense of product. Note that this approach preserves commutativity and associativity.

Notable examples of bipolar semicopulas are the standard product, $a \cdot b$ and the symmetric minimum [1, 2],

$$a \oslash b = sign(a \cdot b)(|a| \wedge |b|).$$

Proposition 1. *$\otimes_b : [-1,1]^2 \to [-1,1]$ is a bipolar semicopula if and only if there exists a semicopula $\otimes : [0,1]^2 \to [0,1]$ such that for all $a, b \in [-1,1]$*

$$a \otimes_b b = sign(a \cdot b)\,(|a| \otimes |b|)\,. \tag{1}$$

Proof. Sufficient part. Suppose there exists a semicopula $\otimes$ such that (1) holds. If $|a_1| \le |a_2|$ and $|b_1| \le |b_2|$, then $|a_1| \otimes |b_1| \le |a_2| \otimes |b_2|$, i.e. $|a_1 \otimes_b b_1| \le |a_2 \otimes_b b_2|$. Moreover, $a \otimes_b \pm 1 = sign(a \cdot (\pm 1))\,(|a| \otimes 1) = \pm a$ and $\pm 1 \otimes_b a = sign(\pm 1 \cdot a)\,(1 \otimes |a|) = \pm a$. Proof of $(A3)$ is trivial and then, we conclude that $\otimes_b$ is a bipolar semicopula. Necessary part. Suppose $\otimes_b$ is a bipolar semicopula and define $a \otimes b = a \otimes_b b$ for all $a, b \in [0,1]$, then $a \otimes_b b = sign(a \cdot b)|a \otimes_b b| = sign(a \cdot b)\,(|a| \otimes_b |b|) = sign(a \cdot b)\,(|a| \otimes |b|)$.

We call $\otimes_b$ the bipolar semicopula induced by the semicopula $\otimes$ whenever the (1) holds. For example, the semicopula product induces the bipolar semicopula product, the semicopula minimum induces the bipolar semicopula symmetric minimum. Finally let us note that the concept of bipolar semicopula is closely related to that of *symmetric pseudo-multiplication* in [3].

2 Bipolar semicopula and bipolar universal integral

For the sake of simplicity in this note we present the result in a multiple criteria decision making setting. Let $N = \{1, \ldots, n\}$ be the set of criteria and let us identify the set of possible alternatives with $[-1,1]^n$. The definition of bipolar semicopulas can be used in order to define the bipolar universal integral [7] which is a generalization of the universal integral [9] from the scale $[0,1]$ to the symmetric scale $[-1,1]$. Let us consider the set of all disjoint pairs of subsets of N, i.e. $\mathcal{Q} = \left\{(A,B) \in 2^N \times 2^N \ : \ A \cap B = \emptyset\right\}$.

Definition 3. *A function $m_b : \mathcal{Q} \to [-1,1]$ is a normalized bi-capacity ([4], [5], [6]) on N if*

– $m_b(\emptyset,\emptyset) = 0$, $m_b(N,\emptyset) = 1$ *and* $m_b(\emptyset,N) = -1$;
– $m_b(A,B) \le m_b(C,D)\ \forall\ (A,B),(C,D) \in Q\ :\ A \subseteq C$ *and* $B \supseteq D$.

Definition 4. *Let F_b be the set of functions $f : N \to [-1,1]$ and M_b the set of bi-capacities on Q. A function $I_b : M_b \times F_b \to [-1,1]$ is a bipolar universal integral on the scale $[-1,1]$ (or bipolar fuzzy integral) if the following axioms hold:*

(I1) $I_b(m_b, f)$ *is nondecreasing with respect to* m_b *and with respect to* f*;*
(I2) There exists a bipolar semicopula $\otimes_b$ *such that for any* $m_b \in M_b$, $c \in [0,1]$ *and* $(A,B) \in Q$, $I(m_b, c \cdot 1_{(A,B)}) = c \otimes_b m_b(A,B)$*;*
(I3) for all pairs $(m_{b_1}, f_1), (m_{b_2}, f_2) \in M_b \times F_b$*, such that for all* $t \in [0,1]$,
$m_{b_1}(\{i \in N : f_1(i) \ge t\}, \{i \in N : f_1(i) \le -t\}) =$
$= m_{b_2}(\{i \in N : f_2(i) \ge t\}, \{i \in N : f_2(i) \le -t\})$, $I(m_{b_1}, f_1) = I(m_{b_2}, f_2)$.

Clearly, in definition 4, F_b can be identified with $[-1,1]^n$, such that a function $f : N \to [-1,1]$ can be regarded as a vector $\boldsymbol{x} \in [-1,1]^n$. Note that the bipolar Choquet, Shilkret and Sugeno integrals [8] are bipolar universal integrals in the sense of Definition 4. Observe that the underlying bipolar semicopula $\otimes_b$ is the standard product in the case of the bipolar Choquet and Shilkret integrals, while $\otimes_b$ is the symmetric minimum for the bipolar Sugeno integral.

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h-k-aggregation functions, measures and integrals

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1 Introduction

We consider h-intervals $[a_1,\dots,a_h], a_1,\dots,a_h \in \mathbb{R}$ such that $a_1 \leq \dots \leq a_h$ that express evaluations with respect to a considered point of view by means of the h-values $a_1,\dots,a_h$. For example, if $h=2$, then evaluations are 2-intervals assigning to each criterion two evaluations corresponding to a pessimistic and an optimistic evaluation. If $h=3$, then evaluations are 3-intervals $[a_1,a_2,a_3]$ assigning to each criterion three evaluations such that a_1 corresponds to a pessimistic evaluation, a_2 corresponds to an average evaluation and a_3 corresponds to an optimistic evaluation. If $h=4$, then evaluations are 4-intervals $[a_1,a_2,a_3,a_4]$ assigning to each criterion four evaluations such that a_1 corresponds to a pessimistic evaluation, a_2 and a_3 to two evaluations defining an interval $[a_2,a_3]$ of average evaluation and a_4 corresponds to an optimistic evaluation. Observe that 2-interval evaluations can be seen as usual intervals of evaluations, 3-interval evaluations can be seen as triangular fuzzy numbers and 4-intervals evaluations can be seen as trapezoidal fuzzy numbers. Similar situations we have with $h \geq 5$. Let us denote by I_h the set of all h-intervals, i.e.

$$I_h = \{[a_1,\dots,a_h] \mid a_1,\dots,a_h \in \mathbb{R}, a_1 \leq \dots \leq a_h\}.$$

A general framework for the comparison of h-intervals has been presented in [2]. Here we introduce h-k-aggregation functions that assigns to vectors

$$\boldsymbol{x} = ([x_{11},\dots,x_{1h}],\dots,[x_{n1}\dots,x_{nh}]) \in I_h^n$$

of h-interval evaluations with respect to a set $N=\{1,\dots,n\}$ of considered criteria an overall evaluation in terms of a $k-$interval. Formally an h-k-aggregation function is a function $g: I_h^n \to I_k$ satisfying the following properties:

- monotonicity: for all $\boldsymbol{x},\boldsymbol{y} \in I_h^n$, if $x_{i,j} \geq y_{i,j}$ for all $i \in N$ and for all $j=1,\dots,h$, then $g_r(\boldsymbol{x}) \geq g_r(\boldsymbol{y})$ for all $r=1,\dots,k$;
- left boundary condition: if $x_{i,h} \to -\infty$ for all $i=1,\dots,n$, then $g_r(\boldsymbol{x}) \to -\infty$ for all $r=1,\dots,k$;
- right boundary condition if $x_{i,1} \to +\infty$ for all $i=1,\dots,n$, then $g_r(\boldsymbol{x}) \to +\infty$ for all $r=1,\dots,k$;

2 The h-k-weighted average

Let us consider a vector $\mathbf{a} = [a_{i,j,r}], \mathbf{a} \in [0,1]^{n\times h\times k}$ such that

- $\sum_{j=h-t}^{h} a_{i,j,r_1} \geq \sum_{j=h-t}^{h} a_{i,j,r_2}$, for all $i = 1,\ldots,n$, $t = 1,\ldots h-1$ and $r_1, r_2 = 1,\ldots,k$, such that $r_1 \geq r_2$;
- $\sum_{i=1}^{n} \sum_{j=1}^{h} a_{i,j,r} = 1$, for all $r = 1,\ldots,k$.

The h-k-weighted average with respect to the weights $\mathbf{a} = [a_{i,j,r}]$ is the h-k-aggregation function $WA_{\mathbf{a}} : I_h^n \to I_k$ defined as follows: for all $\boldsymbol{x} \in I_h^n$ and $r = 1,\ldots,k$,

$$WA_{\mathbf{a},r}(\boldsymbol{x}) = \sum_{i=1}^{n} \sum_{j=1}^{h} a_{i,j,r} x_{i,j}. \tag{1}$$

The h-k-weighted average can be formulated also as follows. Let us consider a vector $\mathbf{a}' = [a'_{i,j,r}], \mathbf{a}' \in [0,1]^{n\times h\times k}$ such that

- $a'_{i,1,r} \geq a'_{i,2,r} \geq \ldots \geq a'_{i,h,r} \geq 0$, for all $i = 1,\ldots,n$ and $r = 1,\ldots,k$;
- $a'_{i,j,1} \geq a'_{i,j,2} \geq \ldots \geq a'_{i,j,k} \geq 0$, for all $i = 1,\ldots,n$ and $j = 1,\ldots,h$;
- $\sum_{i=1}^{n} a'_{i,1,r} = 1$, for all $i = 1,\ldots,n$ and $r = 1,\ldots,k$.

The h-k-weighted average with respect to weights $\mathbf{a}' = [a'_{i,j,r}]$ is the h-k-aggregation function $WA_{\mathbf{a}} : I_h^n \to I_k$ defined as follows: for all $\boldsymbol{x} \in I_h^n$ and $r = 1,\ldots,k$,

$$WA_{\mathbf{a},r}(\boldsymbol{x}) = \sum_{i=1}^{n} a'_{i,1,r} x_{i,1} + \sum_{i=1}^{n} \sum_{j=2}^{h} a'_{i,j,r}(x_{i,j} - x_{i,j-1}). \tag{2}$$

There is the following relation between weights $a'_{ij,r}$ and $a_{i,j,r}$: for all $i = 1,\ldots,n$; $j = 1,\ldots,h-1$, and $r = 1,\ldots,k$,

$$\begin{cases} a_{i,j,r} = a'_{i,j,r} - a'_{i,j+1,r} \\ a_{i,h,r} = a'_{i,h,r}. \end{cases} \tag{3}$$

Two very natural conditions for h-k-aggregation functions are the following

- additivity: for all $\boldsymbol{x}, \boldsymbol{y} \in I_h^n$, $g(\boldsymbol{x}+\boldsymbol{y}) = g(\boldsymbol{x}) + g(\boldsymbol{y})$, where $\boldsymbol{x}+\boldsymbol{y} = \boldsymbol{z}$ with $z_{i,j} = x_{i,j} + y_{i,j}$ for all $i \in N$ and for all $j = 1,\ldots,h$;
- idempotence: for all $a \in \mathbb{R}$, $g(\boldsymbol{a}) = a$, where $\boldsymbol{a} \in I_h^n$ is $\boldsymbol{a} = [a,\ldots,a]$.

Theorem 1. *An h-k-aggregation function is additive and idempotent if and only if it is the h-k-weighted average.*

3 Non-additive h-k-aggregation functions

Let us consider the set $\mathcal{Q} = \{(A_1,\ldots,A_h) \mid A_1 \subseteq A_2 \subseteq \ldots \subseteq A_h \subseteq N\}$. With a slight abuse of notation we extend to $\mathcal{Q}$ the relation of set inclusion and the operations of union and intersection by defining for all $(A_1,\ldots,A_h)$ and $(B_1,\ldots,B_h) \in \mathcal{Q}$,

$$(A_1,\ldots,A_h) \subseteq (B_1,\ldots,B_h) \text{ if and only if } A_i \subseteq B_i \text{ for all } i = 1,\ldots,h;$$

$$(A_1,\ldots,A_h)\cup(B_1,\ldots,B_h)=(A_1\cup B_1,\ldots,A_h\cup B_h);$$
$$(A_1,\ldots,A_h)\cap(B_1,\ldots,B_h)=(A_1\cap B_1,\ldots,A_h\cap B_h).$$

Regarding the algebraic structure of $\mathcal{Q}$, we can observe that with respect to the relation $\subseteq$, $\mathcal{Q}$ is a lattice.

Definition 1. *A function $\mu_h:\mathcal{Q}\to[0,1]$ is an h-interval-capacity on $\mathcal{Q}$ if*

- *$\mu_r(\emptyset,\ldots,\emptyset)=0$, and $\mu_h(N,\ldots,N)=1$; and*
- *$\mu_h(A_1,\ldots,A_h)\le\mu_h(B_1,\ldots,B_h)$ for all $(A_1,\ldots,A_h),(B_1,\ldots,B_h)\in\mathcal{Q}$ such that $A_i\subseteq B_i$ for all $i=1,\ldots,h$.*

Definition 2. *An h-k-interval capacity is a vector $\left(\mu_{h_1},\ldots,\mu_{h_k}\right)$ such that*

- *for every $i=1,\ldots,k$, $\mu_{h_i}:\mathcal{Q}\to[0,1]$ is an h-interval capacity; and*
- *for all $(A_1,\ldots,A_h)\in\mathcal{Q}$, $\mu_{h_i}(A_1,\ldots,A_h)\le\mu_{h_{i+1}}(A_1,\ldots,A_h)$, for all $i=1,\ldots,k-1$.*

Definition 3. *An h-interval-capacity μ_h is an additive h-interval-capacity on $\mathcal{Q}$ if for all $(A_1,\ldots,A_h)\in\mathcal{Q}$, for any $i=1,\ldots,h-1$, for any $B_i\subseteq N$ such that $A_i\cap B_i=\emptyset$ and $A_i\cup B_i\subseteq A_{i+1}$,*

$$\mu_h(A_1,\ldots,A_i\cup B_i,\ldots,A_h)=\mu_h(A_1,\ldots,A_h)+\mu_h\left(\emptyset,\emptyset,\ldots,\overbrace{B_i,B_i,\ldots,B_i}^{h-i+1}\right).$$

An h-k-interval capacity $\left(\mu_{h_1},\ldots,\mu_{h_k}\right)$ is additive if all h-interval-capacities $\mu_{h_1},\ldots,\mu_{h_k}$ are additive.

Definition 4. *The h-k-Choquet Integral of*

$$\boldsymbol{x}=([x_{1,1},\ldots,x_{1,h}],\ldots,[x_{n,1}\ldots,x_{n,h}])$$

with respect to the h-k-interval capacity $\left(\mu_{h_1},\ldots,\mu_{h_k}\right)$ is given by

$$Ch_{h-k}\left(\boldsymbol{x},\left(\mu_{h_1},\ldots,\mu_{h_k}\right)\right)=:\left[Ch_h\left(\boldsymbol{x},\mu_{h_1}\right),\ldots,Ch_h\left(\boldsymbol{x},\mu_{h_k}\right)\right],\tag{4}$$

being for all $r=1,\ldots,k$

$$Ch_h\left(\boldsymbol{x},\mu_{h_r}\right)=\int_{\min\limits_{i\in N}x_{1,i}}^{\max\limits_{i\in N}x_{i,h}}\mu_{h_r}(\{i\in N|x_{1,i}\ge t\},\ldots,\{i\in N|x_{n,i}\ge t\})dt+\min_{i\in N}x_{1,i}.\tag{5}$$

Note that the $2-1-$Choquet integral is the robust Choquet integral presented in [1]

Definition 5. *The two vectors of I_h^n*

$$\boldsymbol{x}=([x_{1,1},\ldots,x_{1,h}],\ldots,[x_{n,1}\ldots,x_{n,h}])\,,\boldsymbol{y}=([y_{1,1},\ldots,y_{1,h}],\ldots,[y_{n,1}\ldots,y_{n,h}])$$

are comonotone if the two vectors of $\mathbb{R}^{nh}$ $\boldsymbol{x}^*=(x_{1,1},\ldots,x_{1,h},\ldots,x_{n,1},\ldots,x_{n,h})$ and $\boldsymbol{y}^*=(y_{1,1},\ldots,y_{1,h},\ldots,y_{n,1}\ldots,y_{n,h})$ are comonotone.

We extend the property of comonotone additivity for a standard aggregation function to an *h*-*k*-aggregation function: an *h*-*k*-aggregation function is comonotone additive if it is additive for comonotone vectors.

Theorem 2. *An h-k-aggregation function is comonotone additive and idempotent if and only if it is the h-k-Choquet integral.*

Theorem 3. *The h-k-Choquet integral* $Ch_{h-k}\left(\boldsymbol{x},\left(\mu_{h_1},\ldots,\mu_{h_k}\right)\right)$ *is the h-k-weighted average if and only if the h-k-interval capacity* $\left(\mu_{h_1},\ldots,\mu_{h_k}\right)$ *is additive.*

In [1] the robust Shilkret and Sugeno integrals have been presented. These are $2-1-$aggregation functions which can be generalized to the case of *h*-*k*-aggregation functions.

Definition 6. *The h-k-Shilkret integral of* $\boldsymbol{x} = ([x_{1,1},\ldots,x_{1,h}],\ldots,[x_{n,1}\ldots,x_{n,h}]) \in I_h^n$ *with respect to the the h-k-interval capacity* $\left(\mu_{h_1},\ldots,\mu_{h_k}\right)$ *is given by*

$$Sh_{h-k}\left(\boldsymbol{x},\left(\mu_{h_1},\ldots,\mu_{h_k}\right)\right) =: \left[Sh_h\left(\boldsymbol{x},\mu_{h_1}\right),\ldots,Sh_h\left(\boldsymbol{x},\mu_{h_k}\right)\right], \tag{6}$$

being for all $r = 1,\ldots,k$

$$Sh_h\left(\boldsymbol{x},\mu_{h_r}\right) = \bigvee_{(A_1,\ldots,A_h)\in Q} \left\{ \bigwedge \left\{ \bigwedge_{i\in A_1} x_{1,i},\ldots,\bigwedge_{i\in A_h} x_{h,i} \right\} \cdot \mu_{h,r}(A_1,\ldots,A_h) \right\}. \tag{7}$$

Definition 7. *The h-k-Sugeno integral of* $\boldsymbol{x} = ([x_{1,1},\ldots,x_{1,h}],\ldots,[x_{n,1}\ldots,x_{n,h}]) \in I_h^n$ *with respect to the the h-k-interval capacity* $\left(\mu_{h_1},\ldots,\mu_{h_k}\right)$ *is given by*

$$Su_{h-k}\left(\boldsymbol{x},\left(\mu_{h_1},\ldots,\mu_{h_k}\right)\right) =: \left[Su_h\left(\boldsymbol{x},\mu_{h_1}\right),\ldots,Su_h\left(\boldsymbol{x},\mu_{h_k}\right)\right], \tag{8}$$

being for all $r = 1,\ldots,k$

$$Su_h\left(\boldsymbol{x},\mu_{h_r}\right) = \bigvee_{(A_1,\ldots,A_h)\in Q} \bigwedge \left\{ \mu_{h,r}(A_1,\ldots,A_h),\ \bigwedge_{i\in A_1} x_{1,i},\ldots,\bigwedge_{i\in A_h} x_{h,i} \right\}. \tag{9}$$

Finally, in [1] several non-additive $2-1-$aggregation functions have been presented, i.e. the robust Choquet integral with respect to a bipolar interval-capacity, the robust Choquet integral with respect to an interval capacity level dependent, the robust concave integral and the robust universal integral. All these integrals admit a natural generalization to the case of *h*-*k*-aggregation functions presented here.

4 A motivating example

Let us provide an example where 2-interval numbers need to be aggregated into a triangular number. The director of a university decides on students who are applying for graduate studies in management. Since some prerequisites from school are required, the students are indeed evaluated according to mathematics, literature and language skills.

	Mathematics	Literature	Language
student A	6	$[5,7]$	$[7,8]$
student B	7	$[6,7]$	9
student C	$[6,8]$	7	7

Table 1. Students evaluation

All the marks with respect to the scores are given on the scale from 0 to 10. The director receives the candidates evaluations serving as a basis for the selection. He notes that some judgments are expressed as intervals (corresponding to some evaluators doubts, see Table 1). At the university the freshmen are initially divided into three groups, depending on the starting level. The assignment of a student to a group is not just decided on the basis of his average evaluation, but more properly, depends on the potentiality of the student. This means that the director prefers that every student is represented by a triangular number (E_p, E_a, E_o), where E_p corresponds to a pessimistic evaluation, E_a corresponds to an average evaluation and E_o corresponds to an optimistic evaluation. On the basis of this triple information the director will decide, for each student, the pertinent group. This is a realistic example where 2-interval numbers need to be aggregated into a triangular number.

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Partial orders on the truth value algebra of type-2 fuzzy sets

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1 Introduction

The algebra of truth values of type-2 fuzzy sets, as described below, has two primary binary operations. These operations are idempotent, commutative, and associative, so each induces a partial order on the elements of the algebra. These partial orders are not equal. The basic goal in this paper is the study of these partial orders, and the partial order given by their intersection. Some principal results are that neither partial order is a lattice order, and under the partial order given by the intersection of these two partial orders, the set of convex elements is a disjoint union of complete lattices.

2 The Algebra $([0,1]^{[0,1]}, \sqcup, \sqcap, ^*, \bar{0}, \bar{1})$

The algebra of truth values of type-2 fuzzy sets is the set $[0,1]^{[0,1]}$ of all functions from the unit interval into itself furnished with the operations given below: the binary operations $\sqcup$ and $\sqcap$, the unary operation *, and the nullary operations $\bar{1}$ and $\bar{0}$.

$$\begin{aligned}
(f \sqcup g)(x) &= \sup\{f(y) \wedge g(z) : y \vee z = x\} \\
(f \sqcap g)(x) &= \sup\{f(y) \wedge g(z) : y \wedge z = x\} \\
f^*(x) &= \sup\{f(y) : 1 - y = x\} = f(1-x) \\
\bar{1}(x) &= \begin{cases} 0 \text{ if } x \neq 1 \\ 1 \text{ if } x = 1 \end{cases} \quad \bar{0}(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}
\end{aligned}$$

The resulting algebra $([0,1]^{[0,1]}, \sqcup, \sqcap, ^*, \bar{0}, \bar{1})$ is denoted $\mathbb{M}$. This algebra was introduced by Zadeh in [8], and have been heavily investigated.

The pointwise operations $\max$ and $\min$, denoted $\vee$ and $\wedge$ respectively, help in determining the properties of the algebra $\mathbb{M}$ via the auxiliary operations $f^L(i) = \vee_{j \leq i} f(j)$ and $f^R(i) = \vee_{j \geq i} f(j)$.

The operations $\sqcup$ and $\sqcap$ in $\mathbb{M}$ can be expressed in terms of the pointwise max and min of functions in two different ways, as follows.

Theorem 1. *The following hold for all $f, g \in [0,1]^{[0,1]}$.*

$$f \sqcup g = \left(f \wedge g^L\right) \vee \left(f^L \wedge g\right) = (f \vee g) \wedge \left(f^L \wedge g^L\right)$$

$$f \sqcap g = \left(f \wedge g^R\right) \vee \left(f^R \wedge g\right) = (f \vee g) \wedge \left(f^R \wedge g^R\right)$$

It has been shown [7, 1] that $\mathbb{M}$ satisfies the following equations.

Proposition 1. *Let* $f,\ g,\ h \in [0,1]^{[0,1]}$.

1. $f \sqcup f = f$; $f \sqcap f = f$
2. $f \sqcup g = g \sqcup f$; $f \sqcap g = g \sqcap f$
3. $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h$; $f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$
4. $f \sqcup (f \sqcap g) = f \sqcap (f \sqcup g)$
5. $\bar{1} \sqcap f = f$; $\bar{0} \sqcup f = f$
6. $f^{**} = f$
7. $(f \sqcup g)^* = f^* \sqcap g^*$; $(f \sqcap g)^* = f^* \sqcup g^*$

Since the operations $\sqcap$ and $\sqcup$ are idempotent, commutative and associative, they each induce a partial order on the set A as follows:

$$f \sqsubseteq_\sqcap g \text{ if } f \sqcap g = f$$
$$f \sqsubseteq_\sqcup g \text{ if } f \sqcup g = g$$

It is easy to see that the operations $\sqcup$ and $\sqcap$ do not give the same partial orders. *Our basic objective is to investigate the properties of these partial orders and their intersection.* This was partially motivated by considering maps between objects in the category of subsets of type-2 fuzzy sets. We will state some of the principal results. While most of the proofs are straightforward, some are a bit laborious.

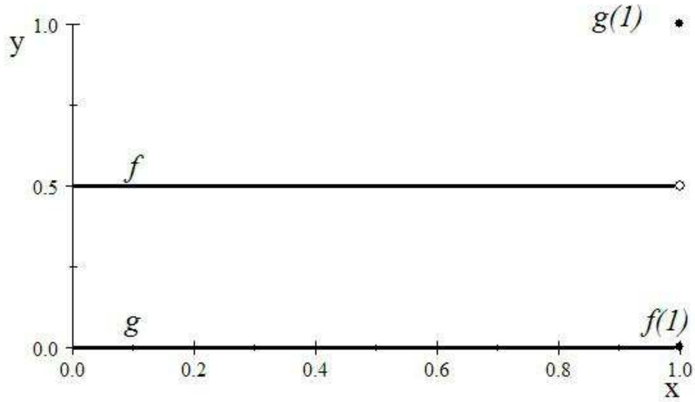
3 General Properties

Proposition 2. *The partial orders* $\sqsubseteq_\sqcup$ *and* $\sqsubseteq_\sqcap$ *each induces a semilattice on* $[0,1]^{[0,1]}$. *That is,*

1. $f \sqcap g$ in the order $\sqsubseteq_\sqcap$ is the $\inf$ of f and g.
2. $f \sqcup g$ in the order $\sqsubseteq_\sqcup$ is the $\sup$ of f and g.

Theorem 2. $([0,1]^{[0,1]}, \sqcup)$ *is not a lattice under the partial order* $\leq_\sqcup$*, and* $([0,1]^{[0,1]}, \sqcap)$ *is not a lattice under the partial order* $\leq_\sqcap$.

The functions f and g in the picture below, have no greatest lower bound under the partial order $\leq_\sqcup$.



Corollary 1. *Infinite* $\sup$*'s do not exist under* $\sqsubseteq_{\sqcup}$*, and infinite* $\inf$*'s do not exist under* $\sqsubseteq_{\sqcap}$*.*

Theorem 3. *Let* C *be the continuous functions in* $[0,1]^{[0,1]}$*. Then the subalgebra* $(C, \sqcup)$ *is not a lattice under the order* $\leq_{\sqcup}$*.*

Remark 1. There are various interesting subalgebras and subsets of $[0,1]^{[0,1]}$ that are lattices under $\sqsubseteq_{\sqcup}$ or under $\sqsubseteq_{\sqcap}$.

Problem 1. Is the subalgebra of upper semicontinuous functions a lattice under the order $\sqsubseteq_{\sqcup}$?

4 The Double Order

It is clear that the intersection of two partial orders on a set is a partial order.

Definition 1. *The intersection of the partial orders* $\sqsubseteq_{\sqcap}$ *and* $\sqsubseteq_{\sqcup}$ *on* $[0,1]^{[0,1]}$ *is denoted* $\sqsubseteq$*. Thus if* $f \sqsubseteq_{\sqcap} g$ *and* $f \sqsubseteq_{\sqcup} g$*, we write* $f \sqsubseteq g$*.*

Theorem 4. *For any* f *and* g*,* $(f \sqcap g) \sqsubseteq (f \sqcup g)$*.*

Definition 2. *For* $f \in [0,1]^{[0,1]}$*, the* ***height*** *of* f *is* $\vee_{x \in [0,1]} f(x)$*. If the height of* f *is* 1*, then* f *is* ***normal****.*

Theorem 5. *The set of elements of height* h *is a subalgebra of the reduct* $([0,1]^{[0,1]}, \sqcup, \sqcap, \neg)$*. For* $h \neq 0$*, these algebras are isomorphic to each other.*

Theorem 6. *Two elements are incomparable with respect to the partial order* $\sqsubseteq$ *if they have different heights.*

Corollary 2. *The algebra* $([0,1]^{[0,1]}, \sqcup, \sqcap, \neg)$ *with the partial order* $\sqsubseteq$ *is the disjoint union of its subalgebras of elements of height* h*,* $h \in [0,1]$*.*

Definition 3. *A function* $f \in [0,1]^{[0,1]}$ *is* ***convex*** *if for all* $x \leq y \leq z$ *in* $[0,1]$*,* $f(y) \geq f(x) \wedge f(z)$*. Equivalently,* $f = f^L \wedge f^R$*.*

Proposition 3. *The set of convex elements of* $[0,1]^{[0,1]}$ *forms a subalgebra of* $([0,1]^{[0,1]}), \sqcap, \sqcup, \neg)$*.*

Theorem 7. *The set of normal convex elements of* $[0,1]^{[0,1]}$ *forms a subalgebra of* $([0,1]^{[0,1]}), \sqcap, \sqcup, \neg, \overline{1}, \overline{0})$ *on which the partial orders* $\sqsubseteq_\sqcap$*,* $\sqsubseteq_\sqcup$ *and* $\sqsubseteq$ *coincide, and this subalgebra is a complete lattice, in fact, a complete De Morgan algebra, under these common partial orders.*

Corollary 3. *Let* $\mathbb{C}_h$ *be the subalgebra of the reduct* $([0,1]^{[0,1]}), \sqcap, \sqcup, \neg)$ *of convex elements of height* h*. Then on* $\mathbb{C}_h$ *the partial orders* $\sqsubseteq_\sqcap$*,* $\sqsubseteq_\sqcup$ *and* $\sqsubseteq$ *coincide, and this subalgebra is a complete lattice, in fact, a complete De Morgan algebra, under these common partial orders.*

Corollary 4. *The poset* $(\mathbb{C}, \sqsubseteq)$ *of convex elements of* $[0,1]^{[0,1]}$ *is the disjoint union* $\coprod_h \mathbb{C}_h$ *of the complete lattices* $\mathbb{C}_h$*.*

– We do not have an adequate description of the partial order $\sqsubseteq$ on the totality of the algebra of normal functions.

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Chebyshev and Jensen inequalities, weak convergence, and convergence in distribution in a non-additive setting

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1 Preliminaries

Given a non-empty set Ω, let $\mathcal{F}$ be such that $\emptyset, \Omega \in \mathcal{F} \subset 2^{\Omega}$. We denote by μ (with or without indices) any (real) monotone measure $\mu : \mathcal{F} \to [0, +\infty[$, i.e. $\mu(\emptyset) = 0$ and $\mu(F_1) \leq \mu(F_2)$, if $F_1 \subset F_2$.

We call μ *additive*, if $\mu(F_1 \cup F_2) = \mu(F_1) + \mu(F_2)$, *subadditive*, if $\mu(F_1 \cup F_2) \leq \mu(F_1) + \mu(F_2)$, *superadditive*, if $\mu(F_1 \cup F_2) \geq \mu(F_1) + \mu(F_2)$, whenever $F_1 \cap F_2 = \emptyset$. Moreover, for any μ, we consider the corresponding *conjugate* monotone measure defined as: $\overline{\mu}(F) = \|\mu\| - \mu(F^c)$, where $\|\mu\| = \mu(\Omega)$. Finally, μ is a *monotone probability* if $\|\mu\| = 1$.

Given $X : \Omega \to \mathbb{R}$, we put $\{X > t\} = \{\omega : X(\omega) > t\}$ and $\{X \geq t\} = \{\omega : X(\omega) \geq t\}$ for any real t. Moreover, X is called *$\mathcal{F}$-measurable* if $\{X > t\}, \{X \geq t\} \in \mathcal{F}$ for all real t. Henceforth, X, Y are assumed to be $\mathcal{F}$-measurable functions.

We recall that X, Y are said to be *comonotonic* if $(X(\omega_1) - X(\omega_2))\,(Y(\omega_1) - Y(\omega_2)) \geq 0$ for any ω_1, ω_2.

Now, given the monotone space $(\Omega, \mathcal{F}, \mu)$:

• the *Choquet integral* of X (w.r.t. μ) is defined as:

$$\mathrm{C}\int_{\Omega} X\, d\mu = \int_{-\infty}^{0} [\mu(\{X > t\}) - \|\mu\|]\, dt + \int_{0}^{+\infty} \mu(\{X > t\})\, dt,$$

whenever at least one of the Riemann integrals is finite. In the sequel, we denote by $\Im_{\mu}(X)$ the Choquet integral of X (w.r.t. μ). We call X *Choquet summable*, if $\Im_{\mu}(X)$ exists, and *Choquet integrable*, whenever $\Im_{\mu}(X)$ is finite;
• the *Sugeno integral* of $X \geq 0$ (w.r.t. μ) is defined as:

$$\mathrm{S}\int_{\Omega} X\, d\mu = \sup_{\alpha \geq 0} \min\,(\alpha, \mu(\{X \geq \alpha\}))\,.$$

2 Chebyshev inequality

We link comonotonicity and Chebyshev inequality for Choquet and Sugeno integrals. In this section, we assume that $\mathcal{F}$ is a **σ-field**.

2.1 Choquet integral (GH [4],[5])

We start with the following version of Chebyshev inequality for Choquet integral:

$$\|\mu\| \,\mathrm{C}\int_\Omega X\,Y\,d\mu \geq \left(\mathrm{C}\int_\Omega X\,d\mu\right)\left(\mathrm{C}\int_\Omega Y\,d\mu\right). \tag{1}$$

Theorem 1. *Let X,Y be comonotonic such that X,Y and XY are Choquet summable. Assume one of the following conditions being valid:*

(i) $X,Y \geq 0$*;*
(ii) Let μ *be additive and assume one of the following conditions being valid:*
  (ii1) X,Y *are Choquet integrable;*
  (ii2) $\Im_\mu(X) = +\infty$ *and there is* ω_0 *such that* $Y(\omega_0) > 0$*;*
  (ii3) $\Im_\mu(X) = -\infty$ *and there is* ω_0 *such that* $Y(\omega_0) < 0$.

Then, the Chebyshev inequality (1) *holds.*

Theorem 2 (Characterization theorem). *The following statements hold:*

(i) Let X,Y be non negative. Then, X,Y are comonotonic iff the Chebyshev inequality (1) *holds for any monotone* μ*;*
(ii) Let X,Y be bounded. Then, X,Y are comonotonic iff the Chebyshev inequality (1) *holds for any additive* μ.

Remark 1. Without additivity assumption, non negativity hypothesis in Theorem 2.1(i) can not be dropped. To this end, consider $\Omega = [0,1]$, $\mathcal{F} = 2^\Omega$, $\mu(F) = \inf f(F^c)$ with

$$f(\omega) = \begin{cases} 1 & \text{if } \omega = 0 \\ \frac{3-2\omega}{4} & \text{if } \omega \in]0,\frac{1}{2}[\\ \frac{1-\omega}{2} & \text{if } \omega \in [\frac{1}{2},1] \end{cases}$$

and

$$X(\omega) = \begin{cases} -2 & \text{if } \omega \in [0,\frac{1}{4}[\\ 5 & \text{if } \omega \in [\frac{1}{4},1] \end{cases}, \quad Y(\omega) = \begin{cases} -2 & \text{if } \omega \in [0,\frac{1}{2}[\\ 1 & \text{if } \omega \in [\frac{1}{2},1] \end{cases}.$$

2.2 Sugeno integral (GH [3])

Now, in the setting of **non-negative functions**, we consider the following version of Chebyshev inequality for Sugeno integral:

$$\|\mu\| \,\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}\,X\,Y\,d\mu \geq \left(\mathrm{S}\int_\Omega X\,d\mu\right)\left(\mathrm{S}\int_\Omega Y\,d\mu\right), \tag{2}$$

for any non-null μ.

Theorem 3. *Let X, Y be comonotonic. Then, the following inequalities:*

$$\max(\|\mu\|,1)\,\mathrm{S}\int_\Omega X\;Y\,d\mu \geq \|\mu\|\,\mathrm{S}\int_\Omega \frac{1}{\|\mu\|}\,X\,Y\,d\mu \geq \left(\mathrm{S}\int_\Omega X\,d\mu\right)\left(\mathrm{S}\int_\Omega Y\,d\mu\right)$$

hold for any non-null monotone μ.

Remark 2. Both inequalities may be strict, either $\|\mu\| < 1$ or $\|\mu\| > 1$. To this end, let Ω be a real interval, $\mathcal{F}$ the Borel sets on Ω, λ the Lebesgue measure and $X(\omega) = \omega$, $Y(\omega) = 2\,\omega$ for all ω. Then, consider $\Omega = [0,1]$ with $\mu = \frac{1}{2}\lambda$ and $\Omega = [0,3]$ with $\mu = \lambda$.

Theorem 4 (Characterization theorem). *The following statements are equivalent:*

(i) X, Y are comonotonic;
(ii) The Chebyshev inequality (2) *holds for any non-null monotone μ.*

The following sections concern the **Choquet integral** only.

3 Jensen inequality (GH [5])

In this section, we assume that $\mathcal{F}$ is a **σ-field**.

Theorem 5. *Given $I =]i_0, i_1[\subset \mathbb{R}$ (bounded or not), let $X : \Omega \to I$ and $g : I \to \mathbb{R}$ such that $X, g \circ X$ are Choquet summable. If g is a convex function and μ is a monotone probability, then we have:*

$$g(\Im_\mu(X)) \leq \max(\Im_\mu(g \circ X), \Im_{\overline{\mu}}(g \circ X)),$$

where $g(\Im_\mu(X)) = g(i_1^-)$, if $\Im_\mu(X) = i_1$, and $g(\Im_\mu(X)) = g(i_0^+)$, if $\Im_\mu(X) = i_0$. More precisely:
- when $\Im_\mu(X) \in I$, then $g(\Im_\mu(X)) \leq \Im_\mu(g \circ X)$, if $g'_+(\Im_\mu(X)) \geq 0$, and $g(\Im_\mu(X)) \leq \Im_{\overline{\mu}}(g \circ X)$, if $g'_+(\Im_\mu(X)) < 0$;
- when $\Im_\mu(X) = i_1$, then $g(\Im_\mu(X)) \leq \Im_\mu(g \circ X)$.

Finally, $g(\Im_\mu(X)) \leq \Im_\mu(g \circ X)$, if μ is subadditive, and $g(\Im_\mu(X)) \leq \Im_{\overline{\mu}}(g \circ X)$, if μ is superadditive.

Remark 3. The inequality can be strict. To this end, consider $\Omega =]0,2[$, $\mathcal{F} = 2^\Omega$, $S =]0, \frac{3}{2}]$ and $X(\omega) = 2 - \omega$, $g(\omega) = (\omega - 1)^2$ for all ω. Then, given the monotone probability (unanimity game):

$$u_S(F) = \begin{cases} 1 & \text{if } F \supset S \\ 0 & \text{otherwise} \end{cases},$$

we have $\Im_{u_S}(g \circ X) < g(\Im_{u_S}(X)) < \Im_{\overline{u}_S}(g \circ X)$.

4 Weak convergence (GH [1])

The following abstract treatment of the weak convergence for monotone measures allows us to obtain some basic results generalizing well known theorems regarding classical weak and vague convergences.

Let $\mathcal{F} = \mathcal{F}^c$ (i.e. $\mathcal{F}$ closed under complementation). Moreover, let $\mathcal{C}, \mathcal{U} \subset \mathcal{F}$ such that $\emptyset \in \mathcal{C}$ and $\Omega \in \mathcal{U}$; we denote by C, U elements of $\mathcal{C}$, $\mathcal{U}$, respectively.

Definition 1. A function X is called $(\mathcal{C}, \mathcal{U})$-*measurable* if:

• for any $t > 0$, we have $\{X \geq t\} \in \mathcal{C}$ and $\{X > t\} \in \mathcal{U}$, when $X \geq 0$;
• X^+, X^- are $(\mathcal{C}, \mathcal{U})$-measurable, otherwise.

Now, we denote by $\mathbf{M}$ the set of all bounded $(\mathcal{C}, \mathcal{U})$-measurable functions which vanish outside some element of $\mathcal{C}$ (i.e. with support in $\mathcal{C}$), i.e.:

$$\mathbf{M} = \{X : X \text{ is } (\mathcal{C}, \mathcal{U})\text{-measurable and } \exists k(\sup |X| \leq k) \\ \text{and } \exists C \forall \omega(\omega \notin C \Rightarrow X(\omega) = 0\}.$$

In this section we assume the following property:

Example 1. For all C, U such that $C \subset U$ there is $X \in \mathbf{M}$, with values in $[0, 1]$, such that $X(\omega) = 1$, if $\omega \in C$, and $X(\omega) = 0$, if $\omega \notin U$.

The next definition introduces a notion of convergence which generalizes the classical weak convergence and vague convergence for countable additive measures. Henceforth, we assume D to be a directed set.

Definition 2. A net $\{\mu_d; d \in D\}$ *weakly converges* to μ (w.r.t. $(\mathcal{C}, \mathcal{U})$) (briefly $\mu_d \stackrel{\text{w}}{\to} \mu$) if $\mathrm{C}\int_\Omega X \, d\mu_d \to \mathrm{C}\int_\Omega X \, d\mu$ for all $X \in \mathbf{M}$.

Remark 4. Let Ω be a metric space, $\mathcal{F}$ the Borel σ-field, $\mathcal{C}$ the set of closed sets and $\mathcal{U}$ the set of open sets; then, $\mathbf{M}$ is the set $C_b(\Omega)$ of bounded continuous functions on Ω and Separation Axiom holds. Consequently, in the setting of (finite) countable additive measures, $\stackrel{\text{w}}{\to}$ coincides with the usual weak convergence.

Let Ω be a locally compact Hausdorff space, $\mathcal{F}$ the Borel σ-field, $\mathcal{C}$ the set of compact sets and $\mathcal{U}$ the set of open sets; then, $\mathbf{M}$ is the set of continuous functions with compact support and Separation Axiom holds. Consequently, in the setting of countable additive measures, $\stackrel{\text{w}}{\to}$ coincides with the usual vague convergence.

Now, we introduce two basic notions of regularity for elements of $\mathcal{F}$ (w.r.t. $(\mathcal{C}, \mathcal{U})$) in order to characterize this kind of weak convergence.

Definition 3. We say that F is a:

• *μ-regular set* if it satisfies the "approximation property":

$$\sup\{\mu(C) : C \subset F\} = \mu(F) = \inf\{\mu(U) : U \supset F\}.$$

We denote by $\mathcal{R}_\mu$ (*μ-regularity system*) the family of μ-regular sets;
• *μ-strongly regular set* if for all $\varepsilon > 0$ there are $C, U \in \mathcal{R}_\mu$ such that $U \subset F \subset C$ and $\mu(C) - \mu(U) < \varepsilon$. We denote by $\mathcal{R}_\mu^0$ (*μ-strong regularity system*) the family of μ-strongly regular sets.

Theorem 6 (Portmanteau type theorem I). *Given μ and a net $\{\mu_d; d \in D\}$, the following statements are equivalent:*

(i) $\mu_d \stackrel{\text{w}}{\to} \mu$;
(ii) $\overline{\mu}_d \stackrel{\text{w}}{\to} \overline{\mu}$;

(iii) $\limsup_{d\in D}\mu_d^{(i)}(C) \le \mu^{(i)}(C)$, $\liminf_{d\in D}\mu_d^{(i)}(U) \ge \mu^{(i)}(U)$ $\forall C,U \in \mathcal{R}_{\mu^{(i)}}$ *$(i = 1,2)$;*

(iv) $\mu_d^{(i)}(F) \to \mu^{(i)}(F)$ $\forall F \in \mathcal{R}^{(0)}_{\mu^{(i)}}$ *$(i = 1,2)$,*

where $\mu_d^{(1)} = \mu_d$, $\mu^{(1)} = \mu$ *and* $\mu_d^{(2)} = \overline{\mu}_d$, $\mu^{(2)} = \overline{\mu}$.

Theorem 7 (Portmanteau type theorem II). *Let $\mathcal{C} = \mathcal{U}^c$. Then, given μ and a net $\{\mu_d; d \in D\}$, the following statements are equivalent:*

(i) $\mu_d \xrightarrow{\text{w}} \mu$;
(ii) $\overline{\mu}_d \xrightarrow{\text{w}} \overline{\mu}$;
(iii) $\limsup_{d\in D}\mu_d(C) \le \mu(C)$, $\liminf_{d\in D}\mu_d(U) \ge \mu(U)$ *for all* $C,U \in \mathcal{R}_\mu$;
(iv) $\mu_d(F) \to \mu(F)$ *for all* $F \in \mathcal{R}^{(0)}_\mu$.

Theorem 8 (Portmanteau type theorem III). *Let $\mathcal{C} = \mathcal{U}^c$. Moreover, given μ and a net $\{\mu_d; d \in D\}$, let μ, μ_d be additive for any $d \in D$. Then, the following statements are equivalent:*

(i) $\mu_d \xrightarrow{\text{w}} \mu$;
(ii) $\|\mu_d\| \to \|\mu\|$ *and* $\limsup_{d\in D}\mu_d(C) \le \mu(C)$ *for all* $C \in \mathcal{R}_\mu$;
(iii) $\|\mu_d\| \to \|\mu\|$ *and* $\liminf_{d\in D}\mu_d(U) \ge \mu(U)$ *for all* $U \in \mathcal{R}_\mu$;
(iv) $\mu_d(F) \to \mu(F)$ *for all* $F \in \mathcal{R}^{(0)}_\mu$.

The *Lévy topology* on the set $\mathrm{bm}(\Omega,\mathcal{F})$ of monotone measures on $\mathcal{F}$ is the topology such that, for any μ, the basic neighborhoods of μ are the sets of the form:

$$\{\mu' : |\mu'(F_i) - \mu(F_i)| < \varepsilon \text{ and } |\overline{\mu}'(F_i) - \overline{\mu}(F_i)| < \varepsilon \quad (i = 1,\dots,n)\},$$

where $\varepsilon > 0$ and $F_1,\dots,F_n \in \mathcal{R}^{(0)}_\mu \cap \mathcal{R}^{(0)}_{\overline{\mu}}$ $(n \ge 1)$.

Theorem 9. *Given μ and a net $\{\mu_d; d \in D\}$, we have $\mu_d \xrightarrow{\text{w}} \mu$ iff μ_d converges to μ under the Lévy topology.*

Assuming Ω to be a separable metric space, Kawabe [6] supplies interesting results regarding the metrizability of the Lévy topology of suitable subspaces of $\mathrm{bm}(\Omega,\mathcal{F})$.

5 Convergence in distribution (GH [2])

Let $\Omega = \mathbb{R}$ and $\mathcal{F}$ include all upper half-lines. Moreover, for any μ, let G_μ be the decreasing distribution function of μ, i.e. $G_\mu(x) = \mu(]x,+\infty[)$ for any real x.

Definition 4. The net $\{\mu_d; d \in D\}$ *converges in distribution* to μ (briefly $\mu_d \xrightarrow{\text{d}} \mu$) if $\|\mu_d\| \to \|\mu\|$ and $G_{\mu_d}(x) \to G_\mu(x)$ at all continuity points x of G_μ.

Theorem 10 (Characterization theorem I). *Given μ and a net $\{\mu_d; d \in D\}$, the following statements are equivalent:*

(i) $\mu_d \xrightarrow{\mathrm{d}} \mu$*;*
(ii) $\mathrm{C}\int_{\mathbb{R}} X\,d\mu_d \to \mathrm{C}\int_{\mathbb{R}} X\,d\mu$ *for all increasing functions* $X \in C_b(\mathbb{R})$.

Remark 5. The convergence in distribution of a net does not assure the convergence in distribution of the net of the corresponding conjugates. To this end, let $\mathcal{F}$ be the Borel σ-field, $\mu_n(]x,+\infty[) = 1 = \|\mu_n\|$ and $\mu_n(]-\infty,x]) = |\sin n|$ for all real x and natural n. Then, $G_{\mu_n}(x) = 1$, $G_{\overline{\mu}_n}(x) = 1-|\sin n|$ for all x,n. Consequently, $\mu_n \xrightarrow{\mathrm{d}} \mu_1$ but not $\overline{\mu}_n \xrightarrow{\mathrm{d}} \overline{\mu}_1$.

The convergence in distribution is not equivalent to the classical weak convergence. To this end, let $\mathcal{F}$ be the Borel σ-field and $\mathbb{Z}$ the set of integer numbers. Denoting by $\mathcal{A}$ the smallest field containing all intervals (bounded or not), let ν be any additive probability on $\mathcal{A}$ such that $\nu(A) = 0$ for all bounded sets $A \in \mathcal{A}$. Moreover, let:

$$B = \bigcup_{t\in\mathbb{Z}}]t-\frac{1}{2}, t+\frac{1}{2}[\notin \mathcal{A}$$

and μ be an additive extension of ν such that $\mu(B) = 0$. Finally, let μ' be an additive extension of ν such that $\mu'(\mathbb{Z}) = 1$. Then, by putting $\mu_n = \mu'$ for all n, we have $\mu_n \xrightarrow{\mathrm{d}} \mu$ but μ_n does not weakly converge to μ in the classical sense.

The following theorem characterizes, when $\mathcal{F} = \mathcal{F}^c$, the convergence in distribution for both a net of measures and the net of corresponding conjugate measures.

Theorem 11 (Characterization theorem II). *Let* $\mathcal{F} = \mathcal{F}^c$*. Then, given* μ *and a net* $\{\mu_d; d \in D\}$*, the following statements are equivalent:*

(i) $\mu_d \xrightarrow{\mathrm{d}} \mu$ *and* $\overline{\mu}_d \xrightarrow{\mathrm{d}} \overline{\mu}$*;*
(ii) $\mathrm{C}\int_{\mathbb{R}} X\,d\mu_d \to \mathrm{C}\int_{\mathbb{R}} X\,d\mu$ *for all monotone functions* $X \in C_b(\mathbb{R})$*;*
(iii) $\mu_d \xrightarrow{\mathrm{w}} \mu$ *w.r.t.* $(\mathcal{C},\mathcal{U})$*, where* $\mathcal{U}$ *is the set of open half-lines together with* $\emptyset, \mathbb{R}$ *and* $\mathcal{C} = \mathcal{U}^c$.

Finally, for all rational numbers a,b $(a<b)$, let:

$$\phi_{ab}(x) = \begin{cases} 0 & \text{if } x<a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } a > b \end{cases}.$$

Since the set of all these functions is countable, we can enumerate them as $\phi_1,\dots,\phi_n,\dots$; moreover, let ϕ_0 be the constant function with value 1. Then, the *topology of convergence in distribution* on the set $\mathrm{bm}(\mathbb{R},\mathcal{F})$ is the topology such that, for any μ, the basic neighborhoods of μ are the sets of the form:

$$\{\mu' : |\mathrm{C}\int_{\mathbb{R}} \phi_{h_i} d\mu' - \mathrm{C}\int_{\mathbb{R}} \phi_{h_i} d\mu| < \varepsilon \quad (i = 1,\dots,m)\},$$

where $\varepsilon > 0$ and $\phi_{h_1},\dots,\phi_{h_m} \in \{\phi_0,\phi_1,\dots\}$ $(m \ge 1)$.

Theorem 12. *Given μ and a net $\{\mu_d; d \in D\}$, we have $\mu_d \overset{\mathrm{d}}{\to} \mu$ iff μ_d converges to μ under the topology of convergence in distribution.*

We conclude on noting that the topological space $\mathrm{bm}(\mathbb{R}, \mathcal{F})$ with the topology of convergence in distribution is metrizable by the pseudo-metric:

$$\delta(\mu,\mu') = \sum_{n=0}^{+\infty} 2^{-n} \, |\mathrm{C}\int_{\mathbb{R}} \phi_n d\mu' - \mathrm{C}\int_{\mathbb{R}} \phi_n d\mu|$$

and, in this way, it turns out to be a σ-compact Polish space.

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Some notes on ordinal strategic interaction with possibilistic expectation

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Games with ordinal utilities have been studied extensively in the literature (see, for instance, [1, 4, 5]). Here, we try a different approach relying on the possibilistic concept of expectation based on the Sugeno integral (see [2]). In particular, we investigate ordinal games, and their related notion of equilibrium, based on optimistic and pessimistic expectations derived from Possibility Decision Theory [3].

In the following, L denotes a linearly ordered set with minimum 0 and maximum 1.

Definition 1 (Game). *We define a n-person game in normal form as a tuple*

$$\mathcal{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}$$

where:

1. *each $S_i = \{s_{i1}, \ldots, s_{iK}\}$ denotes the set of pure strategies available to player i, also called the* strategy space*, and*
2. *each $u_i : S_1 \times \cdots \times S_n \to L$ is a function that sets player i's payoff for each combination of the players' strategies.*

The above definition of a game corresponds to the standard one for pure strategies. We introduce a concept of mixed strategy based on (normalized) possibility distributions.

Definition 2 (Possibilistic Mixed Strategy). *Given a normal-form game*

$$\mathcal{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\},$$

a possibilistic mixed strategy *for a player i is a possibility distribution $\pi_i = S_i \to L$ such that* $\sup_k \pi_i(s_k) = 1$*, for $k = 1, \ldots, K$.*

In order to study games of the above form with possibilistic mixed strategies, we need a notion of expected utility. Possibilistic Decision Theory offers two basic notions of expectation, an optimistic and a pessimistic one. We investigate the main properties of games under both concepts.

1 Optimistic Expectation

Suppose we have 2-player game $\mathcal{G} = \{S_1, S_2; u_1, u_2\}$, where $S_1 = \{s_{11}, \ldots, s_{1J}\}$, $S_2 = \{s_{21}, \ldots, s_{2K}\}$. Player 1's optimistic payoff from playing the possibilistic mixed strategy π_1 is given by

$$E_1^+(\pi_1, \pi_2) = \bigvee_{j=1}^{J} \left(\pi_1(s_{1j}) \wedge \left(\bigvee_{k=1}^{K} (\pi_2(s_{2k}) \wedge u_1(s_{1j}, s_{2k})) \right) \right) \tag{1}$$

Similarly, Player 2's expected optimistic payoff from playing the possibilistic mixed strategy π_2 is

$$E_2^+(\pi_1, \pi_2) = \bigvee_{k=1}^{K} \left(\pi_2(s_{2k}) \wedge \left(\bigvee_{j=1}^{J} (\pi_1(s_{1j}) \wedge u_2(s_{1j}, s_{2k})) \right) \right) \tag{2}$$

2 Pessimistic Expectation

Suppose we have 2-player game $\mathcal{G} = \{S_1, S_2; u_1, u_2\}$, where $S_1 = \{s_{11}, \ldots, s_{1J}\}$, $S_2 = \{s_{21}, \ldots, s_{2K}\}$. Player 1's expected pessimistic payoff from playing the possibilistic mixed strategy π_1 is

$$E_1^-(\pi_1, \pi_2) = \bigwedge_{j=1}^{J} \left(\left((1 - \pi_1(s_{1j})) \vee \left(\bigwedge_{k=1}^{K} (((1 - \pi_2(s_{2k})) \vee u_1(s_{1j}, s_{2k}))) \right) \right) \right) \tag{3}$$

Similarly, Player 2's expected pessimistic payoff from playing the possibilistic mixed strategy π_2 is

$$E_2^-(\pi_1, \pi_2) = \bigwedge_{k=1}^{K} \left(\left((1 - \pi_2(s_{2k})) \vee \left(\bigwedge_{j=1}^{J} (((1 - \pi_1(s_{1j})) \vee u_2(s_{1j}, s_{2k}))) \right) \right) \right) \tag{4}$$

3 Equilibrium

The above definitions are given for two-player games for the sake of simplicity, but can be generalized to games with n players.

Let Σ_i denote the set of mixed strategies of player i. Player i's *best optimistic response* (*best pessimistic response*) to player j's mixed strategy π_j is a mixed strategy π_i such that $E_i^+(\pi_i, \pi_j) \geq E_i^+(\pi_i', \pi_j)$ ($E_i^-(\pi_i, \pi_j) \geq E_i^-(\pi_i', \pi_j)$) for all strategies $\pi_i' \in \Sigma_i$.

Definition 3 (Optimistic and Pessimistic Equilibria). *In an n-person game*

$$\mathcal{G} = \{S_1, \ldots, S_n; u_1, \ldots, u_n\},$$

we call a tuple of possibilistic mixed strategies $(\pi_1^*, \ldots, \pi_n^*)$*:*

1. *an* optimistic equilibrium *if each player's mixed strategy is a best optimistic response to the other players' mixed strategy;*
2. *a* pessimistic equilibrium *if each player's mixed strategy is a best pessimistic response to the other players' mixed strategy;*

We discuss the meaning of both the optimistic and pessimistic approach and, in particular, we examine the existence of possibilistic equilibria and its consequences for ordinal interactions.

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Classification results on residuated lattices – a survey with new results

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Ward and Dilworth, to investigate ideal theory of commutative rings with unit, introduced residuated lattices in the 30s of the last century. Examples of residuated lattices include Boolean algebras, Heyting algebras, MV-algebras, BL-algebras, and lattice-ordered groups; a variety of other algebraic structures can be rendered as residuated lattices. Ono introduced substructural logics; they encompass classical logic, intuitionistic logic, relevance logics, many-valued logics, mathematical fuzzy logics, linear logic and their non-commutative versions. The theory of substructural logics has put all these logics, along with many others, under the same motivational and methodological umbrella. Residuated lattices, being the algebraic counterpart of substructural logics just like Boolean algebras are for classical logic, have been the key component in this remarkable unification. Applications of substructural logics and residuated lattices span across proof theory, algebra, and computer science.

In this talk classification results on residuated lattices will be surveyed ranging from Hölder's precursor via Aczél, Clifford, Mostert, and Shields to the most recent ones. Also the latest findings will be presented.

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Games with fuzzy coalitions: modelling partial coalitional memberships

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Games with fuzzy coalitions [1, 5, 3] are cooperative games in which the formation of coalitions with partial memberships of players is explicitly modelled. Let $N = \{1,\dots,n\}$ be a finite set of *players*. A *fuzzy coalition* is a vector $a = (a_1,\dots,a_n) \in [0,1]^n$, where each coordinate a_i represents a degree of participation of player i in the coalition a. A *game with fuzzy coalitions* is a pair (N, v), where N is a finite set of n players and v is a bounded function $[0,1]^n \to \mathbb{R}$ vanishing at the zero vector (empty coalition). There are several generalizations of this model, games with fuzzy coalitions over an infinite player set [2, 5] being one of them. Interestingly, a game with fuzzy coalitions can be associated already with a classical cooperative game w over the coalitional system 2^N [10] by taking an appropriate extension of w over $[0,1]^n$. For example, Owen [9] extends w to $[0,1]^n$ as a multilinear function and Tsurumi et al. [11] uses the Choquet integral with respect to w.

A *solution* on a class of games with fuzzy coalitions can be defined as a multifunction sending each game v to a subset $\sigma(v)$ of payoff vectors in some $\mathbb{R}^n$. The shape of $\sigma(v)$ depends on the postulated principles of economic rationality or axioms governing the behavior of players. In this way we arrive at the core solution, Shapley value, or other solution concepts—see [4] for a survey.

While there have been developed many solution concepts and there exists even numerical procedures for their approximation [6], the difficulties may arise when we want to give the behavioral interpretation to the membership degrees. What does it mean that a player participates in a fuzzy coalition a to a degree 0.7? The aim of this contribution is thus twofold. First, we will show that most games with fuzzy coalitions can be studied with the help of MV-algebras [7]—the many-valued analogues of Boolean algebras—in the unified way. This generalization makes possible to study some natural generalizations [8] of important classes of classical games, such as simple games corresponding to voting in committees. In particular, we will present an example of a game where the players' memberships degrees are proportions of affirmative votes for some proposal. Second, we try to interpret the players' membership degrees in coalitions by adopting the minimax characterization of a (superadditive) coalition game, which was considered already by von Neumann and Morgenstern [12].

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Convergence theorems in monotone measure theory

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In classical measure theory, Egoroff's theorem, Lebesgue's theorem and Riesz's theorem, etc., are important convergence theorems. They describe implication relationship between three convergence concepts:almost everywhere convergence, almost uniform convergence, and convergence in measure of measurable function sequence. These well-known theorems are discussed and generalized in monotone measure theory and a lot of results are obtained [5-32].

Since monotone measures generally lose additivity, the theorems in classical measure theory do not hold for monotone measures without additional conditions. On the other hand, if only three concepts (a.e. convergence, a.u. convergence, and convergence in measure) are considered in classical measure theory, we should discuss six implication relations among them. Three of these relations are described by the above mentioned theorems. But now, in monotone measure theory, since each convergence concept splits into two (e.g, the concept "almost everywhere" splits into two different concepts "almost everywhere" and "pseudo-almost everywhere"), there are 30 implication relations we should discuss. Thus, for Egoroff's theorem, Lebesgue's theorem and Riesz's theorem, each of them derive four different forms. We shall present the most important relations in this contribution.

Let X be a non-empty set, $\mathcal{F}$ be a σ-algebra of subsets of X. Unless stated otherwise, all the subsets mentioned are supposed to belong to $\mathcal{F}$.

Definition 1. ([19, 28]) *A monotone measure on $\mathcal{F}$ is an extended real valued set function $\mu : \mathcal{F} \to [0,+\infty]$ satisfying the following conditions:*
(1) $\mu(\emptyset) = 0$ *(vanishing at $\emptyset$) and* $\mu(X) > 0$*;*
(2) $\mu(A) \leq \mu(B)$ *whenever $A \subset B$ and $A, B \in \mathcal{F}$ (monotonicity).*

When μ is a monotone measure, the triple $(X, \mathcal{F}, \mu)$ is called a monotone measure space ([19]).

In this paper, we always assume that μ is a monotone measure on $\mathcal{F}$.

When μ is finite, we define the conjugate $\overline{\mu}$ of μ by $\overline{\mu}(A) = \mu(X) - \mu(X \setminus A)$, $A \in \mathcal{F}$.

A monotone measure $\mu : \mathcal{F} \to [0,+\infty]$ is said to be *continuous from below*, if $\lim_{n\to\infty}\mu(A_n) = \mu(A)$ whenever $A_n \nearrow A$. μ is called to have *property* (S) [24], if for any $\{A_n\}_n$ with $\lim_{n\to+\infty}\mu(A_n) = 0$, there exists a subsequence $\{A_{n_i}\}_i$ of $\{A_n\}_n$ such that $\mu(\limsup A_{n_i}) = 0$; *property* (PS), if for any $A \in \mathcal{F}$, $\{A_n\}_n \subset A \cap \mathcal{F}$ with $\lim\limits_{n\to+\infty}\mu(A \setminus A_n) = \mu(A)$, there exists a subsequence $\{A_{n_i}\}_i$ of $\{A_n\}_n$ such that $\mu(A \setminus \limsup A_{n_i}) = \mu(A)$.

Definition 2. ([14]) *A set function $\mu : \mathcal{F} \to [0,+\infty]$ is said to be strongly order continuous, if* $\lim_{n\to\infty}\mu(A_n) = 0$ *whenever* $A_n \searrow A$ *and* $\mu(A) = 0$.

Definition 3. ([11, 15]) *A set function $\mu : \mathcal{F} \to [0,+\infty]$ is said to fulfil condition (E), if for every double sequence $\{E_n^{(m)}\} \subset \mathcal{F}$ $(m,n \in N)$ satisfying the conditions: for any fixed $m = 1,2,\ldots$,*

$$E_n^{(m)} \searrow E^{(m)} \ (n \to \infty) \text{ and } \mu\left(\bigcup_{m=1}^{+\infty} E^{(m)}\right) = 0$$

there exist increasing sequences $\{n_i\}_{i\in N}$ and $\{m_i\}_{i\in N}$ of natural numbers, such that

$$\lim_{k\to+\infty}\mu\left(\bigcup_{i=k}^{+\infty} E_{n_i}^{(m_i)}\right) = 0.$$

Let $\mathbf{F}$ be the class of all finite real-valued measurable functions on $(X,\mathcal{F},\mu)$, and let $f, f_n \in \mathbf{F}$ $(n = 1,2,\ldots)$. We say that $\{f_n\}$ *converges almost everywhere to f on X* , and denote it by $f_n \xrightarrow{a.e.} f$, if there is subset $E \subset X$ such that $\mu(E) = 0$ and $f_n \to f$ on $X \setminus E$; $\{f_n\}_n$ *converges pseudo-almost everywhere to f on A* if there is a subset $F \subset A$ such that $\mu(A\setminus F) = \mu(A)$ and $f_n \to f$ on $A\setminus F$, and denote it by $f_n \xrightarrow{p.a.e.} f$ on A; $\{f_n\}$ *converges almost uniformly to f on X*, and denote it by $f_n \xrightarrow{a.u.} f$, if for any $\varepsilon > 0$ there is a subset $E_\varepsilon \in \mathcal{F}$ such that $\mu(X\setminus E_\varepsilon) < \varepsilon$ and f_n converges to f uniformly on E_ε; $\{f_n\}$ *converges to f pseudo-almost uniformly on A* and, denote it by $f_n \xrightarrow{p.a.u.} f$ on A, if there exists $\{F_k\} \subset \mathcal{F}$ with $\lim\limits_{k\to+\infty}\mu(A\setminus F_k) = \mu(A)$ such that f_n converges to f on $A\setminus F_k$ uniformly for any fixed $k = 1,2,\ldots$; $\{f_n\}$ *converge to f in measure μ* (resp. *pseudo-in measure μ*) on A, in symbols $f_n \xrightarrow{\mu} f$ (resp. $f_n \xrightarrow{p.\mu} f$), if for any $\sigma > 0$, $\lim_{n\to\infty}\mu(\{x : |f_n(x) - f(x)| \geq \sigma\} \cap A) = 0$ (resp. $\lim\limits_{n\to+\infty}\mu(\{x : |f_n - f| < \sigma\} \cap A) = \mu(A)$).

In the following we present Egoroff's theorem, Lebesgue's theorem and Riesz's theorem on monotone measure space, respectively.

Theorem 1. (Egoroff's theorem) ([11, 15]) *Let μ be a finite monotone measure. Then,*

(1) *μ fulfils condition (E) iff for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$,*

$$f_n \xrightarrow{a.e.} f \implies f_n \xrightarrow{a.u.} f.$$

(2) *$\overline{\mu}$ fulfils condition (E) iff for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$,*

$$f_n \xrightarrow{p.a.e.} f \implies f_n \xrightarrow{p.a.u.} f.$$

Theorem 2. (Lebesgue's theorem) ([10, 23]) *Let μ be a monotone measure. Then,*

(1) *μ is strongly order continuous iff, for any $A \in \mathcal{F}$, $f \in \mathbf{F}$ and $\{f_n\} \subset \mathbf{F}$,*

$$f_n \xrightarrow[A]{a.e.} f \implies f_n \xrightarrow[A]{\mu} f;$$

(2) *μ is continuous from below iff, for any $A \in \mathcal{F}$, $f \in \mathbf{F}$ and $\{f_n\} \subset \mathbf{F}$,*

$$f_n \xrightarrow[A]{p.a.e.} f \implies f_n \xrightarrow[A]{p.\mu} f.$$

Theorem 3. (Riesz's theorem) ([13, 16, 24]) *Let μ be a monotone measure. Then,*

(1) *μ has property (S) iff for any $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{\mu} f$, there exists a subsequence $\{f_{n_i}\}_i$ of $\{f_n\}_n$ such that $f_{n_i} \xrightarrow{a.e.} f$;*

(2) *μ has property (PS) iff for any $A \in \mathcal{F}$, $f \in \mathbf{F}$ and $\{f_n\}_n \subset \mathbf{F}$, $f_n \xrightarrow{p.\mu} f$ on A, there exists a subsequence $\{f_{n_i}\}_i$ of $\{f_n\}_n$ such that $f_{n_i} \xrightarrow{p.a.e.} f$ on A.*

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On the cardinality index of fuzzy measures and the signatures of coherent systems

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Boolean and pseudo-Boolean functions play a central role in various areas of applied mathematics. We will focus here on their use in decision making, cooperative game theory, and engineering reliability theory.

A *discrete fuzzy measure* on the finite set $X = \{1,\ldots,n\}$ is a nondecreasing set function $\mu\colon 2^X \to [0,1]$ satisfying the boundary conditions $\mu(\varnothing) = 0$ and $\mu(X) = 1$. For any subset $S \subseteq X$, the number $\mu(S)$ can be interpreted as the certitude that we have that a variable will take on its value in the set $S \subseteq X$.

A *cooperative game* on a finite set of players $N = \{1,\ldots,n\}$ is a set function $v\colon 2^N \to \mathbb{R}$ which assigns to each coalition S of players a real number $v(S)$. This number represents the *worth* of S. (Even though the condition $v(\varnothing) = 0$ is often required for v to define a game, here we do not need this restriction.)

A *system* is defined by a finite set of components $C = \{1,\ldots,n\}$ that are interconnected according to a certain structure. The components are either in function or in a failed state, and the same holds for the whole system. It is common to associate the Boolean value 0 with a failed state and the value 1 with a component that is in function. Therefore the *structure function* of a system is the function ϕ from 2^C to $\mathbb{B} = \{0,1\}$ which associates with any set A of components that are in function the corresponding state of the system. The system is semicoherent if the structure function is nondecreasing and satisfies the conditions $\phi(\varnothing) = 0$ and $\phi(C) = 1$. It is coherent if in addition all the components are essential.

We identify any subset S of $\{1,\ldots,n\}$ with its characteristic vector $\mathbf{1}_S \in \{0,1\}^n$ (defined by $(\mathbf{1}_S)_k = 1$ if and only if k is in S). This identification allows us to identify set functions and pseudo-Boolean functions, i.e., functions from $\mathbb{B}^n$ to $\mathbb{R}$. Therefore discrete fuzzy measures, cooperative games, and structure functions of coherent systems are all described by pseudo-Boolean functions.

The use of discrete fuzzy measures allows us to model real situations where additivity is not suitable since the set of such measures is richer than the set of classical additive measures. In the same way, cooperative games allow us to take into account possible interactions between the players and need not be additive. Finally, the set of all increasing pseudo-Boolean functions is necessary to describe all the possible semicoherent systems. However, the variousness of this set of functions also has the drawback that a general set function (fuzzy measure, cooperative game, or structure function of a system) might be difficult to interpret or analyze.

Various kinds of power indexes, or values, are used in cooperative game theory to overcome this problem. They measure the influence that a given player has on the outcome of the game or define a way of sharing the benefits of the game among the players. The best known values, due to Shapley [10] and Banzhaf [1], are defined in the following way. The Shapley value of player k in a game v on the set of players $[n]=\{1,\ldots,n\}$ is defined by

$$\phi_{\mathrm{Sh}}(v,k)=\sum_{S\subseteq[n]\setminus\{k\}}\frac{(n-s-1)!s!}{n!}(v(S\cup\{k\})-v(S)), \tag{1}$$

while the Banzhaf value is given by

$$\phi_B(v,k) \;=\; \frac{1}{2^{n-1}}\sum_{S\subseteq[n]\setminus\{k\}}\big(v(S\cup\{k\})-v(S)\big) \;=\; \frac{1}{2^{n-1}}\sum_{S\ni k}v(S)-\frac{1}{2^{n-1}}\sum_{S\not\ni k}v(S). \tag{2}$$

There are several axiomatic characterizations of these values. They are also used to analyze fuzzy measures and were generalized by the concepts of Shapley or Banzhaf *interaction indexes*; see, e.g., [5].

In reliability theory of coherent systems, the importance of component k for system S can also be measured in various ways. Assuming that the components of the system have continuous i.i.d. lifetimes $T_1,\ldots,T_n$, Barlow and Proschan [2] introduced in 1975 the n-tuple $\mathbf{I}_{\mathrm{BP}}$ (the Barlow-Proschan index) whose kth coordinate ($k\in[n]$) is the probability that the failure of component k causes the system to fail; that is,

$$I_{\mathrm{BP}}^{(k)} \;=\; \Pr(T_S=T_k)\,,$$

where T_S denotes the system lifetime. It turns out that for continuous i.i.d. component lifetimes, this index reduces to the Shapley value of the system structure function.

In this note we consider slightly different importance indexes that do not measure the influence of a given variable over a function but rather the influence of adding a variable to a given subset of variables. These indexes are the *cardinality index* introduced in 2002 by Yager [11] in the context of fuzzy measures and the *signature* of coherent systems introduced in 1985 by Samaniego [8, 9].

The cardinality index associated with a fuzzy measure μ on $X=\{1,\ldots,n\}$ is the n-tuple $(C_0,\ldots,C_{n-1})$, where C_k is the average gain in certitude that we obtain by adding an arbitrary element to an arbitrary k-element subset, that is,

$$C_k=\frac{1}{(n-k)\binom{n}{k}}\sum_{|S|=k}\sum_{x\notin S}\big(\mu(S\cup\{x\})-\mu(S)\big).$$

We observe that this expression, which resembles the Banzhaf value (2), could be used in cooperative game theory to measure the marginal contribution of an additional player to a k-element coalition. It is also clear that this index can be written as

$$C_k=\frac{1}{\binom{n}{k+1}}\sum_{|S|=k+1}\mu(S)-\frac{1}{\binom{n}{k}}\sum_{|S|=k}\mu(S)\,. \tag{3}$$

The signature of a *system* consisting of n interconnected components having continuous and i.i.d. lifetimes $T_1,\ldots,T_n$ is defined as the n-tuple $(s_1,\ldots,s_n)\in[0,1]^n$ with $s_k=\Pr(T_S=T_{(k)})$, where T_S denotes the system lifetime and $T_{(k)}$ is the kth order statistic derived from $T_1,\ldots,T_n$, i.e., the kth smallest lifetime. Thus, s_k is the probability that the kth failure causes the system to fail. It was proved [3] that

$$s_k = \frac{1}{\binom{n}{n-k+1}} \sum_{|\mathbf{x}|=n-k+1} \phi(\mathbf{x}) - \frac{1}{\binom{n}{n-k}} \sum_{|\mathbf{x}|=n-k} \phi(\mathbf{x})\,, \tag{4}$$

where $\phi\colon \mathbb{B}^n \to \mathbb{B}$ is the structure function of the system.

Clearly, Equations (3) and (4) show that for a given pseudo-Boolean function, the cardinality index and the signature are related by the formula $s_k = C_{n-k}$ for $1\leqslant k\leqslant n$.

We now show in detail some properties of the cardinality index and signatures that are similar to properties of the Banzhaf and Shapley values. First we show how this index can be computed from the multilinear extension of the pseudo-Boolean function (or set function) under consideration.

Recall that any set function $f\colon 2^{[n]}\to\mathbb{R}$ can be represented in a unique way as a multilinear polynomial. This representation is given by

$$f(x_1,\ldots,x_n)=\sum_{S\subseteq[n]} f(S)\prod_{i\in S}x_i\prod_{i\notin S}(1-x_i), \qquad x_i\in\mathbb{B}.$$

The multilinear extension $\hat{f}$ of f is then given by the same polynomial expression but for variables in $[0,1]$. For a function $g\colon [0,1]^n\to\mathbb{R}$, we simply denote by $g(x)$ the polynomial function $g(x,\ldots,x)$. The Banzhaf and Shapley indexes for f can be computed easily from the multilinear extension of f. In fact, the Shapley index of player k in f is given by

$$\phi_{\mathrm{Sh}}(f,k)=\int_0^1\Big(\frac{\partial}{\partial x_k}\hat{f}\Big)(x)\,dx.$$

The Banzhaf index of player k in f is given by $\phi_{\mathrm{B}}(f,k)=(\frac{\partial}{\partial x_k}\hat{f})(\frac{1}{2})$.

It is possible to obtain a similar formula for the *tail signature*, or the *cumulative cardinality index* that we now introduce. These are the $(n+1)$-tuples $\overline{\mathbf{S}}=(\overline{S}_0,\ldots,\overline{S}_n)$ and $\overline{\mathbf{C}}=(\overline{C}_0,\ldots,\overline{C}_n)$, respectively, defined by (see (4) and (3))

$$\overline{S}_k \;=\; \sum_{i=k+1}^{n} s_i \;=\; \frac{1}{\binom{n}{n-k}} \sum_{|A|=n-k} \phi(A) = \overline{C}_{n-k}\,. \tag{5}$$

With any pseudo-Boolean function f on $\{0,1\}^n$, we associate the polynomial function p_f defined by $p_f(x)=x^n\hat{f}(1/x)$.

Theorem 1. *The cumulative cardinality index and the tail signature are obtain from p_f by*

$$p_f(x) \;=\; \sum_{k=0}^{n}\binom{n}{k}\overline{C}_{n-k}\,(x-1)^k \;=\; \sum_{k=0}^{n}\binom{n}{k}\overline{S}_k\,(x-1)^k.$$

The Banzhaf and Shapley values can be obtained via (weighted) least squares approximations of the pseudo-Boolean function by a pseudo-Boolean function of degree 1; see, e.g., [4, 6]. A similar property holds for the cardinality index and signature : they can be obtained via least squares approximations of the given pseudo-Boolean function by a symmetric pseudo-Boolean function. Recall that for $k \in \{1,\ldots,n\}$ the *kth order statistic function* is the function $\mathrm{os}_k \colon \mathbb{B}^n \to \mathbb{B}$, defined by the condition $\mathrm{os}_k(\mathbf{x}) = 1$, if $|\mathbf{x}| = \sum_{i=1}^n x_i \geqslant n-k+1$, and 0, otherwise. We also formally define $\mathrm{os}_{n+1} \equiv 1$. Then it can be showed that the space of symmetric pseudo-Boolean functions is spanned by the order statistic functions. The *best symmetric approximation* of a pseudo-Boolean function f is the unique symmetric pseudo-Boolean function f_S that minimizes the weighted squared distance

$$\|f-g\|^2 = \sum_{\mathbf{x}\in\mathbb{B}^n} \frac{1}{\binom{n}{|\mathbf{x}|}} \big(f(\mathbf{x}) - g(\mathbf{x})\big)^2$$

from among all symmetric functions pseudo-Boolean functions g.

Theorem 2. *The best symmetric approximation of a pseudo-Boolean function f such that $f(0,\ldots,0) = 0$ is given by*

$$f_S = \sum_{k=1}^n s_k \, \mathrm{os}_k = \sum_{k=1}^n C_{n-k} \, \mathrm{os}_k \tag{6}$$

where s_k and C_k are derived from f by using (4) and (3), respectively.

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Pivotal decompositions of aggregation functions

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1 Preliminaries

A remarkable (though immediate) property of Boolean functions is the so-called *Shannon decomposition* [9], also called *pivotal decomposition* [1]. This property states that, for every n-ary Boolean function $f\colon \{0,1\}^n \to \{0,1\}$ and every $k \in [n] = \{1,\ldots,n\}$, the following decomposition formula holds

$$f(\mathbf{x}) \ = \ \overline{x}_k\, f(\mathbf{x}_k^0) + x_k\, f(\mathbf{x}_k^1)\,, \qquad \mathbf{x} \in \{0,1\}^n, \tag{1}$$

where $\overline{x}_k = 1 - x_k$ and $\mathbf{x}_k^0$ (resp. $\mathbf{x}_k^1$) is the n-tuple whose i-th coordinate is 0 (resp. 1), if $i = k$, and x_i, otherwise. Here the '$+$' sign represents the classical addition for real numbers.

As it is well known, repeated applications of (1) show that any n-ary Boolean function can always be expressed as the multilinear polynomial function

$$f(\mathbf{x}) \ = \ \sum_{S \subseteq [n]} f(\mathbf{1}_S) \prod_{i \in S} x_i \prod_{i \in [n] \setminus S} \overline{x}_i\,, \qquad \mathbf{x} \in \{0,1\}^n, \tag{2}$$

where $\mathbf{1}_S$ is the characteristic vector of S in $\{0,1\}^n$, that is, the n-tuple whose i-th coordinate is 1, if $i \in S$, and 0, otherwise.

One can easily show that, if f is nondecreasing (in each variable), decomposition formula (1) reduces to

$$f(\mathbf{x}) \ = \ \mathrm{med}(f(\mathbf{x}_k^0), x_k, f(\mathbf{x}_k^1))\,, \qquad \mathbf{x} \in \{0,1\}^n, \tag{3}$$

or, equivalently,

$$f(\mathbf{x}) \ = \ \overline{x}_k\,(f(\mathbf{x}_k^0) \wedge f(\mathbf{x}_k^1)) + x_k\,(f(\mathbf{x}_k^0) \vee f(\mathbf{x}_k^1))\,, \qquad \mathbf{x} \in \{0,1\}^n. \tag{4}$$

where $\wedge$ (resp. $\vee$) is the minimum (resp. maximum) operation and med is the ternary median operation.

Actually, any of the decomposition formulas (3)–(4) exactly expresses the fact that f should be nondecreasing and hence characterizes the subclass of nondecreasing n-ary Boolean functions.

Decomposition property (1) also holds for functions $f\colon \{0,1\}^n \to \mathbb{R}$, called *$n$-ary pseudo-Boolean functions*. As a consequence, these functions also have the representation given in (2). Moreover, formula (4) clearly characterizes the subclass of nondecreasing n-ary pseudo-Boolean functions.

The *multilinear extension* of an n-ary pseudo-Boolean function $f\colon \{0,1\}^n \to \mathbb{R}$ is the function $\hat{f}\colon [0,1]^n \to \mathbb{R}$ defined by (see Owen [7, 8])

$$\hat{f}(\mathbf{x}) \;=\; \sum_{S\subseteq[n]} f(\mathbf{1}_S)\prod_{i\in S}x_i \prod_{i\in[n]\setminus S}(1-x_i)\,, \qquad \mathbf{x}\in[0,1]^n.$$

Thus defined, one can easily see that the class of multilinear extensions and that of nondecreasing multilinear extensions can be characterized as follows.

Proposition 1. *A function $f\colon [0,1]^n \to \mathbb{R}$ is a multilinear extension if and only if it satisfies*

$$f(\mathbf{x}) \;=\; (1-x_k)\,f(\mathbf{x}_k^0)+x_k\,f(\mathbf{x}_k^1)\,, \qquad \mathbf{x}\in[0,1]^n,\ k\in[n].$$

Proposition 2. *A function $f\colon [0,1]^n \to \mathbb{R}$ is a nondecreasing multilinear extension if and only if it satisfies*

$$f(\mathbf{x}) \;=\; \overline{x}_k\,(f(\mathbf{x}_k^0)\wedge f(\mathbf{x}_k^1))+x_k\,(f(\mathbf{x}_k^0)\vee f(\mathbf{x}_k^1))\,, \qquad \mathbf{x}\in[0,1]^n,\ k\in[n].$$

The decomposition formulas considered in this introduction share an interesting common feature, namely the fact that any variable, here denoted x_k and called *pivot*, can be isolated from the others in the evaluation of functions. This feature may be useful when for instance the values $f(\mathbf{x}_k^0)$ and $f(\mathbf{x}_k^1)$ are much easier to compute than that of $f(\mathbf{x})$. In addition to this, such (pivotal) decompositions may facilitate inductive proofs and may lead to canonical forms such as (2).

In this note we define a more general concept of pivotal decomposition for various functions $f\colon [0,1]^n \to \mathbb{R}$, including certain aggregation functions. We also introduce pivotal characterizations of classes of such functions.

2 Pivotal decompositions of functions

The examples presented in the previous section motivate the following definition.

Definition 1. *We say that a function $f\colon [0,1]^n \to \mathbb{R}$ is* pivotally decomposable *if there exists a subset D of $\mathbb{R}^3$ and a function $\Phi\colon D\to\mathbb{R}$, called* pivotal function, *such that*

$$D \supseteq \{(f(\mathbf{x}_k^0),z,f(\mathbf{x}_k^1)) : z\in[0,1],\mathbf{x}\in[0,1]^n\}\,, \qquad k\in[n]$$

and

$$f(\mathbf{x}) \;=\; \Phi(f(\mathbf{x}_k^0),x_k,f(\mathbf{x}_k^1))\,, \qquad \mathbf{x}\in[0,1]^n,\ k\in[n].$$

In this case, we say that f is Φ-decomposable.

Example 1 (Lattice polynomial functions). Recall that a *lattice polynomial function* is simply a composition of projections, constant functions, and the fundamental lattice operations $\wedge$ and $\vee$; see, e.g., [3, 4]. An n-ary function $f\colon [0,1]^n \to [0,1]$ is a lattice polynomial function if and only if it can be written in the (disjunctive normal) form

$$f(\mathbf{x}) \;=\; \bigvee_{S\subseteq[n]} f(\mathbf{1}_S)\wedge\bigwedge_{i\in S}x_i\,, \qquad \mathbf{x}\in[0,1]^n.$$

The so-called *discrete Sugeno integrals* are exactly those lattice polynomial functions which are idempotent (i.e., $f(x,\ldots,x)=x$ for all $x\in[0,1]$).

Every lattice polynomial function is Φ-decomposable with $\Phi\colon [0,1]^3\to\mathbb{R}$ defined by $\Phi(r,z,s)=\mathrm{med}(r,z,s)$; see, e.g., [6].

Example 2 (Lovász extensions). Recall that the *Lovász extension* of a pseudo-Boolean function $f\colon \{0,1\}^n\to\mathbb{R}$ is the function

$$L_f(\mathbf{x}) \;=\; \sum_{S\subseteq[n]} a(S)\bigwedge_{i\in S} x_i\,,$$

where the set function $a\colon 2^{[n]}\to\mathbb{R}$ is defined by $a(S)=\sum_{T\subseteq S}(-1)^{|S|-|T|}f(\mathbf{1}_T)$; see, e.g., [5]. The so-called *discrete Choquet integrals* are exactly those Lovász extensions which are nondecreasing and idempotent.

There are ternary Lovász extensions $L_f\colon [0,1]^3\to\mathbb{R}$ that are not pivotally decomposable, e.g., $L_f(x_1,x_2,x_3)=x_1\wedge x_2+x_2\wedge x_3$.

Example 3 (T-norms). A *t-norm* is a binary function $T\colon [0,1]^2\to[0,1]$ that is symmetric, nondecreasing, associative, and such that $T(1,x)=x$. Every t-norm $T\colon [0,1]^2\to[0,1]$ is Φ-decomposable with $\Phi\colon [0,1]^3\to\mathbb{R}$ defined by $\Phi(r,z,s)=T(z,s)$.

Example 4 (Conjugate functions). Given a function $f\colon [0,1]^n\to[0,1]$ and a strictly increasing bijection $\phi\colon [0,1]\to[0,1]$, the ϕ-conjugate of f is the function $f_\phi=\phi^{-1}\circ f\circ(\phi,\ldots,\phi)$. One can easily show that f is Φ-decomposable for some pivotal function Φ if and only if f_ϕ is Φ_ϕ-decomposable, where $\Phi_\phi=\phi^{-1}\circ\Phi\circ(\phi,\phi,\phi)$. Combining this for instance with Proposition 2 shows that every *quasi-linear mean function* (i.e., ϕ-conjugate of a weighted arithmetic mean) is pivotally decomposable.

For every $k\in[n]$, and every $\mathbf{a}\in[0,1]^n$, we define the *unary section* $f_k^{\mathbf{a}}\colon [0,1]\to\mathbb{R}$ of f by setting $f_k^{\mathbf{a}}(x)=f(\mathbf{a}_k^x)$. The kth argument of f is said to be *inessential* if $f_k^{\mathbf{a}}$ is constant for every $\mathbf{a}\in[0,1]^n$. Otherwise, it is said to be *essential*. We say that a unary section $f_k^{\mathbf{a}}$ of f is *essential* if the kth argument of f is essential.

For every function $f\colon X^n\to Y$ and every map $\sigma\colon [n]\to[m]$, we define the function $f_\sigma\colon X^m\to Y$ by $f_\sigma(\mathbf{a})=f(\mathbf{a}\sigma)$, where $\mathbf{a}\sigma$ denotes the n-tuple $(a_{\sigma(1)},\ldots,a_{\sigma(n)})$.

Define on the set $U=\cup_{n\geqslant 1}\mathbb{R}^{[0,1]^n}$ the equivalence relation $\equiv$ as follows: For functions $f\colon [0,1]^n\to\mathbb{R}$ and $g\colon [0,1]^m\to\mathbb{R}$, we write $f\equiv g$ if there exist maps $\sigma\colon [m]\to[n]$ and $\mu\colon [n]\to[m]$ such that $f=g_\sigma$ and $g=f_\mu$. Equivalently, $f\equiv g$ means that f can be obtained from g by permuting arguments or by adding or deleting inessential arguments.

Definition 2. *Let $\Phi\colon D\to\mathbb{R}$ be a pivotal function. We denote by C_Φ the class of all the functions $f\colon [0,1]^n\to\mathbb{R}$ (where $n\geqslant 0$) that are $\equiv$-equivalent to a Φ-decomposable function with no essential argument or no inessential argument. We say that a class $C\subseteq U$ is* pivotally characterizable *if there exists a pivotal function Φ such that $C=C_\Phi$. In that case, we say that C is* Φ-characterized.

Proposition 3. *Let Φ be a pivotal function.*

(i) A nonconstant function $f\colon [0,1]^n\to\mathbb{R}$ is in C_Φ if and only if so are its essential unary sections.

(ii) A constant function $f\colon [0,1]^n \to \{c\}$ is in C_Φ if and only if $\Phi(c,z,c) = c$ for every $z \in [0,1]$.

Example 5 (Lattice polynomial functions). The class of lattice polynomial functions is Φ-characterized for the pivotal function $\Phi\colon [0,1]^3 \to \mathbb{R}$ defined by $\Phi(r,z,s) = \mathrm{med}(r,z,s)$.

3 Classes characterized by their unary members

Proposition 3 shows that a class C_Φ is characterized by the essential unary sections of its members. This observation motivates the following definition, which is inspired from [2].

Definition 3. *A class $C \subseteq U$ is* characterized by its unary members *if it satisfies the following conditions:*

(i) A nonconstant function f is in C if and only if so are its essential unary sections.
(ii) If f is a constant function in C and $g \equiv f$, then g is in C.

We denote by CU *the family of classes characterized by their unary members.*

Theorem 1. *Let Φ be a pivotal function. A nonempty subclass of C_Φ is characterized by its unary members if and only if it is pivotally characterizable.*

Theorem 2. *The family* CU *can be endowed with a complete and atomic Boolean algebra structure.*

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Generalized concave integral

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Integration of simple functions is the base of general integration theory. The aim of this paper is to find a common framework for many integrals using pseudo-operations ([1,3,12,14,16]), for more details see the paper [11].

Let $N = \{1,\dots,n\}$. A set function $\mu : 2^N \to [0,\infty]$ is called a *monotone* measure if $\mu(\varnothing) = 0$, $\mu(N) > 0$, and $\mu(E) \leqslant \mu(F)$ whenever $E \subseteq F \subseteq N$. A *basic function* $b(c,E) : N \to [0,\infty]$ is given by

$$b(c,E)(i) = \begin{cases} c \text{ if } i \in E \\ 0 \text{ else,} \end{cases},$$

where $E \subseteq N$ and $c \in [0,\infty]$. A binary operation $\oplus : [0,\infty]^2 \to [0,\infty]$ is called a *pseudo-addition* whenever it is increasing in both coordinates, 0 is its neutral element, associative, and continuous. An operation $\odot : [0,\infty]^2 \to [0,\infty]$ is called a *pseudo-multiplication* whenever it is increasing in both coordinates, 0 is its annihilator, and for each $x \in]0,\infty[$ there are $y,z \in]0,\infty[$ so that $0 < x \odot y < \infty$ and $0 < z \odot x < \infty$.

Definition 1. *Let $\oplus : [0,\infty]^2 \to [0,\infty]$ be a fixed pseudo-addition. Let $f : N \to [0,\infty]$ be given. A $(\oplus,\odot)$-integral of f with respect to μ is given by*

$$I_{\oplus}^{\odot}(f,\mu) = \sup\left\{ \bigoplus_{i\in I} (a_i \odot \mu(E_i)) \mid (b(a_i,E_i))_{i\in I} \textit{ is a } \oplus-\textit{decomposition of } f \right\}$$

where system $(b(a_i,E_i))_{i\in I}$ of basic functions is called a $\oplus$-decomposition of f if

$$f = \bigoplus_{i\in I} b(a_i,E_i).$$

Lehrer [6] has introduced his concave integral as minimum over all concave and positively homogeneous functionals $H : \mathcal{F} \to [0,\infty]$ satisfying $H(1_E) \geqslant \mu(E)$ for all $E \subseteq N$, where $\mathcal{F}$ is the set of all $N \to [0,\infty]$ functions,

$$I_+^{\cdot}(f,\mu) = \min\{H(f)\}.$$

We introduce the following generalization of Lehrer concave integral.

Definition 2. *Let $\oplus, \odot : [0,\infty]^2 \to [0,\infty]$ be a fixed pseudo-addition and pseudo-multiplication, respectively, such that e is unique left-neutral element of $\odot$. For any monotone measure $\mu : 2^N \to [0,\infty]$, the integral $I_\odot^\oplus : \mathcal{F} \to [0,\infty]$ is given by*

$$I_\odot^\oplus(f,\mu) = \inf\Big\{H(f) \mid H : \mathcal{F} \to [0,\infty] \text{ is } \oplus-\text{superadditive and } \odot-\text{homogeneous}, H(e,E) \geqslant \mu(E) \text{ for all } E \subseteq N\Big\}$$

where the $\oplus$-superadditivity of H means $H(f \oplus g) \geqslant H(f) \oplus H(g)$ for all $f,g \in \mathcal{F}$, and $\odot$-homogeneity of H means $H(\alpha \odot f) = \alpha \odot H(f)$ for all $\alpha > 0$ and $f \in \mathcal{F}$.

In general the integral $I_\odot^\oplus$ can differ from $I_\oplus^\odot$.

Theorem 1. *Let $([0,\infty],\oplus,\odot)$ be a semiring, i.e., $\oplus : [0,\infty]^2 \to [0,\infty]$ is a given pseudo-addition, and $\odot : [0,\infty]^2 \to [0,\infty]$ is an associative commutative $\oplus$-distributive pseudo-multiplication with neutral element $e \in]0,\infty]$ which is continuous on $[0,\infty]^2 \setminus \{(0,\infty), (\infty,0)\}$. Then $I_\odot^\oplus = I_\oplus^\odot$ and this integral will be called pseudo-concave integral.*

Further we consider different kinds of decomposition and subdecomposition of simple functions into basic functions sums, as well as different kinds of pseudo-operations, with the aim to cover many well known integrals. Many well-known integrals assign to a basic function $b(c,E)$ a value dependent on c and measure value $m(E)$ only. This approach was recently exploited as a basic axiom for the universal integrals [4], generalizing many integrals. However, there are integrals violating this rule, namely PAN-integrals [18, 19] and as we have seen concave integral of Lehrer [6]. For these approaches, the integral of a basic function $b(c,E)$ depends on c and measure $m(F)$ of all non-empty subsets $F \subseteq E$.

During the finalization of this paper, a closely related approach to integration (based on $+$ and $\cdot$ and dealing with subdecomposition additivity) appeared in [7]. Finally, recall that the proposal of $(\oplus,\odot)$-integrals generalizing the Lehrer integral appeared for the first time in [10] and it was presented at NLMUA'2011 conference. The future work will be devoted to applications in several domains, especially in decision making and game theory.

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Universal decomposition integrals

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Several distinguished integrals are substantially linked to special set systems. So, for example, the classical Lebesgue integral is based on disjoint set systems (partitions), Choquet integral [1] deals with chains of sets, Shilkret integral [8] considers only single sets. Lehrer in [7] has proposed to deal with arbitrary set systems. All above mentioned integrals for non-negative functions exploit the common addition and multiplication of reals. As a common framework for all of them, Even and Lehrer [2] have proposed a new concept of decomposition integral, see also [9]. Decomposition integrals can be seen as solution of optimization problems under given constraints on the allowed set systems. To define decomposition integrals, we deal with a finite set X and with systems $\mathcal{H}$ of set systems, $\mathcal{H} \in \mathbb{X} \equiv \left(2^{2^{2^X \setminus \{\emptyset\}}}\right) \setminus \{\emptyset\}$.

Definition 1. *[2] Let $\mathcal{H} \in \mathbb{X}$ be fixed. A mapping $I_{\mathcal{H}} : \mathcal{M} \times \mathcal{F} \to [0,\infty]$ given by*

$$I_{\mathcal{H}}(m,f) = \sup\left\{ \sum_{i\in J} a_i \cdot m(A_i) \,\middle|\, (A_i)_{i\in J} \in \mathcal{H}, a_i \geq 0 \text{ for each } i \in J, \sum_{i\in J} a_i \mathbf{1}_{A_i} \leq f \right\} \quad (1)$$

is called an $\mathcal{H}$-decomposition integral. Here $\mathcal{M}$ is the set of all monotone measures on subsets of X and $\mathcal{F}$ is the set of all non-negative functions on X.

We introduce some results for decomposition integrals, especially with respect to their relationship to universal integrals introduced in [5].

Proposition 1. *Let $\mathcal{H} \in \mathbb{X}$. Then the $\mathcal{H}$-decomposition integral gives back the underlying monotone measure, i.e., $I_{\mathcal{H}}(m, \mathbf{1}_A) = m(A)$ for each $m \in \mathcal{M}$ and $A \subseteq X$, if and only if $\mathcal{H}$ is complete and each system $\mathcal{A} \in \mathcal{H}$ is logically independent (i.e., it has a nonempty intersection).*

Proposition 2. *For a fixed $i \in X = \{1,\dots,n\}$, let $\mathcal{H}^{(i)} = \{\mathcal{B} | \mathcal{B}$ is a chain in $2^X \setminus \{\emptyset\}$ with cardinality $i\}$. Then $\mathcal{H}^{(i)} \in \mathbb{X}$ and $I_{\mathcal{H}^{(i)}}$ is a universal integral.*

The last result can be extended for any abstract measurable space.

Definition 2. *Let $n \in \mathbb{N}$ be fixed. The mapping $I^{(n)} : \bigcup_{(X,\mathcal{A})\in\mathcal{S}} \left(\mathcal{M}^{(X,\mathcal{A})} \times \mathcal{F}^{(X,\mathcal{A})}\right) \to [0,\infty]$, where $\mathcal{S}$ is the class of all measurable spaces, $\mathcal{M}^{(X,\mathcal{A})}$ is the set of all monotone* *measures on* $(X,\mathcal{A})$ *and* $\mathcal{F}^{(X,\mathcal{A})}$ *is the set of all measurable functions* $f : X \to [0,\infty]$*, is given by*

$$I^{(n)}(m,f) \;=\;$$
$$\sup\left\{\sum_{i=1}^{n} a_i \cdot m(A_i)\,\middle|\, a_1,\dots,a_n \ge 0, \{A_1,\dots,A_n\} \in \mathcal{A} \text{ is a chain, } \sum_{i=1}^{n} a_i\, 1_{A_i} \le f \right\}.$$

Theorem 1. *For each* $n \in \mathbb{N}$ *the mapping* $I^{(n)}$ *is a universal integral in the sense of [5]. Moreover, it holds*

$$I^{(n)}(m,f) \;=\; \sup\left\{\sum_{i=1}^{n} a_i \cdot m(\{f \ge a_1 + \cdots + a_i\})\,|a_1,\dots,a_n \ge 0\right\}.$$

Remark 1. It is not difficult to check that for $X = \{1,\dots,n\}$, $I^{(i)}|(X,2^X) = I_{\mathcal{H}^{(i)}}$, $i = 1,\dots,n$. Moreover $I^{(1)} = Sh$ is the Shilkret integral acting on any $(X,\mathcal{A}) \in \mathcal{S}$. On the other side, if $X = \{1,\dots,n\}$ is finite, $I^{(n)}|(X,2^X) = I^{(n+j)}|(X,2^X) = Ch$ is the Choquet integral for all $j = 1,2,\dots$. Finally, if X is infinite, then

$$Ch = \sup\{I^{(n)}, n \in \mathbb{N}\} = \lim_{n\to\infty} I^{(n)}.$$

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A decomposition theorem for the fuzzy Henstock integral

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The *n-dimensional fuzzy number space* E^n is defined as the set

$$E^n = \{u\colon R^n \to [0,1]\colon \quad u \text{ satisfies conditions (1)–(4) below}\} :$$

(1) u is a normal fuzzy set, i.e. there exists $x_0 \in R^n$, such that $u(x_0) = 1$;
(2) u is a convex fuzzy set, i.e. $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in R^n$, $t \in [0,1]$;
(3) u is upper semi-continuous;
(4) $\operatorname{supp} u = \overline{\{x \in R^n : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of A.

The main result of my presentation is the following theorem:

A fuzzy-number valued function $\widetilde{\Gamma}\colon [a,b] \to E^n$ is fuzzy Henstock integrable if and only if $\widetilde{\Gamma}$ can be represented as $\widetilde{\Gamma}(t) = \widetilde{G}(t) + \widetilde{f}(t)$, where $\widetilde{G}\colon [a,b] \to E^n$ is fuzzy McShane integrable and $\widetilde{f}$ is a fuzzy Henstock integrable fuzzy number valued function generated by a Henstock integrable selection of Γ.

Generalization of L^p space and related convergences

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We present here, in the framework of the pseudo-analysis, generalization of L^p space and we consider three types of convergence of sequences of measurable functions, for more details see [15]. The main tools will be the Hölder, Minkowski and Markov inequalities for the pseudo-integral, see [2]. The inequalities for integrals based on non-additive measures, e.g., Choquet, Sugeno, pseudo-integral have been recently given, see for an overview [14]. Specially, in [1, 2, 12–14] inequalities with respect to pseudo-integrals were considered. Using idempotent measure of Maslov an analog of L^p space and the convergence of decision variables were presented in [4]. Based on the pseudo-analysis, a generalization of the classical L^p space is constructed in [3]. Several convergence concepts based on the Sugeno and Choquet integrals are observed, see [17, 18].

Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty,\infty]$. The full order on $[a,b]$ will be denoted by $\preceq$. This can be the usual order of the real line, but it can be another order. The operation $\oplus$ (pseudo-addition) is a commutative, non-decreasing (with respect to $\preceq$), associative function $\oplus : [a,b]\times[a,b] \to [a,b]$ with a zero (neutral) element denoted by $\mathbf{0}$. Denote $[a,b]_+ = \{x : x\in[a,b], \mathbf{0}\preceq x\}$. The operation $\odot$ (pseudo-multiplication) is a function $\odot : [a,b]\times[a,b]\to[a,b]$ which is commutative, positively non-decreasing, i.e., $x\preceq y$ implies $x\odot z\preceq y\odot z$, $z\in[a,b]_+$, associative and for which there exist a unit element $\mathbf{1}\in[a,b]$, i.e., for each $x\in[a,b]$, $1\odot x = x$. We assume $\mathbf{0}\odot x = \mathbf{0}$ and that $\odot$ is distributive over $\oplus$. The structure $([a,b],\oplus,\odot)$ is called a *semiring* (see [10]). For $x\in[a,b]_+$ and $p\in\,]0,\infty[$, the pseudo-power $x_\odot^{(p)}$ is defined in the following way. If $p=n$ is an integer then $x_\odot^{(n)} = \underbrace{x\odot x\odot\ldots\odot x}_{n-\text{times}}$. Moreover, $x_\odot^{(\frac{1}{n})} = \sup\left\{y \mid y_\odot^{(n)}\le x\right\}$. Then $x_\odot^{(\frac{m}{n})} = x_\odot^{(r)}$ is well defined for any rational $r\in\,]0,\infty[$, independently of representation $r=\frac{m}{n}=\frac{m_1}{n_1}$, m, n, m_1, n_1 being positive integers (the result follows from the continuity and monotonicity of $\odot$). Due to continuity of $\odot$, if p is not rational, then $x_\odot^{(p)} = \sup\left\{x_\odot^{(r)} \mid r\in\,]0,p[,\ r \text{ is rational}\right\}$. If $\odot$ is idempotent, then $x_\odot^{(p)} = x$ for any $x\in[a,b]$ and $p\in\,]0,\infty[$.

Definition 1. *Let A be a non-empty set. A function $d_\oplus : A\times A\to[a,b]_+$ is a pseudo-metric on A if there hold*

(PM1) $d_\oplus(x,y)=\mathbf{0}$ *iff* $x=y$, *for all* $x,y\in A$,

(PM2) $d_\oplus(x,y) = d_\oplus(y,x)$, *for all* $x,y \in A$

(PM3) *there exists* $c \in [a,b]_+$ *such that for all* $x,y,z \in A$ *it holds*

$$d_\oplus(x,y) \preceq c \odot (d_\oplus(x,z) \oplus d_\oplus(z,y)).$$

Example 1. (i) Let $([a,b], \oplus, \odot)$ be the semiring with generated pseudo-operations by an increasing and continuous function g. Here we have $x \odot y = g^{-1}(g(x) \cdot g(y))$, and therefore $x_\odot^{(p)} = g^{-1}(g^p(x))$. If the function $d_\oplus : [a,b] \times [a,b] \to [a,b]$ is defined by

$$d_\oplus(x,y) = g^{-1}(|g(x) - g(y)|), \tag{1}$$

then $d_\oplus$ is the pseudo-metric on $[a,b]$ and $c = \mathbf{1}$.

(ii) In the semiring $([a,b], \oplus, \odot)$ where $x \oplus y = \sup(x,y)$, $x \odot y = g^{-1}(g(x)g(y))$ and g is an increasing and continuous function. The function $d_\oplus : [a,b] \times [a,b] \to [a,b]$ defined also by (1) is the pseudo-metric on $[a,b]$ and $c = g^{-1}(2)$. In [4] the semiring $([-\infty, \infty[, \sup, +)$ are considered. The pseudo-metric is defined by $d_\oplus(x,y) = \log|e^x - e^y|$. The condition (PM3) is fulfilled for $c = \log 2$.

Let $([a,b], \oplus, \odot)$ be a semiring and $(X, \mathcal{A})$ be a measurable space, m a σ-$\oplus$-measure with the corresponding pseudo-integral, see [5–11].

We define for $0 < p < \infty$ and $u,v : X \to [a,b]$ measurable functions

$$D_{p\oplus}(u,v) = \left(\int_X^\oplus (d_\oplus(u,v))_\odot^{(p)} \odot dm \right)_\odot^{\left(\frac{1}{p}\right)}.$$

If $p = \infty$, then

$$D_{\infty\oplus}(u,v) = \inf\{\alpha \in [a,b] \mid m(\{x \in X \mid d_\oplus(u(x), v(x)) \geq \alpha\}) = \mathbf{0}\}.$$

Definition 2. *Let* $f : X \to [a,b]$ *be a measurable function and* $D_{p\oplus}(f, \mathbf{0})$ *has a finite value in the sense of a given semiring, i.e., if the operation* $\oplus$ *induces the usual order (opposite to the usual order) on the interval* $[a,b]$ *it means that* $D_{p\oplus}(f, \mathbf{0}) < b$, $(D_{p\oplus}(f, \mathbf{0}) > a)$. *Set of all those functions we denote by* $\mathcal{L}_\oplus^p$, *and set of the functions such that* $D_{\infty\oplus}(f, \mathbf{0})$ *has a finite value in the sense of a given semiring we denote by* $\mathcal{L}_\oplus^\infty$.

We say that measurable functions u and v are equal almost everywhere with respect to σ-$\oplus$-measure m on X (written $u = v$ *m-a.e.*), if

$$m(\{x \mid u(x) \neq v(x)\}) = \mathbf{0}.$$

The function $D_{p\oplus}$ does not satisfy the property (PM1). Namely, due to properties of the pseudo-integral we have that if $u = v$ *m-a.e.*, then $D_{p\oplus}(u,v) = \mathbf{0}$. Similarly as in the classical measure theory, we will consider the equivalence classes containing all the functions from $\mathcal{L}_\oplus^p$ which are equal almost everywhere with respect to σ-$\oplus$-measure m on X. The set of equivalence classes we will denote by $L_\oplus^p$. If U and V are two equivalence classes, then

$$D_{p\oplus}(U,V) = D_{p\oplus}(u,v),$$

where $u \in U$ and $v \in V$. In the following we will talk about the elements of $L^p_\oplus$ as a functions which represent a equivalence class.

By Minkowski inequality [2] the function $D_{p\oplus}$ is a pseudo-metric on $L^p_\oplus$. Due to Hölder inequalities [2] we have the following theorem.

Theorem 1. *Let $x \oplus y = \sup(x,y)$ and $x \odot y = g^{-1}(g(x)g(y))$. Let m be a σ-⊕-measure and p and q be conjugate exponents with $1 \leq p \leq \infty$. If $u \in L^p_\oplus$ and $v \in L^q_\oplus$, and generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition ⊕ and pseudo-multiplication ⊙ is an increasing function, then $u \odot v \in L^1_\oplus$.*

We introduce various notions of convergence related to a σ-⊕-measure and pseudo-integral in the pseudo-L^p space. Let $(X,\mathcal{A})$ be a measurable space and m a σ-⊕-measure.

Definition 3. *Let $\{f_n\}$ be a sequence in $L^p_\oplus$ and $f \in L^p_\oplus$.*

(i) $\{f_n\}$ converges to f in the mean of order p ($p > 0$) if

$$\lim_{n\to\infty} D_{p\oplus}(f_n, f) = \mathbf{0}.$$

(ii) $\{f_n\}$ fundamentally converges in the mean of order p ($p > 0$) if

$$\lim_{n,m\to\infty} D_{p\oplus}(f_n, f_m) = \mathbf{0}.$$

Definition 4. *A sequence $\{f_n\}$ of measurable functions converges to the function f in σ-⊕-measure m in the mean of order p ($p > 0$) if*

$$\lim_{n\to\infty} m\left(\left\{x \mid (d_\oplus(f_n, f))^{(p)}_\odot \geq \varepsilon\right\}\right) = \mathbf{0}$$

for any $\varepsilon > \mathbf{0}$.

The relationships among these types of convergence were considered in [15]. We state here only two results.

As a consequence of the Minkowski type inequality [2] we have the following:

Theorem 2. *Let $([a,b], \oplus, \odot)$ be the semiring with pseudo-operations defined by an increasing and continuous generator g. If $\{f_n\}$ converges to f in the mean of order p in $L^p_\oplus$, $1 \leq p < \infty$, then it holds:*

$$\lim_{n\to\infty} \int_X^\oplus (f_n)^{(p)}_\odot \odot dm = \int_X^\oplus f^{(p)}_\odot \odot dm.$$

Theorem 3. *Let $x \oplus y = \sup(x,y)$ and $x \odot y = g^{-1}(g(x)g(y))$, where g is an increasing and continuous function, and m is a complete σ-*sup*-measure. If $\{f_n\}$ converges to f in the mean of order p in $L^p_\oplus$, $0 < p < \infty$, then it converges in σ-*sup*-measure m in the mean of order p in $L^p_\oplus$.*

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On the extensions of Di Nola's Theorem

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Abstract. The main aim of this paper is to present a direct proof of Di Nola's representation Theorem for MV-algebras and to extend his results to the restriction of the standard MV-algebra on rational numbers. The results are based on a direct proof of the theorem which says that any finite partial subalgebra of a linearly ordered MV-algebra can be embedded into $\mathbb{Q}\cap[0,1]$.

1 Introduction

The representation theory of MV-algebras is based on Chang's representation Theorem [4], McNaughton's Theorem and Di Nola's representation Theorem [5]. Chang's representation Theorem yields a subdirect representation of all MV-algebras via linearly ordered MV-algebras. McNaughton's Theorem characterizes free MV-algebras as algebras of continuous, piece-wise linear functions with integer coefficients on $[0,1]$. Finally, Di Nola's representation Theorem describes MV-algebras as sub-algebras of algebras of functions with values into a non-standard ultrapower of the MV-algebra $[0,1]$.

The main motivation for our paper comes from the fact that although the proofs of both Chang's representation Theorem [4] and McNaughton's Theorem are of algebraic nature the proof of Di Nola's representation Theorem is based on model-theoretical considerations. We give a simple, purely algebraic, proof of it and its variants based on the Farkas' Lemma for rationals [6] and General finite embedding theorem [3].

1.1 Generalized finite embedding theorem

By an *ultrafilter* on a set I we mean an ultrafilter of the Boolean algebra $\mathcal{P}(I)$ of the subsets of I.

Let $\{\mathbf{A}_i;\ i\in I\}$ be a system of algebras of the same type F for $i\in I$. We denote for any $x,y\in\prod_{i\in I}A_i$ the set

$$[\![x=y]\!]=\{j\in I; x(j)=y(j)\}.$$

If F is a filter of $\mathcal{P}(I)$ then the relation θ_F defined by

$$\theta_F=\{\langle x,y\rangle\in(\prod_{i\in I}A_i)^2; [\![x=y]\!]\in F\}$$

is a congruence on $\prod_{i\in I}\mathbf{A}_i$. For an ultrafilter U of $\mathcal{P}(I)$, an algebra $(\prod_{i\in I}\mathbf{A}_i)/U := (\prod_{i\in I}\mathbf{A}_i)/\theta_U$ is said to be an *ultraproduct* of algebras $\{\mathbf{A}_i;\ i\in I\}$. Any ultraproduct of an algebra $\mathbf{A}$ is called an ultrapower of $\mathbf{A}$. The class of all ultraproducts (products, isomorphic images) of algebras from some class of algebras $\mathcal{K}$ is denoted by $\mathrm{P_U}(\mathcal{K})$ $(\mathrm{P}(\mathcal{K}), \mathrm{I}(\mathcal{K}))$. The class of all finite algebras from some class of algebras $\mathcal{K}$ is denoted by $\mathcal{K}_{Fin}$.

Definition 1. Let $\mathbf{A} = (A, \mathrm{F})$ be a partial algebra and $X \subseteq A$. Denote the partial algebra $\mathbf{A}|_X = (X, \mathrm{F})$, where for any $f \in F_n$ and all $x_1, \ldots, x_n \in X$, $f^{\mathbf{A}|_X}(x_1, \ldots, x_n)$ is defined if and only if $f^{\mathbf{A}}(x_1, \ldots, x_n) \in X$ holds. Moreover, then we put

$$f^{\mathbf{A}|_X}(x_1, \ldots, x_n) := f^{\mathbf{A}}(x_1, \ldots, x_n).$$

Definition 2. An algebra $\mathbf{A} = (A, \mathrm{F})$ satisfies *the general finite embedding* (*finite embedding property*) *property for the class* $\mathcal{K}$ of algebras of the same type if for any finite subset $X \subseteq A$ there are an (finite) algebra $\mathbf{B} \in \mathcal{K}_{\mathrm{E}}$ and an embedding $\rho : \mathbf{A}|_X \hookrightarrow \mathbf{B}$, i.e. an injective mapping $\rho : X \to B$ satisfying the property $\rho(f^{\mathbf{A}|_X}(x_1, \ldots, x_n)) = f^{\mathbf{B}}(\rho(x_1), \ldots, \rho(x_n))$ if $x_1, \ldots, x_n \in X$, $f \in \mathrm{F}_n$ and $f^{\mathbf{A}|_X}(x_1, \ldots, x_n)$ is defined.

Finite embedding property is usually denoted by (FEP). Note also that a quasivariety $\mathcal{K}$ has the FEP if and only if $\mathcal{K} = \mathrm{ISPP_U}(\mathcal{K}_{Fin})$ (see [2, Theorem 1.1] or [1]).

Theorem 1. [3, Theorem 6] *Let* $\mathbf{A} = (A, \mathrm{F})$ *be a algebra and let* $\mathcal{K}$ *be a class of algebras of the same type. If* $\mathbf{A}$ *satisfies the general finite embedding property for* $\mathcal{K}$ *then* $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$.

Theorem 2. [3, Theorem 7] *Let* $\mathbf{A} = (A, \mathrm{F})$ *be an algebra such that* F *is finite and let* $\mathcal{K}$ *be a class of algebras of the same type. If* $\mathbf{A} \in \mathrm{ISP_U}(\mathcal{K})$ *then* $\mathbf{A}$ *satisfies the general finite embedding property for* $\mathcal{K}$.

1.2 Farkas' lemma

Let us recall the original formulation of Farkas' lemma [6, 7] on rationals:

Theorem 3 (Farkas' lemma). *Given a matrix* A *in* $\mathbb{Q}^{m \times n}$ *and* $\boldsymbol{c}$ *a column vector in* $\mathbb{Q}^m$*, then there exists a column vector* $\boldsymbol{x} \in \mathbb{Q}^n$*,* $\boldsymbol{x} \geq \mathbf{0}_n$ *and* $A \cdot \boldsymbol{x} = \boldsymbol{c}$ *if and only if, for all row vectors* $\boldsymbol{y} \in \mathbb{Q}^m$*,* $\boldsymbol{y} \cdot A \geq \mathbf{0}_m$ *implies* $\boldsymbol{y} \cdot \boldsymbol{c} \geq 0$.

In what follows, we will use the following equivalent formulation:

Theorem 4 (Theorem of alternatives). *Let* A *be a matrix in* $\mathbb{Q}^{m \times n}$ *and* $\boldsymbol{b}$ *a column vector in* $\mathbb{Q}^n$*. The system* $A \cdot \boldsymbol{x} \leq \boldsymbol{b}$ *has no solution if and only if there exists a row vector* $\boldsymbol{\lambda} \in \mathbb{Q}^m$ *such that* $\boldsymbol{\lambda} \geq \mathbf{0}_m$*,* $\boldsymbol{\lambda} \cdot A = \mathbf{0}_n$ *and* $\boldsymbol{\lambda} \cdot \boldsymbol{b} < 0$.

2 The Embedding Lemma

In this section, we use the Farkas' lemma on rationals to prove that any finite partial subalgebra of a linearly ordered MV-algebra can be embedded into $\mathbb{Q} \cap [0, 1]$ and hence into the finite MV-chain $\mathbf{L}_k \subseteq [0, 1]$ for a suitable $k \in \mathbb{N}$.

Lemma 1. *Let* $\mathbf{M} = (M; \oplus, \neg, 0)$ *be a linearly ordered MV-algebra,* $X \subseteq M \setminus \{0\}$ *be a finite subset. Then there is a rationally valued map* $s : X \cup \{0,1\} \longrightarrow [0,1] \cap \mathbb{Q}$ *such that*

1. $s(0) = 0$, $s(1) = 1$,
2. *if* $x, y, x \oplus y \in X \cup \{0,1\}$ *such that* $x \leq \neg y$ *and* $x, y \in X \cup \{0,1\}$ *then* $s(x \oplus y) = s(x) + s(y)$.
3. *if* $x \in X$ *then* $s(x) > 0$.

Lemma 2 (Embedding Lemma). *Let us have a linearly ordered MV-algebra* $\mathbf{M} = (M; \oplus, \neg, 0)$ *and let* $X \subseteq M$ *be a finite set. Then there exists an embedding* $f : \mathbf{X} \hookrightarrow \mathbf{L}_k$, *where* $\mathbf{X}$ *is a partial MV-algebra obtained by the restriction of* $\mathbf{M}$ *to the set* X *and* $\mathbf{L}_k \subseteq [0,1]$ *is the linearly ordered finite MV-algebra on the set* $\{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$.

3 Extensions of Di Nola's Theorem

In this section, we are going to show Di Nola's representation Theorem and its several variants not only via standard MV-algebra $[0,1]$ but also via its rational part $\mathbb{Q} \cap [0,1]$ and finite MV-chains. To prove it, we use the Embedding Lemma obtained in the previous section. First, we establish the FEP for linearly ordered MV-algebras.

Theorem 5. *1. The class* $\mathcal{LMV}$ *of linearly ordered MV-algebras has the FEP.*
2. The class $\mathcal{MV}$ *of MV-algebras has the FEP.*

Note that the part (1) of the preceding theorem for subdirectly irreducible MV-algebras can be easily deduced from the result that the class of subdirectly irreducible Wajsberg hoops has the FEP (see [1, Theorem 3.9]). The well-known part (2) then follows from [1, Lemma 3.7,Theorem 3.9]. We are now ready to establish a variant of Di Nola's representation Theorem for finite MV-chains (finite MV-algebras).

Theorem 6. *1. Any linearly ordered MV-algebra can be embedded into an ultraproduct of finite MV-chains.*
2. Any MV-algebra can be embedded into a product of ultraproducts of finite MV-chains.
3. Any MV-algebra can be embedded into an ultraproduct of finite MV-algebras (which are embeddable into powers of finite MV-chains).

The next two theorems cover Di Nola's representation Theorem and its respective variants both for rationals and reals.

Theorem 7. *1. Any linearly ordered MV-algebra can be embedded into an ultrapower of* $\mathbb{Q} \cap [0,1]$.
2. Any MV-algebra can be embedded into a product of ultrapowers of $\mathbb{Q} \cap [0,1]$.
3. Any MV-algebra can be embedded into an ultrapower of the countable power of $\mathbb{Q} \cap [0,1]$.
4. Any MV-algebra can be embedded into an ultraproduct of finite powers of $\mathbb{Q} \cap [0,1]$.

Theorem 8. *1. Any linearly ordered MV-algebra can be embedded into an ultrapower of* $[0,1]$.

2. *Any MV-algebra can be embedded into a product of ultrapowers of* $[0,1]$.
3. *Any MV-algebra can be embedded into an ultrapower of the countable power of* $[0,1]$.
4. *Any MV-algebra can be embedded into an ultraproduct of finite powers of* $[0,1]$.

Proof. (1)-(4) It is a corollary of Theorem 6.

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On non-additive probability theory

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There are some theories about non-additive measure and integration. On the base of these models we want to obtain some assertions concerning probability such as laws of large numbers, central limit theorem etc.

First as a motivation we mention the Šipoš's integration theory with its remarkable applications in economy. Then we present a metod of local representation of sequences of observables by the help of sequences of random variables. These random variables are defined on a Boolean algebra probability space, hence the Kolmogorov probability results may be applied. Finally we use the method to the case of non-additive set-functions.

1 Motivation

In [10] J. Šipoš started a new direction in the integration theory. The main idea was in considering partitions in y - axis instead of x - axis. Let Ω be a non-empty set and $\mathcal{S}$ be a family of subsets of Ω obtaining $\emptyset$. Further a function $\mu : \mathcal{S} \to [0,\infty)$ is given such that $\mu(\emptyset) = 0$. Consider a non-negative real function f on Ω and $0 = a_0 < a_1 < ... < a_{n-1} < a_n$. Then the integral sum is defined as the number

$$\Sigma_{i-1}^{n}(a_i - a_{i-1})\mu(\{\omega \in \Omega; f(\omega) \geq a_i\}).$$

Recall that a similar construction has been used by G. Choquet [3, 6]. Their definition can be different in the case that f has also negative values. Evidently the mapping μ need not be additive and it is remarkable that the Šipoš's integral has been used as a theoretical background of the Tversky and Kahneman economic theory [11]. From our point of view it is important that the best Šipoš's results has been obtained under the assumption that the mapping μ is continuous. Therefore in our probability theory instead of σ-additivity we shall use the more general assumption of the continuity.

2 Local representation method

Our main idea is in applying some results from the Kolmogorov theory to a non-Kolmogorov case. We use a method which was succesfully applied for σ-additive but not-Boolean case ([2, 4, 7–9]).

Consider the simplest MV-algebra (multivalued algebra) $M = [0,1]$ with usual ordering $\leq$, two binary operations $\oplus, \odot$ and one unary operation $\neg$:

$$a \oplus b = \min(a+b, 1),$$

$$a \odot b = \max(a+b-1, 0),$$

$$\neg a = 1 - a.$$

The operation $\oplus$ corresponds to disjunction, $\odot$ to conjunction, $\neg$ to negation. The probability on MV-algebras has been studied in [9] and systematically in [8]. In an alternative terminology instead of probability the term state is used and instead of random variable the term observable. A state m is a mapping $m : M \to [0,1]$ satisfying the following conditions:

$$m(1) = 1,$$

$$a \odot b = 0 \Longrightarrow m(a \oplus b) = m(a) + m(b),$$

$$a_n \nearrow a \Longrightarrow m(a_n) \nearrow m(a).$$

An observable is a mapping $x : \mathcal{B}(R) \to M$ ($\mathcal{B}(R)$ is the σ-algebra of Borel sets) satisfying the following conditions:

$$x(R) = 1,$$

$$A \cap B = \emptyset \Longrightarrow x(A) \odot x(B) = 0, x(A \cup B) = x(A) \oplus x(B),$$

$$A_n \nearrow A \Longrightarrow x(A_n) \nearrow x(A).$$

Now to a given sequence $(x_n)_n$ of observables it can be constructed a probability space $(\Omega, \mathcal{S}, P)$ and a sequence $(f_n)_n$ of random variables on Ω such that from the independence of $(x_n)_n$ the independence of $(f_n)_n$ follows and from the convergence of $(f_n)_n$ (of some type) the corresponding convergence of $(x_n)_n$ follows.

By the method some basic results of probability theory has been achieved in MV-algebras, and also on some other structures as D-posets = effect algebras ([4, 5]), Atanassov intuitionistic fuzzy sets ([1, 7]) etc. Therefore we apply the method also in non-additive case.

3 Continuous measures

We start with a measurable space $(\Omega, \mathcal{S})$ and a mapping $\mu : \mathcal{S} \to [0,1]$ satisfying the following conditions:

$$\mu(\Omega) = 1, \mu(\emptyset) = 0,$$

$$A_n \nearrow A \Longrightarrow \mu(A_n) \nearrow \mu(A),$$

$$A_n \searrow A \Longrightarrow \mu(A_n) \searrow \mu(A).$$

Now let $\xi : \Omega \to R$ be a measurable function. Define $F : R \to [0,1]$ by the help of equality

$$F(x) = \mu(\{\omega \in \Omega; \xi(\omega) < x\}).$$

Then F is a distribution function, hence there exists exactly one probability measure $\lambda_F : \mathcal{B}(R) \to [0,1]$ such that

$$\lambda_F([a,b)) = F(b) - F(a), a,b \in R, a < b.$$

If we have random variables $\xi_1, \xi_2, ..., \xi_n$, then $T_n = (\xi_1, \xi_2, ..., \xi_n) : \Omega \to R^n$ is a random vector, we can construct the mapping $\mu_{T_n} : \mathcal{B}(R_n) \to [0,1]$ by the equality

$$\mu_{T_n}(C) = \mu(T_n^{-1}(C), C \in \mathcal{B}(R^n).$$

On the other hand, if λ_{F_i} is the Lebesgue - Stieltjes measure corresponding to F_i we can contruct the product measure

$$\lambda_{F_1} \times \lambda_{F_2} \times ... \times \lambda_{F_n}$$

We shall say that a sequence $(\xi_n)_n$ of random variables is independent, if

$$\mu_{T_n} = \lambda_{F_1} \times \lambda_{F_2} \times ... \times \lambda_{F_n}$$

for any n. Put $P_n = \mu_{T_n} : \mathcal{B}(R^n) \to [0,1]$. Then $(P_n)_n$ satisfy the Kolmogorov consistency condition, hence we can construct the probability space

$$(R^N, \sigma(\mathcal{C}), P)$$

such that

$$P(\Pi_n^{-1}(C)) = P_n(C)$$

for any $C \in \mathcal{B}(R^n)(\Pi_n : R^N \to R^n$ is the projection). The corresponding sequence $(f_n)_n$ of random variables $f_n : R^N \to R$ can be defined by the equality

$$f_n((u_i)_i) = u_n.$$

Of course, to various sequences $(\xi_n)_n$ various spaces $(R^N, \sigma(\mathcal{C}), P)$ are obtained. On the other hand, for every independent sequence $(\xi_n)_n$ the central limit theorem can be proved as well as the law of large numbers.

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A remark on Archimax copulas

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In 2000 Capéraà, Fougères and Genest [1] introduced a family of bivariate copulas that includes both the extreme–value and the Archimedean copulas, and which they called *Archimax copulas*.

Let φ be an *inner generator*, viz., a strictly decreasing and convex function $\varphi : \mathbb{I} \to [0, +\infty]$ such that $\varphi(1) = 0$; moreover, one sets $\varphi(0) := \lim_{t \to 0+} \varphi(t)$. Also let A be a Pickands *dependence function*, namely a convex function $A : \mathbb{I} \to [1/2, 1]$ such that, for every $t \in \mathbb{I}$

$$\max\{t, 1-t\} \leq A(t) \leq 1\,.$$

Then an Archimax copula is defined, for $(u, v) \in \mathbb{I}^2$ by

$$C_{\varphi,A}(u,v) := \varphi^{(-1)} \left[\{\varphi(u) + \varphi(v)\} A \left(\frac{\varphi(u)}{\varphi(u) + \varphi(v)} \right) \right]. \tag{1}$$

Their proof that eq. (1) actually defines a *bona fide* copula is extremely terse, and can be followed with real difficulty, at least by the present author; he is probably not alone in this, although several authors have dealt with this family of copulas (see, e.g., [3]). In a forthcoming paper Wysocki [4] gives necessary and sufficient conditions for a copula to be Archimax.

Here we present a sufficient condition in the form of a differential inequality derived from the theory of copula transforms (see, e.g., [2]).

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On a constructive approach to complicated probability measures

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In several important statistical problems we are of evident lack of data which makes asymptotics not-applicable. Therefore small sample asymptotics or exact distribution theory should be used. In the latter one should be able to evaluate non-trivial probabilities. Here we will illustrate how geometric integration (see e.g. [1]) can help to solve such a problems. We illustrate the application of geometric integration in several examples.

In particular, we discuss the complexity of exact distribution of Kullback-Leibler divergence which can be obtained by convolution of variables derived in [2]. These convolutions are tractable only in very small samples and involve a complicated functions, e.g. complete elliptic integrals of 1st, 2nd and 3rd kind (see [5]). Validation of statistical usefulness of such a statistics has been made in [3] mainly by means of simulations. In this setup, geometric integration approach can serve as a good substitute to convolution.

Second nice example is evaluation of probability of given sets of interest when ratios of random variables are involved. Such a problem can appear by popular tests for normality, e.g. Jarque-Berra test. By its robustification against Pareto tails (see e.g. [4]) we need to compare probabilities of given sets for several ratios of complicated random variables. Not surprisingly, geometric integration can shed a light on such a comparisons.

Some more examples, together with specific elements of geometric integration theory will be discussed.

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Interval-valued pseudo-integral based measures

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1 Introduction

The interval-valued measures have been successfully applied in many different areas such as economic uncertainty, interval-valued probability, multi-valued fuzzy entropy, fuzzy random variables, fixed point theory and so on. This interval form has imposed itself as worth of studying due to the fact that often, while working with uncertainty, instead of the actual values we use intervals that incorporate some errors. Also, reasoning in the presence of uncertainty often involves some sort of measurement, which essentially boils down to integration of certain functions. Therefore, the focus of this research is on the interval-valued measure obtained by integration, with pseudo-analysis as the background.

Due to the course of research, this problem has diversified into two directions. The first one is based on the pseudo-integral of interval-valued function that is a generalization of the Aumann's integral. The second direction as the base has an interval-valued measure obtained through pseudo-integrals of real-valued functions.

The starting notion, that is the frame for later on obtained integral forms, is the interval-valued $\sigma-\oplus$-measure.

Let $[a,b]$ be a closed subinterval of $[-\infty,+\infty]$, let $\oplus,\odot : [a,b]\times[a,b]\to[a,b]$ be a pseudo-addition and a pseudo-multiplication, let $\preceq$ be a total order on $[a,b]$ and let $([a,b],\oplus,\odot)$ be a semiring from one of three basic classes from [6]. Let X be a nonempty set, Σ a σ-algebra and (X,Σ,μ) measure space where μ is a $\oplus$-measure ([6]). Let I be the class of all closed subintervals of $[a,b]$.

Definition 1. *Let Σ be a σ-algebra of subsets of X. A function $\overline{\mu}:\Sigma\to I$ is the* interval-valued $\sigma-\oplus$-measure *if*

i) $\overline{\mu}(\emptyset)=\{\mathbf{0}\}=[\mathbf{0},\mathbf{0}]$,

ii) $\overline{\mu}(\bigcup_{i=1}^{\infty}A_i)=\bigoplus_{i=1}^{\infty}\overline{\mu}(A_i)=\lim_{n\to\infty}\bigoplus_{i=1}^{n}\overline{\mu}(A_i)$,

where $(A_i)_{i\in N}$ is a sequence of pairwise disjoint sets from Σ. If $\oplus$ is an idempotent operation, the disjointness of sets can be omitted.

2 Case I: Interval-valued measure via pseudo-integral of interval-valued function

This section as a base has the Aumann approach ([1]), that is now applied to the pseudo-integral ([6]). More on this subject can be found in [2, 3].

Let X be a nonempty set, Σ a σ-algebra and (X,Σ,μ) measure space where μ is a $\oplus$-measure ([6]). An interval-valued function F is a function from X to I and, due to its specific range, is represented by border functions $l,r : X \to [a,b]$ as

$$F(x) = [l(x), r(x)] .$$

Further on, notion "less or equal" applied on sets from I is denoted by $\preceq_S$ and is given by the following: let $C,D \in I$, then $C \preceq_S D$ if for all $x \in C$ there exists $y \in D$ such that $x \preceq y$ and for all $y \in D$ there exists $x \in C$ such that $x \preceq y$.

Definition 2. *Let F be an interval-valued function.* The pseudo-integral *of F on $A \in \Sigma$ is*

$$\int_A^{\oplus} F \odot d\mu = \left\{ \int_A^{\oplus} f \odot d\mu \,\middle|\, f \in S(F) \right\},$$

where

$$S(F) = \left\{ f \in L^1_{\oplus}(\mu) \,|\, f(x) \in F(x) \text{ on } X\ \mu - a.e. \right\},$$

$\int_A^{\oplus} f \odot d\mu$ *is the pseudo-integral of f and $L^1_{\oplus}(\mu)$ is family of all functions $f : X \to [a,b]$ that are integrable with respect to the pseudo-integral (see [6]).*

An interval-valued measure based on pseudo-integral of an interval-valued function is given by the following theorem ([3]).

Theorem 1. *Let $F : X \to I$ be a pseudo-integrably bounded interval-valued function with border functions l and r, an interval-valued set-function μ_F based on interval-valued pseudo-integral of F given by*

$$\mu_F(A) = \int_A^{\oplus} F \odot d\mu = \left[\int_A^{\oplus} l \odot d\mu, \int_A^{\oplus} r \odot d\mu \right],$$

where $A \subseteq X$, has the following properties:

i) $\mu_F(\emptyset) = \{\mathbf{0}\} = [\mathbf{0},\mathbf{0}]$;
ii) μ_F *is* monotone with respect to $\preceq_S$.
iii) μ_F *is* $\oplus$-additive.
iv) μ_F *is* $\sigma - \oplus$-additive.

Remark 1. [3] An interval-valued function F is *pseudo-integrably bounded* if there is a function $h \in L^1_{\oplus}(\mu)$ such that:

- $\bigoplus_{\alpha \in F(x)} \alpha \preceq h(x),$ for the idempotent pseudo-addition,
- $\sup_{\alpha \in F(x)} \alpha \preceq h(x),$ for $\oplus$ given by an increasing generator,
- $\inf_{\alpha \in F(x)} \alpha \preceq h(x),$ for $\oplus$ given by a decreasing generator.

3 Case II: Interval-valued measure via pseudo-integrals of real valued functions

This section follows the approach to interval-valued measures from [5] that is now considered in the pseudo-analysis' framework. More on the presented construction can be found in [4].

Let X be a nonempty set, Σ a σ-algebra and (X,Σ,μ) measure space where μ is a $\oplus$-measure. Let $\mathcal{M}$ be an arbitrary nonempty family of $\oplus$-measures μ that contains μ_l and μ_r such that for all μ from $\mathcal{M}$ and all A from Σ the following holds

$$\mu_l(A) \preceq \mu(A) \preceq \mu_r(A).$$

3.1 The first step

The first step is introduction of an interval-valued set-function that is not of the integral form.

Definition 3. *The interval-valued set-function* $\overline{\mu}_{\mathcal{M}} : \Sigma \to I$ *for the family* $\mathcal{M}$ *is*

$$\overline{\mu}_{\mathcal{M}} = [\mu_l, \mu_r], \quad \mu_l, \mu_r \in \mathcal{M}.$$

Proposition 1. $\overline{\mu}_{\mathcal{M}}$ *is an interval-valued* σ-$\oplus$*-measure.*

3.2 The second step

The second step is an extension of the construction from [6] to the interval-valued case.

Connection between interval valued pseudo-integral based on the interval-valued measure from Definition 3 (see [4]) and pseudo-integral ([6]) is given in the next theorem.

Theorem 2. *If* $f : X \to [a,b]$ *is a measurable function, then*

$$\int_X^{\oplus} f \odot d\overline{\mu}_{\mathcal{M}} = \left[\int_X^{\oplus} f \odot d\mu_l, \int_X^{\oplus} f \odot d\mu_r \right].$$

3.3 The third step

An interval-valued measure based on pseudo-integral of a real-valued function is given by the following theorem ([4]).

Theorem 3. *Let* $f : X \to [a,b]$ *be a measurable function. An interval-valued set-function* $\overline{\mu}^f_{\mathcal{M}}$ *based on interval-valued pseudo-integral of* f *given by*

$$\overline{\mu}^f_{\mathcal{M}}(A) = \int_A^{\oplus} f \odot d\overline{\mu}_{\mathcal{M}} = \left[\int_A^{\oplus} f \odot d\mu_l, \int_A^{\oplus} f \odot d\mu_r \right],$$

where $A \subseteq X$, *has the following properties:*

i) $\overline{\mu}^f_{\mathcal{M}}(\emptyset) = \{\mathbf{0}\} = [\mathbf{0},\mathbf{0}]$;
ii) $\overline{\mu}^f_{\mathcal{M}}$ *is* monotone with respect to $\preceq_S$.
iii) $\overline{\mu}^f_{\mathcal{M}}$ *is* $\oplus$-additive.
iv) $\overline{\mu}^f_{\mathcal{M}}$ *is* $\sigma - \oplus$-additive.

Remark 2. The presented research can be further extended towards integration of interval-valued functions with respect to interval-valued measures.

4 Integral inequalities: interval-valued measure form

Some well known integral inequalities can be extended to the interval-valued case and, under certain additional assumptions that correspond to the classical cases, are of the following forms (see [4, 7]):

- Jensen type inequality for Case I

$$\Phi(\mu_F(X)) \preceq_S \mu_{\Phi \circ F}(X).$$

- Jensen type inequality for Case II

$$\overline{\mu}^{\Phi\circ f}_{\mathcal{M}_0}(X) \preceq_S \Phi\left(\overline{\mu}^f_{\mathcal{M}_0}(X)\right).$$

- Chebyshev type integral inequalities for Case I

$$\mu_{F_1}([c,d]) \odot \mu_{F_2}([c,d]) \preceq_S \mu_{F_1 \odot F_2}([c,d]).$$

- Chebyshev type integral inequalities for Case II

$$\overline{\mu}^{f_1}_{\mathcal{M}_0}([c,d]) \odot \overline{\mu}^{f_2}_{\mathcal{M}_0}([c,d]) \preceq_S \overline{\mu}^{f_1 \odot f_2}_{\mathcal{M}_0}([c,d]),$$

where $\Phi : [a,b] \to [a,b]$ is a convex, decreasing and bounded function such that $\Phi(A) = \{\Phi(x) \,|\, x \in A\}$, $A \subseteq X$, and $\mathcal{M}_0$ is a family of $\oplus$-measures that includes the trivial one of the form $\mu_0(A) = \mathbf{0}$ for all $A \in \Sigma$.

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Triangular norms and tomonoid extensions

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1 Introduction

Left-continuous t-norms are binary operations on the real unit interval suitable to interpret the conjunction in fuzzy logic. For an overview, see, e.g., [5]. Considerable effort has been devoted to bringing light into the structure of this type of operation. The crucial property of a t-norm is its associativity. However, associativity alone hardly allows a deep structure theory. In fact, the systematisation of t-norms has turned out to be a difficult project.

Left-continuous t-norms can be studied with two different aims. We might be interested in specific such operations, for instance to fit a given application. In this case we consider a t-norm as a two-place real function and may apply the methods of real analysis. For the purpose of illustration, we often use the graph of a t-norm, that allows a visualisation in three-dimensional space.

A more modest aim is the description of t-norms up to isomorphism. In this case, we do not make a difference between an operation $\odot\colon [0,1]^2 \to [0,1]$ and $\odot'\colon [0,1]^2 \to [0,1],\ (a,b) \mapsto \varphi^{-1}(\varphi(a) \odot \varphi(b))$, where φ is an order automorphism of $[0,1]$. The tools that then suggest themselves come from algebra and we work within an appropriate class of algebraic structures. In fact, we may choose to work with MTL-algebras, or with strictly two-sided commutative quantales, or with totally ordered commutative monoids.

Aiming at a classification of t-norms up to isomorphism, we adopt the latter approach here, and we choose totally ordered monoids as our basic notion. For an overview, see [3]. We note that we could have equally well chosen MTL-algebras; however, we do not find the residual implication as an additional operation useful in the present context. Moreover, we would restrict our possibilities unnecessarily if quantales were our framework. The analysis that we propose cannot be performed within the category of quantales.

Definition 1. *A structure $(L;\leq,\odot,1)$ is a totally ordered monoid, or tomonoid for short, if* (i) *$\leq$ is a total order,* (ii) *$\odot$ is a commutative and associative binary operation,* (iii) *1 is neutral w.r.t. $\odot$, and* (iv) *$\odot$ is translation-invariant, that is, for any $a,b,c \in L$, $a \leq b$ implies $a \odot c \leq b \odot c$.*

Moreover, let $[0,1]$ be the real unit interval and let $\leq$ be its natural order. A tomonoid $([0,1];\leq,\odot,1)$ is then called a t-norm monoid and the monoidal operation $\odot$ is called a t-norm.

Structure theory in algebra usually includes the determination of the quotients of the algebras in question. The congruences of MTL-algebras are induced by their filters [1]; we can proceed analogously for tomonoids.

Definition 2. *Let $(L;\leq,\odot,1)$ be a tomonoid. Then a filter of a tomonoid L is a subtomonoid $(F;\leq,\odot,1)$ of L such that $a \in F$ and $b \geq a$ imply $b \in F$. Let then, for $a,b \in L$,*

$$\begin{aligned} a \sim_F b \quad & \text{if } a = b, \\ & \text{or } a < b \text{ and there is a } c \in F \text{ such that } b \odot c \leq a, \\ & \text{or } b < a \text{ and there is a } c \in F \text{ such that } a \odot c \leq b. \end{aligned}$$

Then we call $\sim_F$ the congruence induced by F.

Clearly, the congruence induced by a filter F of a tomonoid L gives rise to a quotient M of L. We then alternatively say that L is an *extension* of M, where F is the *extending tomonoid*.

Since we deal with a total order, the set of all filters is totally ordered as well. In fact, we may associate with any t-norm monoid the chain of quotients induced by its filters. We consider this chain as the basic means to analyse t-norms.

2 The chain of quotients of a t-norm monoid

It is an almost trivial statement from the point of view of algebra, but has a strong impact on the problem of how to describe t-norms: The basic construction tool for tomonoids are tomonoid extensions. Given a tomonoid, we are looking for a further tomonoid such that the former is a quotient of the latter.

For t-norm monoids, the picture is rather clear if there are only finitely many quotients: Starting with the trivial tomonoid, which consists of a single element only, we construct step by step extensions, being finally led to the t-norm monoid in question.

The task of constructing a tomonoid extension can furthermore be illustrated by means of the function algebra approach, as proposed for t-norms in [6]. In fact, any semigroup can be identified with the semigroup of its (inner, right or left) transformations [2]. Here, a transformation is the mapping from the semigroup to itself given by the multiplication with a fixed element, and the multiplication of two transformations is their functional composition. The situation can be described as follows; cf. [6]:

Proposition 1. *Let $(L;\leq,\odot,1)$ be a tomonoid. For each $a \in L$, put*

$$\lambda_a\colon L \to L, \quad x \mapsto x \odot a,$$

and let $\Lambda = \{\lambda_a\colon a \in L\}$. Then Λ is closed under the functional composition $\circ$, and the pointwise order $\leq$ on Λ is a total order with the identity id as its top element. Moreover, the mapping

$$L \to \Lambda, \quad a \mapsto \lambda_a$$

is an isomorphism of the tomonoids $(L;\leq,\odot,1)$ and $(\Lambda;\leq,\circ,id)$.

A rough idea of how to determine an extension of a given tomonoid can be guessed from Figure 1. We start with Ł$_5$, the five-element Łukasiewicz chain; the transformation tomonoid of Ł$_5$ is depicted on the left. The extending tomonoid is $((0,1]; \leq, \cdot, 1)$, where $\cdot$ is the usual multiplication. The result is not unique. The figure shows the standard extension, which resembles a t-norm defined by Hájek [4]. Its three quotients induced by filters are indicated as well.

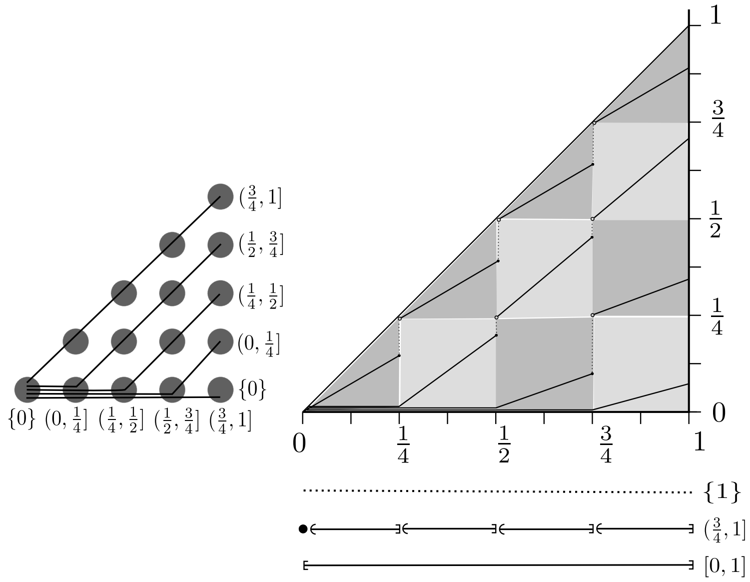

Fig. 1. An extension of the five-element Łukasiewicz chain Ł$_5$ by the tomonoid $((0,1]; \leq, \cdot, 1)$.

3 Types of extensions

Although the examples of t-norms found in the literature typically allow a straightforward construction along its chain of quotients, the general case is certainly involved. In particular, the order type of the chain of quotients is not arbitrary but can be quite complicated. For a discussion, see the forthcoming paper [7].

Here, we shall mention three basic types of extensions occurring with t-norms. For a reasonable description of a t-norm monoid, two contradicting goals must be brought together. On the one hand, we are interested in extensions of low complexity; on the other hand, the total number of extensions needed to describe a t-norm should be as low as possible. We distinguish the following types of quotients.

- *Archimedean extensions.* In this case, the extending tomonoid is of the simplest type we may ask for, namely, it is archimedean. Archimedeanicity for tomonoids means that there are at most two archimedean classes; apart from the always present class $\{1\}$, there may be only one further class.

- *Quasiarchimedean extensions.* In this case, the extending tomonoid is requested to fulfil a relaxed condition with regard to its archimedean classes. It applies to tomonoids that have so-to-say an added zero: A tomonoid L is quasiarchimedean if L possesses a smallest element 0 and $L\backslash\{0\}$ is an archimedean subtomonoid.
- *Semilattice extensions.* Semilattices might be considered as the tomonoids with the simplest possible structure: a semilattice is a totally ordered set together with the minimum as its monoidal operation. Note that then each singleton is an own archimedean class. This type of extension uses semilattices as extending tomonoids.

4 Conclusion

For what is the analysis of a t-norm monoid in terms of its quotients and extensions useful? First of all, it allows a rough classification of all t-norms. With any t-norm, we may associate a chain of increasingly finer equivalence relations on the real unit interval, each element of this chain gives rise to a tomonoid, and for each pair of these tomonoids we may specify by which further tomonoid one extends the other one.

For the exact description of a t-norm, we have to determine all possible extensions of a given tomonoid by a further one and this is still difficult. Nevertheless we are provided a framework within which the specification of the t-norm can take place.

Finally, we have established two cases where an extension can be described in full detail [7]. The first case concerns the case that the following condition is fulfilled: Each congruence class is either a singleton or else order-isomorphic to a real interval (with or without left, right boundary). The second case puts a restriction on the set of idempotents of a t-norm.

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Problems of conditioning

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Measure-free conditioning works in two steps. In a first step, *conditional events* "a given b" are defined as well-defined elements in terms of the events $a, b \in \mathbb{L}$, a suitable lattice. In a second step, their uncertainty is expressed by suitable *measures of conditional events* (a given b) in terms of a given measure of the (unconditional) events a, b.

The aim of my talks at the Linz Seminars 2007-2009 was to see how additivity of measures for events from an MV-algebra $\mathbb{L}$ can be extended to measures for conditional events. For details see [5], Sections 1 (Introduction) and 6 (Biographical and further remarks) and the References. In the present talk, the emphasis lays more on the different types of conditional events.

On the one hand, consider as the lattice $\mathbb{L}$ of events the lattice $\mathbb{F} = [0,1]^\Omega$ of fuzzy subsets of an universe Ω which becomes a (semi-simple) MV-algebra by pointwise extension of the standard MV-chain $[0,1]$. Then, for events $\varphi,\ \psi \in \mathbb{F}$, conditional events $(\varphi \mid \psi)$ can be taken as the special events

$$(\varphi \mid \psi) = C(\varphi \wedge \psi,\ \psi \to \varphi)\ \in \mathbb{F}$$

based on the pointwise extension of any mean value function C on $[0,1]$ which is compatible with the complement. For details see [4], Section 5, and, in a more general setting, [3], Sections 4 and 5.

Particularly, starting with a Boolean algebra $\mathbb{B}$ of subsets of Ω, the corresponding characteristic functions form a Boolean subalgebra $\mathbb{F}_0$ of $\mathbb{F}$. Then, any mean value function C leads to the conditional events

$$(\mathbf{1}_A \mid \mathbf{1}_B) = 1 \cdot \mathbf{1}_{A \cap B} + 0 \cdot \mathbf{1}_{A^c \cap B} + \frac{1}{2} \cdot \mathbf{1}_{B^c}, \quad \text{for events } \mathbf{1}_A,\ \mathbf{1}_B \in \mathbb{F}_0 \quad (\text{i.e. for } A, B \in \mathbb{B}).$$

It can be shown directly that the set $\mathbb{F}_1$ of all such $(\mathbf{1}_A \mid \mathbf{1}_B)$ results to be an MV-subalgebra of $\mathbb{F}$ containing $\mathbb{F}_0$.

On the other hand, consider as the lattice $\mathbb{L}$ of events the Boolean algebra $\mathbb{B}$ of subsets of Ω. Then, it is well known that the set $\tilde{\mathbb{B}}$ of conditional events $(A \parallel B)$ defined as lattice-intervals

$$(A \parallel B) = [A \cap B, B \to A], \quad \text{for events } A, B \in \mathbb{B},$$

of events form a (semi-simple) MV-algebra. For details see [1], Chapter 4, and, in a more general setting, [3], Section 2.

Now, it can also be shown directly that the bijection between the MV-algebras $\mathbb{F}_1$ and $\tilde{\mathbb{B}}$, although at the first look the constructions seem to be very different, really results to be an MV-algebra isomorphism. In the following we see that the same happens for the problem of iterating the two types of conditional events.

On the one hand, the conditional event of conditional events in $\mathbb{F}_1$ always is in $\mathbb{F}$, but even it remains to be in $\mathbb{F}_1$ if and only if $C(\frac{1}{2},1) = (\frac{1}{2} \mid \frac{1}{2}) \in \{\frac{1}{2}\ ,\ 1\}$. Therefore, there result two different conditional events based on two different types of mean value functions C on $[0,1]$. Examples are

$$C_1(\alpha,\beta) = \left\{ \begin{array}{l} \beta \ \mathit{if} \quad \beta < \frac{1}{2} \\ \frac{1}{2} \ \mathit{if}\ \alpha \le \frac{1}{2} \le \beta \\ \alpha \ \mathit{if} \quad \frac{1}{2} < \alpha \end{array} \right\} \text{for}\, C(\frac{1}{2},1) = \frac{1}{2}\,, \quad C_2(\alpha,\beta) = \frac{\alpha}{1+\alpha-\beta}\, \text{for}\, C(\frac{1}{2},1) = 1\,,$$

where $\alpha \le \beta$.

On the other hand, conditional events in $\tilde{\mathbb{B}}$ can be constructed using different mean value functions $\tilde{C}$ on the so called canonical extension of $\mathbb{B}$, as it was shown in a more general setting in [3], Section 5. It can be shown that the two types of such mean value functions presented in [3], Remark 5.2 lead to two different conditional events in $\tilde{\mathbb{B}}$ which are isomorphic to the two respective conditional events in $\mathbb{F}_1$ from above. The proposal from [2], Remark 2.1 (c) corresponds to the first choice.

Finally, we will discuss briefly the problem to find adequate notions of realizations for conditional events, compare with [2], Definition 4.1 and its motivation.

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Multi-source uncertain information fusion using measures

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We are interested in the problem of multi-source uncertain information fusion particularly in the case when the information provided can be both soft and hard information. We note that hard sensor provided information generally has a probabilistic type of uncertainty whereas soft linguistic information typically introduces a possibilistic type of uncertainty. In order to provide a unified framework for the representation of different types of uncertain information we will use a set measure approach for the representation of uncertain information. We shall discuss the set measure representation of uncertain information. In the multi-source fusion problem, in addition to having a collection of pieces of information that must be fused, we need to have some expert provided instructions on how to fuse these pieces of information. Generally these instructions can involve a combination of linguistically and mathematically expressed directions. In the course of this presentation we begin to consider the fundamental task of how to translate these instructions into formal operations that can be applied to our information. This requires us to investigate the important problem of the aggregation of set measures.

A monotonic set measure provides a very general structure for the representation of knowledge about an uncertainty variable. Let V be a variable taking its value in the space X. In using a measure μ to express our knowledge about V we provide the following interpretation. For any subset B of X we have that $\mu(B)$ indicates our anticipation that the value of V lies in B. We see that $\mu(\emptyset) = 0$ reflects the fact that the value of V will not be in the null set. The property $\mu(X) = 1$, indicates the fact that we completely anticipate that the value of V lies in X. Finally the monotonicity of μ reflects the fact that you cant be more confident of finding the value V in the set B than in a set that contains B. We shall use the expression V is μ to denote the situation where knowledge about V is carried by the set measure μ.

An important class of measures are those composed from other measures. Let μ_1 and μ_2 be two measures on X. We can show that the set function μ defined such that $\mu(A) = \mu_1(A)\mu_2(A)$ for all $A \subseteq X$ is also measure. We see that $\mu(\emptyset) = \mu_1(\emptyset)\mu_2(\emptyset) = 0$, $\mu(X) = \mu_1(X)\mu_2(X) = 1$ and if $A \supseteq B$ then $\mu_1(A) \geq \mu_2(A)$ and $\mu_1(B) \geq \mu_2(B)$ and hence $\mu(A) = \mu_1(A)\mu_2(A) \geq \mu_1(B)\mu_2(B) \geq \mu(B)$. Thus μ as defined above is a monotonic measure. This result can easily be extended to the fusion of q monotonic set measures $\mu(A) = \Pi_1^q \mu_i(A)$.

In the following we provide a generalization of this result, which shall form the basis of our approach to the fusion of hard/soft information.

Definition 1. *An aggregation function G is a function of $q > 1$ arguments $G\colon [0,1]^q \to [0,1]$ having the properties: $G(0,0,\dots,0) = 0$, $G(1,1,\dots,1) = 1$ and $G(a_1,\dots,a_q) \geq G(b_1,\dots,b_q)$ if all $a_j \geq b_j$.*

Theorem 1. *Assume for $j = 1,\dots,q$ that μ_j are a collection of monotonic measures on X. Then μ defined such that for all $A \subseteq X$*

$$\mu(A) \;=\; G(\mu_1(A),\dots,\mu_q(A))$$

is a monotonic measure.

In the problem of fusing information from multiple sources we have a collection of n sources each of which is providing information about the variable V. Here we shall assume each of these pieces of information can be expressed in terms of a monotonic set measure. Thus our information is a collection V is μ_j where μ_j is a monotonic measure defined on the domain X of V.

In addition we must have some expert provided instructions on how to fuse these pieces of information so as to obtain a unified view of the value of V. The basis of this expert provided knowledge can be very diverse. It can be based on a human expert's practical experience in processing multiple-sourced information. It can be based on some formal data mining technology. Most generally these instructions can involve a combination of linguistically and mathematical expressed directions. A fundamental task in multi-source information fusion (MSIF) is the translation of these instructions into formal operations that can be applied. The task of operationalizing these expert provided instructions is generally a very complex one, it often involves a tradeoff between precisely following the instructions and functionality, translating the instruction into implementable operations. Here the capacity of Zadehs paradigm of computing with words can become very useful for translating these instructions into formal operations.

The type of aggregation operators previously discussed provides a very useful tool for implementing a wide body of expert provided instructions for fusing multiple pieces of information. One of our interests here is to look at the use of these aggregation operators for the fusion of information expressed via monotonic measures. We shall be particularly concerned with probability and possibility type information as they represent two very important classes of provided information. We note that possibilistic information often arises from a linguistic description of the value of some variable. An example of this is information such as the house is close to the river. Here close is a linguistic term that can be expressed using fuzzy sets which in turn induces a possibility distribution on the variable "the distance of the house to the river." Probabilistic information often appears because it provides an effective model to represent the accuracy of physical sensing devices.

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