LINZ 2014

Abstracts

35th Linz Seminar on Fuzzy Set Theory

Graded Logical Approaches and their Applications

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Abstracts

Tommaso Flaminio
Lluís Godo
Siegfried Gottwald
Erich Peter Klement

Editors

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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2014 will be the 35th seminar carrying on this tradition and is devoted to the theme "Graded logical approaches and their applications". The goal of the seminar is to present and to discuss recent advances of graded logical approaches and their various applications.

A large number of highly interesting contributions were submitted for possible presentation at LINZ 2014. In order to maintain the traditional spirit of the Linz Seminars — no parallel sessions and enough room for discussions — we selected those thirty-one submissions which, in our opinion, fitted best to the focus of this seminar. This volume contains the abstracts of this impressive selection. These regular contributions are complemented by six invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

Tommaso Flaminio
Lluís Godo
Siegfried Gottwald
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Contents

Duality semantics for many-valued logics

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Boolean algebras form the algebraic semantics of Classical Propositional Logic. The celebrated Stone's Representation Theorem states that Boolean algebras and their homomorphisms form a category that is dually equivalent to the category of Stone spaces, that is, compact totally disconnected Hausdorff spaces. The finite slice of the category of Stone spaces is just the category of finite sets and maps between them. A natural understanding of the semantics of Classical Propositional Logic then arises just studying finite sets and their maps. This approach to semantics via categories dually equivalent to the varieties constituting the usual algebraic semantics can be fruitfully applied to several many-valued logics. In this talk we shall focus on the category of finite forests and open maps to show how this category yields a dual semantics for a few different many-valued logics. We shall clarify in which sense those different systems have the same duality semantics, stressing the role of the objects dual to the free singly generated algebras in the primal varieties. We shall exhibit several applications of the duality semantics approach, ranging from construction of free algebras to classification of subvarieties. If time allows we shall propose a notion of many-valued automaton arising naturally from the corresponding duality semantics of a given logic.

Acknowledgement. The content of this talk is based on a set of various jointly authored works, published, submitted or in preparation, with Denisa Diaconescu, Tommaso Flaminio, Brunella Gerla, Enrico Marchioni, and Vincenzo Marra.

Standard completeness: proof-theoretical and algebraic approaches

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In mathematical fuzzy logic, the intended or *standard* semantics is based on algebraic structures over the real interval $[0,1]$ (see [12]). Thus, showing that a logic is *standard complete*, i.e. complete with respect to the standard semantics, is of crucial importance to the field. The usual approach to the problem is algebraic (see, e.g., [5, 8, 11, 2, 13]) and consists of the following steps. Let L be a logic presented as a Hilbert-style system.

1. The completeness of the logic w.r.t. a general class of linearly ordered algebras is established (completeness w.r.t. L-chains).
2. It is shown that any countable L-chain can be embedded into a countable *dense* L-chain by adding countably many new elements to the algebra and extending the operations appropriately. This establishes *rational completeness*: a formula is derivable in L iff it is valid in all countable dense L-chains.
3. Finally, a countable dense L-chain is embedded into a standard L-algebra, that is an L-algebra with lattice reduct $[0,1]$, using a Dedekind-MacNeille-style completion.

The crucial step 2. above (rational completeness) is often the most difficult to establish, as it relies on finding the right embedding, if any. A different method to approach the step 2 was introduced in [15] and is based on *proof-theoretic* techniques. The main idea is to show the admissibility in a logic L of a particular syntactic rule, called *density*. The admissibility of the density rule immediately gives the rational completeness of the logic L.

The density rule was first introduced by Takeuti and Titani [18] and it has the following form in a Hilbert-style system:

$$\frac{(A \to p) \vee (p \to B) \vee C}{(A \to B) \vee C}$$

where p is a propositional variable not occurring in A, B, or C. Ignoring C, this can be read contrapositively as saying "if $A > B$, then $A > p$ and $p > B$ for some p"; hence the name "density" and the intuitive connection with rational completeness.

Density-admissibility, or better, density-elimination, has been first shown in [15] within the proof-theoretic framework of *hypersequents* (see [1, 16, 14] for an overview). The proof of density elimination in [15] is developed in close analogy to Gentzen-style methods for cut-elimination. A more elegant approach to density elimination, *by substitutions*, has been then introduced in [4]. Our contribution will extend the results in [4], proving density elimination, by substitutions, for a wider class of hypersequent calculi. In particular, we will show density elimination, hence standard completeness for classes of axiomatic extensions of: uninorm logic UL, monoidal t-norm logic MTL and its noncommutative variant $psMTL^r$ (see, e.g., [7]). We will also show how to translate the procedure of density elimination by substitutions in an algebraic setting. We will define indeed a method for constructing an embedding from an arbitrary chain to a dense one (see step 2 above), which is closely related to the substitution procedure in the proof of density elimination.

This will be based on *residuated frames* [10], a common abstraction from both the notion of residuated lattices (algebraic) and sequent calculi (proof-theoretic).

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A ground many-valued type theory and its extensions

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Several variants of Fuzzy Type Theory (FTT) over different background logics (in particular, $\mathrm{IMTL}_\triangle$, $Ł_\triangle$, $\mathrm{BL}_\triangle$, ŁΠ, and EQ) have been defined by Novák [6–8]. These theories follow the syntax of Church–Henkin classical type theory (CTT) of [4, 5], differing from the latter only in the choice of logical constants and axioms. Semantically, FTT generalizes CTT by admitting many-valued models over the appropriate algebras of truth degrees and by including fuzzy equality as a primitive logical constant.

In order to facilitate generalizations of FTT (e.g., to partial functions or further background logics), we introduce a minimalistic many-valued theory of types (or higher-order logic), designed by way of isolating a type-theoretical core of FTT, with many-valued equality as the only logical constant. The resulting theory TT_0 is largely independent of the background logic and can be extended in a modular way to FTT as well as other higher-order logics (including, e.g., intuitionistic, relevant, linear, modal, etc.). The strong soundness and completeness theorems for TT_0 with respect to many-valued Henkin models are presented, and some basic extensions of TT_0 with analogous results are introduced. The results presented in this abstract are elaborated in the author's manuscript [2].

1 The Syntax of the Many-Valued Type Theory TT_0

The type theory TT_0 shares the syntax with both CTT and FTT, differing from them only by the choice of the logical vocabulary and the axiomatic system. Thus, TT_0 uses the usual type hierarchy over the primitive types ε (for elements) and o (for truth values); complex types (for functions between the type domains) are obtained by this recursive rule: if α, β are types, then $(\alpha\beta)$ is a type. The set of all types will be denoted by *Types*.

As in CTT and FTT, the primitive symbols of TT_0 are the *variables* $x_\alpha, y_\alpha, \dots$ (forming disjoint infinite countable sets Var_α for each type α) and *constants* $c_\alpha, \dots$ (arbitrarily many for each type α). The list of constants (or the *language* $\mathcal{L}$) is assumed to contain the constant $=_{(o\alpha)\alpha}$ for each type α. *Formulae* (or λ-*terms*) in a given language $\mathcal{L}$ are defined recursively by the usual constructions of λ-abstraction and application:

- Each variable x_α and each constant c_α is a formula of type α
- If A_α is a formula of type α and x_β a variable, then $\lambda x_\beta A_\alpha$ is a formula of type $\alpha\beta$
- If $A_{\alpha\beta}$ and B_β are formulae of types $\alpha\beta$ and β, then $(A_{\alpha\beta}B_\beta)$ is a formula of type α

The set of all formulae in the language $\mathcal{L}$ will be denoted by $Form(\mathcal{L})$. Formulae of type o are called *propositions.* We may use infix notation for $=_{(o\alpha)\alpha}$ and omit type subscripts if they are known from the context or arbitrary (modulo well-typedness).

The notions of *subformula, free* and *bound* (by λ-abstraction) occurrence of a variable, *closed formula,* and *substitutability* are defined as usual. The derivation rule "from $A^1_o, \ldots, A^n_o$ derive B_o" will be written as $A^1_o, \ldots, A^n_o \,/\, B_o$; derivation rules with no premises will be called *axioms.*

The axiomatic system of TT_0 consists of the following rules and axioms, for all propositions A, formulae B, B' of any type β and C of any type γ, variables x of type β, and formulae F, G of type αβ not containing free x:

$A \,/\, A[B/x]$	substitution
$A, B = B' \,/\, A[B' /\!\!/ B]$	equality
$/\, (\lambda x C)B = C[B/x]$	λ-abstraction
$Fx = Gx \,/\, F = G$	extensionality

where $A[B/x]$ denotes the result of substituting the formula B for *all* free occurrences of x in A and $A[B' /\!\!/ B]$ the result of substituting B' for a *single* occurrence of the subformula B in A (assuming substitutability in both cases).

The axioms and rules of TT_0 represent some of the most fundamental principles of type theory. In particular, the substitution rule ensures substitution-invariance for variables; the rule of equality embodies *Leibniz's principle* of indiscernibility of identicals; the axiom of λ-abstraction corresponds to the rule known as β-*conversion* in type theory; and extensionality is equivalent to the type-theoretic axiom of η-*conversion,* $/\, \lambda x(Bx) = B$ if x is not free in B. The axioms and rules of TT_0 also parallel the higher-order machinery of Russell-style fuzzy type theory FCT [3] (namely, the substitution invariance and equality axioms of the background fuzzy logic and the FCT axioms of comprehension and extensionality).

The notions of *proof, theorem,* and *provability* ($\vdash$) in TT_0 are defined as usual in finitary axiomatic systems. A *theory* is any set of propositions in a given language $\mathcal{L}$. A theory is *inconsistent* if it proves all propositions in $\mathcal{L}$, and *consistent* otherwise.

Despite its parsimoniousness, the theory TT_0 proves various type-theoretic principles of CTT. For instance, all λ-conversion steps are derived rules of TT_0; and T-provable equality is (for any theory T over TT_0) a congruence relation on formulae.

2 The Semantics of TT_0

The (Henkin-style) semantics of TT_0 is similar to that of CTT and FTT. Like in FTT, the models of TT_0 admit more than two values of type o. While FTT has only been developed over logics with a single designated truth value, TT_0 admits any (non-exhaustive) set of designated truth values (in order to accommodate extensions to weakening-free logics, including uninorm fuzzy logics, linear, and relevance logics).

Let X, A be non-empty sets and $\mathcal{L}$ a language. Then we define:

- A *basic frame* over (X, A) is a system $M = \{M_\alpha\}_{\alpha \in Types}$ of sets such that $M_\varepsilon = X$, $M_o = A$, and $\emptyset \neq M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$, for all $\alpha, \beta \in Types$.

– A *frame* $\mathbf{M} = (M, D, Eq)$ is a basic frame M equipped with (i) a subset $D \subsetneq M_o$ of designated truth values and (ii) functions $Eq_\alpha \colon M_\alpha^2 \to M_o$ such that $m = m'$ iff $Eq_\alpha(m, m') \in D$, for all $m, m' \in M_\alpha$.
– A *valuation* in a frame $\mathbf{M}$ is a mapping $v = \bigcup_{\alpha \in \mathit{Types}} v_\alpha$, where $v_\alpha \colon \mathit{Var}_\alpha \to M_\alpha$.
– An *interpretation* in a frame $\mathbf{M}$ is an assignment $I \colon \mathcal{L} \to \mathbf{M}$ such that $I(c_\alpha) \in M_\alpha$ for all $c_\alpha \in \mathcal{L}$ and $I(=_{(o\alpha)\alpha}) = Eq_\alpha$ for all $\alpha \in \mathit{Types}$.
– The *semantic value assignment* under an interpretation I and a valuation v in a frame $\mathbf{M}$ is a function $\mathbf{M}_v^I \colon \mathit{Form}(\mathcal{L}) \to \mathbf{M}$ satisfying the Tarski conditions:
  - $\mathbf{M}_v^I(x_\alpha) = v(x_\alpha)$
  - $\mathbf{M}_v^I(c_\alpha) = I(c_\alpha)$
  - $\mathbf{M}_v^I(B_{\beta\alpha} A_\alpha) = \mathbf{M}_v^I(B_{\beta\alpha})\big(\mathbf{M}_v^I(A_\alpha)\big)$
  - $\mathbf{M}_v^I(\lambda x_\alpha B_\beta) = F \colon M_\alpha \to M_\beta$ such that $F(m) = \mathbf{M}^I_{v_{x_\alpha : m}}(B_\beta)$ for all $m \in M_\alpha$

  where $v_{x_\alpha : m}(y_\alpha) = m$ if y_α is the variable x_α and $v_{x_\alpha : m}(y_\alpha) = v(y_\alpha)$ otherwise.
– A *model* is a pair $\mathbf{M}^I = (\mathbf{M}, I)$ of a frame $\mathbf{M}$ and an interpretation I in $\mathbf{M}$ such that for all valuations v in $\mathbf{M}$ there exists a semantic value assignment $\mathbf{M}_v^I$.
– A proposition A_o is *valid* in a model $\mathbf{M}^I$ if $\mathbf{M}_v^I(A_o) \in D$ for all valuations v in $\mathbf{M}$.
– A model $\mathbf{M}^I$ is a *model of a theory* T if all $A_o \in T$ are valid in $\mathbf{M}^I$.
– A theory T *entails* A_o, written $T \models A_o$, if all models of T are also models of A_o.

Theorem 1 (Strong Completeness). *Let T be a theory and A_o a proposition. Then:*

1. *$T \models A_o$ iff $T \vdash A_o$*
2. *T is consistent iff T has a model*
3. *$T \models A_o$ iff $T' \models A_o$ for a finite $T' \subseteq T$ (compactness)*

The proof of the Strong Completeness Theorem is obtained by the standard method of constructing the canonical (closed-term) model for each consistent Henkin theory. The proof requires several modifications to the known completeness proofs for CTT and FTT [5, 1, 6] at such places where they rely on the properties of logical constants absent from TT_0 (e.g., the deduction theorem or the universal closure). For instance, it is the following notion of Henkin completeness which turns out to be suitable for TT_0 (while a weaker notion of *extensional completeness* is sufficient for CTT and FTT, see [1, 6]):

– A theory T is *Henkin complete* if for every closed formula $\lambda x^1 \ldots \lambda x^n A_o$ there are closed formulae $B^1, \ldots, B^n$ such that if $T \vdash \big(\lambda x^1 \ldots \lambda x^n A_o\big) B^1 \ldots B^n$ then $T \vdash A_o$.

Lemma 1 (Henkin completion). *Every consistent theory can be conservatively extended to a consistent Henkin complete theory.*

3 Basic Extensions of TT_0

By the Strong Completeness Theorem for TT_0, the completeness proofs for axiomatic extensions of TT_0 reduce to the characterization of their models among those of TT_0. However, many important extensions of TT_0, including FTT and CTT, cannot be cast as *axiomatic* extensions of TT_0, as they contain additional rules (such as modus ponens, generalization, or $\triangle$-necessitation). Nevertheless, it turns out that these rules can be easily added to TT_0 and the strong completeness proofs for the resulting theories be obtained by just minor modifications of the proof for TT_0. Here we give three sample extensions of TT_0 by derivation rules:

1. The type theory $\mathrm{TT}_{\to}$ extends the logical language of TT_0 by an additional constant $\to$ of type (oo)o, governed by the derivation rule of modus ponens: $A, A \to B \,/\, A$. The frames for $\mathrm{TT}_{\to}$ expand those for TT_0 by an additional function $Imp\colon M_o^2 \to M_o$ such that for all $m, m' \in M_o$, if $m \in D$ and $Imp(m,m') \in D$ then $m' \in D$.
2. The type theory $\mathrm{TT}_{\triangle}$ extends the logical language of TT_0 by an additional constant $\triangle$ of type oo, governed by the derivation rule of necessitation: $A \,/\, \triangle A$. The frames for $\mathrm{TT}_{\to}$ expand those for TT_0 by an additional function $De\colon M_o \to M_o$ such that $De(m) \in D$ whenever $m \in D$.
3. The type theory $\mathrm{TT}_{\forall}$ extends the logical language of TT_0 by additional constants $\forall_{o(o\alpha)}$ of type $o(o\alpha)$ for all types α. The following two rules are added to TT_0: $A \,/\, \forall x A$ (generalization) and $\forall x A \,/\, A$ (specification), where $\forall x A$ abbreviates the formula $\forall_{o(o\alpha)} \lambda x_\alpha A_o$. The frames for $\mathrm{TT}_{\to}$ expand those for TT_0 by functions $All_\alpha\colon M_{o\alpha} \to M_o$, for all types α, such that for every $F \in M_{o\alpha}$: $All_\alpha(F) \in D$ iff $F(m) \in D$ for each $m \in M_\alpha$.

Theorem 2. $\mathrm{TT}_{\to}$, $\mathrm{TT}_{\triangle}$, *and* $\mathrm{TT}_{\forall}$ *enjoy the Strong Completeness Theorem (in the same forms as in Theorem 1, w.r.t. models over frames expanded as described above).*

The type theories $\mathrm{TT}_{\to}$, $\mathrm{TT}_{\forall}$, and $\mathrm{TT}_{\triangle}$ (and combinations thereof) can be used for defining non-classical type theories over a broad class of propositional or first-order logics (whose equivalence connectives satisfy the equality axioms of TT_0), by adding the appropriate logical constants and the translations of their logical axioms for type o. These extensions include intermediary, substructural (incl. relevant, linear, and fuzzy), and modal logics of order ω. A detailed exploration of the landscape of extensions of TT_0 is left for future work.

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Reduction in formal fuzzy contexts

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We consider the following problem: for a given formal fuzzy context $\langle X,Y,I\rangle$, i.e. I is a binary fuzzy relation between a finite set X (of objects) and a finite set Y (of attributes), find a formal fuzzy context $\langle X',Y',I'\rangle$, minimal w.r.t. cardinality of X' and Y', such that the concept lattice of $\langle X',Y',I'\rangle$ is isomorphic to that of $\langle X,Y,I\rangle$.

In the classical, Boolean case, the problem has a well-known solution. Even though this is not explicitly mentioned in the literature, the essence of the problem in the Boolean case may be rephrased as the following problem. Given a system $\mathcal{S}$ of subsets of a set U, find a base of the closure system $[\mathcal{S}]$ generated by $\mathcal{S}$. Such base is unique and consists of intersection-irreducible elements of $\mathcal{S}$. Via the well-known duality between classical closure and interior operator, the problem is equivalent to the problem of finding a base of a Boolean matrix which is known to be unique in Boolean matrix theory [4].

In a fuzzy setting (with complete residuated lattices used as the structures of truth degrees), the problem is more complex, basically because there are two generating operations involved (see [2]): It is well-known that the set $\mathrm{Ext}(X,Y,I)$ of extents of formal fuzzy context forms an **L**-closure system in X, i.e. it is closed under $\rightarrow$-shifts and $\bigwedge$-intersections. A reduction, i.e. finding a base, in a fuzzy case has therefore take both of these operations into account. Conversely, every **L**-closure system in X is in the form $\mathrm{Ext}(X,Y,I)$ [1]; these claims hold for the set $\mathrm{Int}(X,Y,I)$ of intents as well. As a result, the problem behaves differently from the one in the Boolean case, even though it is conceptually of the same character.

In fact, if we consider the fuzzy relation I as a binary matrix in which the entry at row x and column y contains the degree $I(x,y)$, then $\mathrm{Ext}(X,Y,I)$ and $\mathrm{Int}(X,Y,I)$ are just the **L**-closure systems in X and Y generated by the columns and rows of this matrix, i.e. the least **L**-closure system $[\mathcal{S}]$ containing the columns and rows, respectively. Thus, the essence of the considered problem may be rephrased as the problem of finding bases of fuzzy closure systems: Given a system $\mathcal{S}$ of L-sets in U, i.e. $\mathcal{S}\subseteq L^U$, find a $[\,]$-base of the **L**-closure system $[\mathcal{S}]$ generated by $\mathcal{S}$, where a $[\,]$-base of an **L**-closure system $\mathcal{T}$ in U is a set $\mathcal{S}$ of L-sets in U such that $[\mathcal{S}]=\mathcal{T}$ (base generates $\mathcal{T}$), $[\mathcal{P}]\neq\mathcal{T}$ for every $\mathcal{P}\subset\mathcal{S}$ (base is non-redundant). Since **L**-closure systems often occur in fuzzy set theory and fuzzy logic, the ramifications are broad.

We provide a useful description of $[\mathcal{S}]$: we show that $[\,]$ may be seen as a composition of two other, simpler closure operators as follows. Let for $\mathcal{S}\subseteq L^U$,

$$[\mathcal{S}]_\bigwedge=\{\bigwedge\mathcal{T}\,|\,\mathcal{T}\subseteq\mathcal{S}\},$$
$$[\mathcal{S}]_\rightarrow=\{a\rightarrow A\,|\,a\in L, A\in\mathcal{S}\},$$

where $a \to A$ is an L-set in U, called the $\to$-shift of A by a, defined by $(a \to A)(u) = a \to A(u)$. Then we obtain.

Theorem 1. *For any $\mathcal{S} \subseteq L^U$, we have $[\mathcal{S}] = [[\mathcal{S}]_\to]_\bigwedge$.*

Furthermore, bases of $[\ ]_\bigwedge$ and $[\ ]_\to$ are uniquely given by sets of irreducible elements. Namely, define for $\mathcal{V} \subseteq L^U$,

$$\mathrm{irr}_\bigwedge(\mathcal{V}) = \{B \in \mathcal{V} \mid B \notin [\mathcal{V} - \{B\}]_\bigwedge\},$$
$$\mathrm{irr}_\to(\mathcal{V}) = \{B \in \mathcal{V} \mid B' \lhd B \text{ implies } B' = B \text{ for any } B' \in \mathcal{V}\},$$

where $\lhd$ denotes the binary relation in L^U defined by $B_1 \lhd B_2$ if and only if $B_2 = a \to B_1$ for some $a \in L$. Then for every finite set $\mathcal{S}$, $\mathrm{irr}_\bigwedge(\mathcal{S})$ is a unique $[\ ]_\bigwedge$-base of $\mathcal{S}$ and $\mathrm{irr}_\to(\mathcal{V})$ is a unique $[\ ]_\to$-base of $\mathcal{S}$.

Given the descriptions of the unique $[\ ]_\bigwedge$- and $[\ ]_\to$-bases, we propose two simple methods that enable us to obtain for a given finite set $\mathcal{S} \subseteq L^U$ a finite set of generators of $[\mathcal{S}]$:

Theorem 2. *For every $\mathcal{S} \subseteq L^U$, $\mathrm{irr}_\bigwedge(\mathrm{irr}_\to(\mathcal{S}))$ is a $[\ ]_\bigwedge$-non-redundant and $[\ ]_\to$-non-redundant set of generators of $[\mathcal{S}]$ and $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$ is a $[\ ]_\bigwedge$-non-redundant and $[\ ]_\to$-non-redundant set of generators of $[\mathcal{S}]$. Moreover, if $\mathcal{S}$ is closed under $\to$-multiplications, $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$ is a $[\ ]$-base of $[\mathcal{S}]$.*

Figure 1 shows that $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$ and $\mathrm{irr}_\bigwedge(\mathrm{irr}_\to(\mathcal{S}))$ may indeed be different.

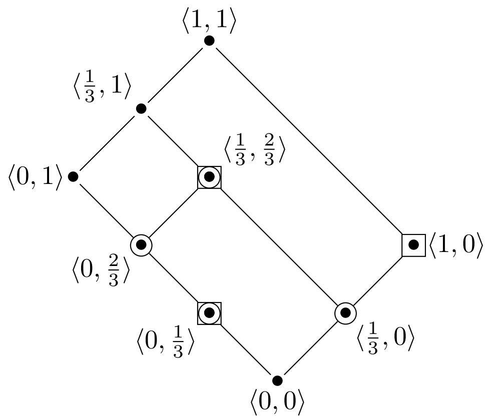
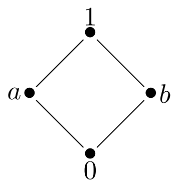


Fig. 1. Hasse diagram of an **L**-closure system $\mathcal{S}$ in a two-element universe U with **L** being the four-element Gödel chain. Circled nodes represent $\mathrm{irr}_\bigwedge(\mathrm{irr}_\to(\mathcal{S}))$. Squared nodes represent $\mathrm{irr}_\to(\mathrm{irr}_\bigwedge(\mathcal{S}))$.

Contrary to the classical case, an **L**-closure system may have different bases. With linearly ordered residuated lattice the bases of the **L**-closure system always equicardinal. For non-linearly ordered residuated lattice, the bases may have different number of elements. For example, consider the residuated lattice in Fig. 2. Then $\{\langle a \rangle, \langle b \rangle\}$ and $\{\langle 0 \rangle\}$ are bases of $\mathcal{S} = \{\langle 0 \rangle, \langle a \rangle, \langle b \rangle, \langle 1 \rangle\}$ (an **L**-closure system in a singleton universe). This shows that bases of the same **L**-closure system can have different size and that two disjoint sets can generate the same **L**-closure system.

In our talk, we present the above results as well as further ones that lead to an algorithm for computing bases of **L**-closure systems which will also be presented.



$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Fig. 2. Residuated lattice $\mathbf{L} = \langle L, \vee, \wedge, \otimes, \rightarrow, 0, 1 \rangle$ (left) and its residuum $\rightarrow$ (right); $\otimes$ is equal to $\wedge$.

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On a graded version of “only knowing” and its relation to fuzzy autoepistemic logic and fuzzy modal logics

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In this talk we will discuss relationships between fuzzy autoepistemic logic and fuzzy modal logics, generalizing well-known links beween autoepistemic logic and several classical modal logic systems. In particular we will generalize Levesque's logic of only knowing [1] and show that when generalizing to the many-valued case the correspondence to autoepistemic logic remains valid. Moreover we provide a sound and complete axiomatization for this many-valued logic of only knowing using the axiomatization we previously proposed for many-valued K45 [2].

Since its introduction in the 1980s, autoepistemic logic [3–5] has been one of the main formalisms for nonmonotonic reasoning. It extends propositional logic by offering the ability to reason about an agent's (lack of) beliefs. More precisely, these beliefs are represented by sets of sentences in a propositional language augmented by a modal operator B. If φ is a formula, then Bφ, which has to be interpreted as “φ is believed”, is a formula as well. Originally, autoepistemic logic was intended to model the beliefs of an ideally rational agent reflecting upon his own beliefs [3]. Given a set of initial premises, the (closed) set of beliefs an agent should adopt is given by the so called stable expansions. In particular, given a set of autoepistemic formulas A, a set of autoepistemic formulas E_A is a stable expansion of A if

$$E_A = \{\varphi \mid A \cup \{\mathrm{B}\psi \mid \psi \in E_A\} \cup \{\neg \mathrm{B}\psi \mid \psi \notin E_A\} \vdash \varphi\},$$

where $\vdash$ denotes derivability in classical propositional logic and each formula Bφ is considered as a new propositional variable (e.g. $B(a \wedge \mathrm{B}b)$ is a variable but $\mathrm{B}a \wedge b$ is the conjunction of the variables Ba and b). In [1], autoepistemic logic is extended such that expressions of the form “φ is all that is believed” (i.e. there are no other relevant beliefs about φ) can be formulated. To do this, the language is augmented with another modal operator O, where Oφ has to be read as "α is *all* that is believed" or "*only* α is believed". In [1] it is then shown that stable expansions correspond to "only knowing" sentences in this logic.

Recently, a fuzzy generalization of autoepistemic logic has been defined in [6]. In particular, given a set of autoepistemic formulas A, a stable fuzzy expansion of A is a fuzzy set of formulas E_A satisfying the following fixpoint condition:

$$E_A(\varphi) = \inf\{v(\varphi) \mid v \in \Omega_k \text{ such that } v(\alpha) = 1 \text{ for all } \alpha \in A, \text{ and } v(\mathrm{B}\phi) = E_A(\phi) \text{ for any formula } \phi\},$$

where Ω_k is the set of all propositional many-valued evaluations treating every formula Bφ as a new propositional variable.

In this talk, we will first recall some generalizations of the main classical propositional modal logics of belief (K45, KD45, S5) based on finitely-valued Łukasiewicz logic with semantics based on Kripke models with crisp accessibility relations from [2]. The approach in [2] is based on the minimal modal logic [7] and generalizes the well known classical modal logics K45, KD45 and S5 [8]. Using these fuzzy modal logics, a graded notion of belief on propositions, in the sense of admitting partial degrees of truth between 0 (fully false) and 1 (fully true), can be modeled. For practical and technical reasons we will consider truth degrees belonging to a finite scale $S_k = \{0, \frac{1}{k}, \ldots, \frac{k-1}{k}, 1\}$.

Then we show how fuzzy autoepistemic logic can be approached using these many-valued modal logics in the following sense. We define a generalization of Levesque's logic of only knowing based on finitely-valued Łukasiewicz logic. As in the classical case we provide a sound and complete axiomatisation for this finitely-valued Łukasiewicz logic of "only knowing" based on finitely-valued Łukasiewicz K45 and show that stable fuzzy expansions correspond to "only knowing" valid sentences.

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The Morita-equivalence between MV-algebras and abelian ℓ-groups with strong unit

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This talk is based on [3]. In this paper, we generalize to a topos-theoretic context the well-known equivalence established by Mundici in 1986 [5] between the category of MV-algebras and the category of lattice ordered abelian groups with strong unit. The key of this generalization is the fact that we can interpret these categories as the categories of set-based models of appropriate theories, namely the theory $\mathbb{MV}$ of MV-algebras and the theory $\mathbb{L}_u$ of abelian ℓ-groups with strong unit. We show that this equivalence generalizes over an arbitrary Grothendieck topos, yielding a Morita-equivalence between the theories $\mathbb{MV}$ and $\mathbb{L}_u$. This allows us to transfer properties and results across the two theories by using the methods of topos theory. Our main applications are:

– a bijective correspondence between the geometric extensions of the two theories;
– a form of compactness and completeness for the theory $\mathbb{L}_u$;
– a logical characterization of the finitely presentable ℓ-groups with strong unit;
– a sheaf-theoretic version of Mundici's equivalence.

A (Grothendieck) topos $\mathscr{E}$ can be considered as a mathematical universe in which one can consider models of any kind of first-order theory. In particular, one can consider models of *geometric theories*, i.e. theories over a first-order signature Σ whose axioms can be presented in the form $(\phi \vdash_x \psi)$, where ϕ and ψ are *geometric formulae*, that is formulae with finite number of free variables in the context x built up from atomic formulae over Σ by only using finitary conjunctions, infinitary disjunctions and existential quantifications.

We observe that the theories $\mathbb{MV}$ and $\mathbb{L}_u$ are geometric theories; $\mathbb{MV}$ is finitary algebraic, whereas $\mathbb{L}_u$ is strictly geometric as we need an infinitary disjunction (over the natural numbers) to describe the property of the strong unit.

Every geometric theory $\mathbb{T}$ has a *classifying topos* (cf. [4]), that is a Grothendieck topos satisfying the universal property that the models of $\mathbb{T}$ in any topos $\mathscr{E}$ are classified by the morphisms from $\mathscr{E}$ to the classifying topos, naturally in $\mathscr{E}$. The classifying topos of a theory is unique (up to categorical equivalence), and it can be seen as a sort of 'semantical core' of the theory, embodying its essential features and providing a 'natural environment' in which the theory can be investigated, both in itself and in relationship with other theories.

Two theories are said to be *Morita-equivalent* if they have the same classifying topos (up to categorical equivalence), that is if the categories of models of the two theories in any topos $\mathscr{E}$ are equivalent, naturally in $\mathscr{E}$.

For every topos $\mathscr{E}$, we build a categorical equivalence

$$\mathbb{MV}\text{-mod}(\mathscr{E}) \simeq \mathbb{L}_u\text{-mod}(\mathscr{E})$$

between the categories of models, respectively, of $\mathbb{MV}$ and $\mathbb{L}_u$ in $\mathscr{E}$. The two functors

$$L_{\mathscr{E}} : \mathbb{MV}\text{-mod}(\mathscr{E}) \to \mathbb{L}_u\text{-mod}(\mathscr{E}) \quad \Gamma_{\mathscr{E}} : \mathbb{L}_u\text{-mod}(\mathscr{E}) \to \mathbb{MV}\text{-mod}(\mathscr{E})$$

of the equivalence generalize the classical functors of Mundici's equivalence.

Further, we observe that this equivalence is natural in $\mathscr{E}$, obtaining the following

Theorem 1. *The functors $L_{\mathscr{E}}$ and $\Gamma_{\mathscr{E}}$ yield a Morita-equivalence between the theories $\mathbb{MV}$ and $\mathbb{L}_u$. In particular, the respective classifying toposes are categorically equivalent.*

This Morita-equivalence is interesting because the theories $\mathbb{MV}$ and $\mathbb{L}_u$ are not bi-interpretable. Indeed, the former is interpretable in the latter, in the sense that we can 'translate' every geometric sequent σ in the language of $\mathbb{MV}$ in a geometric sequent σ' in the language of $\mathbb{L}_u$ in such a way that, in particular, if σ is provable in $\mathbb{MV}$ then σ' is provable in $\mathbb{L}_u$. The opposite does not hold, i.e. the theory $\mathbb{L}_u$ is not interpretable in $\mathbb{MV}$. Nonetheless, the fact that these two theories have equivalent classifying toposes - rather than merely equivalent categories of set-based models - allows us to discover non-trivial connections between the two theories by using appropriate topos-theoretic invariants.

For instance, we can use the invariant notion of subtopos to establish a relationship between the quotients (i.e. geometric extensions over the same signature, c.f. [1]) of the two theories. The duality theorem of [1], by giving a bijective correspondence between the quotients of a geometric theory and the subtoposes of its classifying topos, provides the appropriate characterizations of the notion of subtopos in terms of the syntax of the two theories. This yields at once the following

Theorem 2. *Every quotient of the theory $\mathbb{MV}$ is Morita-equivalent to a quotient of the theory $\mathbb{L}_u$, and conversely. These Morita-equivalences are the restrictions of the one between $\mathbb{MV}$ and $\mathbb{L}_u$.*

By using a different invariant concept, namely the property of compactness of a terminal object in the topos, we obtain that the theory $\mathbb{L}_u$ enjoy a form of compactness, while the fact that its classifying topos is a presheaf topos implies that $\mathbb{L}_u$ satisfies a classical completeness theorem. Thus the Morita-equivalence between the geometric theory $\mathbb{L}_u$ and the finitary algebraic theory $\mathbb{MV}$ also implies a form of compactness and completeness for $\mathbb{L}_u$, properties which are *a priori* not expected as the theory $\mathbb{L}_u$ is infinitary:

Theorem 3. *(i) For any geometric sequent σ in the signature of $\mathbb{L}_u$, σ is valid in all abelian ℓ-groups with strong unit in the topos of sets if and only if it is provable in the theory $\mathbb{L}_u$;*

(ii) For any geometric sentences ϕ_i in the signature of $\mathbb{L}_u$, $\top \vdash \bigvee_{i \in I} \phi_i$ is provable in $\mathbb{L}_u$ (equivalently by (i), every abelian ℓ-group with strong unit in the topos of sets satisfies at least one of the ϕ_i) if and only if there exists a finite subset $J \subseteq I$ such that the sequent $\top \vdash \bigvee_{i \in J} \phi_i$ is provable in $\mathbb{L}_u$ (equivalently by (i), every abelian ℓ-group with strong unit in the topos of sets satisfies at least one of the ϕ_i for $i \in J$).

For any geometric theory $\mathbb{T}$ over Σ we can consider its *syntactic category* $\mathscr{C}_{\mathbb{T}}$, whose objects are geometric formulae-in-context over Σ and arrows are the $\mathbb{T}$-provable classes of geometric formulae which is $\mathbb{T}$-provably functional from the domain formula to the codomain formula (c.f. [4] for more details). We can equip this category with its canonical coverage. If a formula-in-context admits only the trivial covering, we call it an $\mathbb{T}$-*irreducibile* formula.

Let $\mathscr{C}^{irr}_{\mathbb{L}_u}$ be the full subcategory of the syntactic category $\mathscr{C}_{\mathbb{L}_u}$ on the $\mathbb{L}_u$-irreducible formulae. Further, let $\mathscr{C}^{alg}_{\mathbb{MV}}$ be the algebraic syntactic category of $\mathbb{MV}$, whose objects are the finite conjunctions of atomic formulae over the signature of $\mathbb{MV}$ and whose arrows $\{x.\phi\} \to \{y.\psi\}$ are sequences of terms $t_1(x), \ldots, t_m(x)$ such that the sequent $(\phi \vdash_x \psi(t_1(x)), \ldots, t_n(x))$ is provable in $\mathbb{MV}$, modulo the equivalence relation which identifies two such sequences t and t' precisely when the sequent $(\phi \vdash_x t(x) = t'(x))$ is provable in $\mathbb{MV}$.

Theorem 4. *With the above notation, we have an equivalence of categories $\mathscr{C}^{alg}_{\mathbb{MV}} \simeq \mathscr{C}^{irr}_{\mathbb{L}_u}$ representing the syntactic counterpart of the equivalence of categories $\mathbf{MV}_{f.p.} \simeq$ f.p.$\mathbb{L}_u$-mod$(\mathbf{Set})$, where $\mathbf{MV}_{f.p.}$ is the category of finitely presented MV-algebras and homomorphisms between them and f.p.$\mathbb{L}_u$ -mod$(\mathbf{Set})$ is the category of finitely presentable models of the theory $\mathbb{L}_u$.*

In particular, the finitely presentable abelian ℓ-groups with strong unit are precisely the finitely presented abelian ℓ-groups with unit which are presented by a $\mathbb{L}_u$-irreducible formula; the ℓ-group presented by such a formula $\{x.\phi\}$ has as underlying set the set of $\mathbb{L}_u$-provably functional geometric formulae from $\{x.\phi\}$ to $\{z.\top\}$ and as order and operations the obvious ones.

We have defined, for every Grothendieck topos $\mathscr{E}$, a categorical equivalence between the category of models of $\mathbb{L}_u$ in $\mathscr{E}$ and the category of models of $\mathbb{MV}$ in $\mathscr{E}$, which is natural in $\mathscr{E}$. By specializing this result to toposes $\mathbf{Sh}(X)$ of sheaves on a topological space X, we shall obtain a sheaf-theoretic generalization of Mundici's equivalence. The two functors $\Gamma_{\mathbf{Sh}(X)}$ and $L_{\mathbf{Sh}(X)}$ defining the equivalence can be described as follows: $\Gamma_{\mathbf{Sh}(X)}$ sends any sheaf F in $\mathbf{Sh}_{\mathbb{L}_u}(X)$ to the sheaf $\Gamma_{\mathbf{Sh}(X)}(F)$ on X sending every open set U of X to the MV-algebra given by the unit interval in the ℓ-group $F(U)$, and it acts on arrows in the obvious way. In the converse direction, $L_{\mathbf{Sh}(X)}$ assigns to any sheaf G in $\mathbf{Sh}_{\mathbb{MV}}(X)$ the sheaf $L_{\mathbf{Sh}(X)}(G)$ on X whose stalk at any point $x \in X$ is equal to the ℓ-group corresponding via Mundici's equivalence to the MV-algebra G_x.

The naturality in $\mathscr{E}$ of our Morita-equivalence implies in particular that the resulting equivalence

$$\tau_X : \mathbf{Sh}_{\mathbb{MV}}(X) \simeq \mathbf{Sh}_{\mathbb{L}_u}(X)$$

is natural in X. In particular, by taking X to be the one-point space, we obtain that, at the level of stalks, τ_X acts as the classical Mundici's equivalence.

Summarizing, we have the following result.

Corollary 1. *Let X be a topological space. Then, with the above notation, we have a categorical equivalence*

$$\tau_X : \mathbf{Sh}_{\mathbb{MV}}(X) \simeq \mathbf{Sh}_{\mathbb{L}_u}(X)$$

sending any sheaf F in $\mathbf{Sh}_{\mathbb{L}_u}(X)$ to the sheaf $\Gamma_{\mathbf{Sh}(X)}(F)$ on X sending every open set U of X to the MV-algebra given by the unit interval in the ℓ-group $F(U)$, and any sheaf G in $\mathbf{Sh}_{\mathbb{MV}}(X)$ to the sheaf $L_{\mathbf{Sh}(X)}(G)$ in $\mathbf{Sh}_{\mathbb{L}_u}(X)$ whose stalk at any point x of X is the ℓ-group corresponding to the MV-algebra G_x under Mundici's equivalence.

The equivalence τ_X is natural in X, in the sense that for any continuous map $f : X \to Y$ of topological spaces, the diagram

$$\begin{array}{ccc} \mathbf{Sh}_{\mathbb{MV}}(Y) & \xrightarrow{\tau_Y} & \mathbf{Sh}_{\mathbb{L}_u}(Y) \\ \downarrow{\scriptstyle j_f} & & \downarrow{\scriptstyle i_f} \\ \mathbf{Sh}_{\mathbb{MV}}(X) & \xrightarrow{\tau_X} & \mathbf{Sh}_{\mathbb{L}_u}(X) \end{array}$$

commutes, where $i_f : \mathbf{Sh}_{\mathbb{MV}}(Y) \to \mathbf{Sh}_{\mathbb{MV}}(X)$ and $j_f : \mathbf{Sh}_{\mathbb{L}_u}(Y) \to \mathbf{Sh}_{\mathbb{L}_u}(X)$ are the morphisms induced on sheaves by f.

Moreover, τ_X acts, at the level of stalks, as the classical Mundici's equivalence.

This work represents a contribution to the research programme 'toposes as bridges' introduced in [2], which aims at developing the unifying potential of the notion of Grothendieck topos as a means for relating different mathematical theories to each other through topos-theoretic invariants.

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Skolem and Herbrand theorems for uninorm-based fuzzy logics

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1 Introduction

The aim of the research reported here is to provide Herbrand and Skolemization theorems for first-order fuzzy logics (see, e.g., [1–3]). Such logics are often undecidable, but their (decidable) fragments provide the foundations for knowledge representation and reasoning methods such as non-classical logic programming and description logics (see, e.g., [4, 5]). Our goal is to avoid a duplication of research effort by providing a general approach to the development of automated reasoning techniques for first-order fuzzy logics. Herbrand and Skolemization theorems play a pivotal role in this development, reducing first-order problems to propositional problems. These theorems are also helpful for addressing theoretical problems in particular cases such as first-order Łukasiewicz logic.

In classical first-order logic, questions of validity and semantic consequence reduce to the satisfiability of a set of sentences; Skolemization and Herbrand theorems then reduce these questions further to the satisfiability of a set of propositional formulas (see, e.g., [6]). In first-order fuzzy logics, semantic consequence does not (typically) reduce to satisfiability and in the absence of quantifier shifts and a deduction theorem, non-prenex formulas should be considered on both sides of the consequence relation. The general Skolemization and Herbrand theorems obtained here therefore take various forms, applying either to the left or right of the consequence relation, and to restricted sets of formulas. The logics investigated in this paper are defined based on arbitrary classes of complete UL-chains. Herbrand and Skolemization theorems may often be established for such logics proof-theoretically (see [2]) via mid(hyper)sequent theorems proved using permutations of rules tailored to the case at hand. By contrast, the our proposed uniform approach is purely algebraic and applies also to many cases where no calculus has yet been defined.[3]

[3] This work is based on the paper [7], some of our results have also been independently obtained by Terui [8]. However, his approach is narrower and more algebraic in scope (e.g., Skolemization is not really considered); his main result shows rather that algebras for a broad class of logics admit suitable completions and that therefore these logics have a Herbrand theorem.

2 Preliminaries

Algebras Let us recall that a residuated uninorm $*$ is an associative and commutative binary function $*$ on $[0,1]$ that is increasing in both arguments and has a unit e_* and residuum $\to_*$. Fixing an arbitrary element $d \in [0,1]$, any residuated uninorm determines the so-called *standard* UL-chain $\langle [0,1], *, \to_*, \min, \max, d, e_* \rangle$. In this work we will study logics given by classes of linearly ordered algebras from the variety generated by standard UL-chains.[4]

Definition 1. *A* UL-algebra *is an algebra* $A = \langle A, \&, \to, \wedge, \vee, \overline{0}, \overline{1} \rangle$ *such that:*

(a) $\langle A, \wedge, \vee \rangle$ *is a lattice with an order defined by* $x \leq y$ *iff* $x \wedge y = x$.
(b) $\langle A, \&, \overline{1} \rangle$ *is a commutative monoid.*
(c) $\to$ *is the residuum of* $\&$*; i.e., for all* $x, y, z \in A$*:* $x \& y \leq z$ *iff* $x \leq y \to z$.
(d) $((x \to y) \wedge \overline{1}) \vee ((y \to x) \wedge \overline{1}) = \overline{1}$.

The algebra A is complete *if for all* $X \subseteq A$*, both* $\bigvee X$ *and* $\bigwedge X$ *exists in A, and A is an* UL-chain *if for all* $x, y \in A$*, either* $x \leq y$ *or* $y \leq x$.

Logics We assume the reader to be familiar with first order fuzzy logic as described e.g. in [1–3]. Thus in the short preliminary section we concentrate on setting the framework and recalling some denotation. A *(countable) predicate language* $\mathcal{P}$ is defined as usual; for convenience, we call nularry predicate symbols, *propositional atoms*, and a language $\mathcal{P}$ containing only propositional atoms, *propositional*. By model $\mathcal{P}$-$\mathbb{K}$-model of T we understand any predicate structure for the language $\mathcal{P}$ over an algebra from the class $\mathbb{K}$ of UL-chains in which all formulas of T are true.

Definition 2. *Let* $\mathbb{K}$ *be a class of complete* UL*-chain. A* $\mathcal{P}$*-formula* φ *is a* semantic consequence *of a* $\mathcal{P}$*-theory* T *in* $\mathbb{K}$*, written* $T \models^{\mathcal{P}}_{\mathbb{K}} \varphi$*, if for each* $\mathcal{P}$*-*$\mathbb{K}$*-model* $\mathfrak{M}$ *of* T*, is also model of* φ.

A description of *propositional* fuzzy logics is implicit in our definitions. Let $\mathcal{P}_0$ be a propositional language consisting of countably infinitely many propositional atoms. Then we can identify $\models^{\mathcal{P}_0}_{\mathbb{K}}$ with the propositional logic of $\mathbb{K}$. In particular, the propositional logic of all complete UL-algebras is the finitely axiomatizable logic UL and other well-known propositional fuzzy logics are axiomatized by adding finitely many additional (propositional) axioms [9].

To obtain Herbrand theorems, we require a further crucial ingredient. Let us say that $\mathbb{K}$ is *finitary* if each $\mathcal{P}_0$-theory $T \cup \{\varphi\}$:

$$T \models^{\mathcal{P}_0}_{\mathbb{K}} \varphi \qquad \text{iff} \qquad \text{there is a finite } T' \subseteq T \text{ such that } T' \models^{\mathcal{P}_0}_{\mathbb{K}} \varphi.$$

The prototypical examples of finitary classes of UL-chains are any finite class of finite chains or the class of complete chains of a variety whose class of chains admits the *regular completions*; which is the case whenever the variety is axiomatized relative to all UL-algebras by so-called P_3 identities (see [10]), particular examples being e.g. the class of MTL-, UL-, IMTL-, and G-chains.

[4] Broadening the scope to non-commutative (or even non-associative) algebras or algebras with different operation symbols would lead to similar results, but complicate the presentation without adding greatly to our stock of useful examples.

3 Skolemization

Unlike first-order classical logic, we cannot assume the existence of equivalent prenex formulas or reductions of semantic consequence to satisfiability. We therefore obtain separate Skolemization theorems for formulas of a restricted form on the right and left of the consequence relation, where the latter is established only for certain cases.

Theorem 1 (Skolemization Right). *For each $\mathcal{P}$-theory $T \cup \{\varphi(x,y), \psi\}$ and function symbols $f_\varphi \notin \mathcal{P}$ of the same arity as y:*

$$T \models_{\mathbb{K}} \psi \to (\exists y)(\forall x)\varphi(x,y) \quad \text{iff} \quad T \models_{\mathbb{K}} \psi \to (\exists y)\varphi(f_\varphi(y),y)$$

$$T \models_{\mathbb{K}} (\forall y)(\exists x)\varphi(x,y) \to \psi \quad \text{iff} \quad T \models_{\mathbb{K}} (\forall y)\varphi(f_\varphi(y),y) \to \psi.$$

Theorem 2 (Skolemization Left). *Suppose that one of the following holds:*

(a) $\mathbb{K}$ *is the class of complete chains of a variety of* $\mathrm{FL_e}$*-algebras whose class of chains admits regular completions.*

(b) $\max\{V \in A \mid V < \overline{1}^A\}$ *exists for all* $A \in \mathbb{K}$ *(e.g., if each* $A \in \mathbb{K}$ *is finite).*

(c) $\mathbb{K}$ *consists of the standard Łukasiewicz algebra* $[0,1]_{\text{Ł}}$.

Then for each $\mathcal{P}$-theory $T \cup \{\varphi(x,y), \psi\}$ and any function symbol $f_\varphi \notin \mathcal{P}$ of the same arity as y:

$$T \cup \{(\forall y)(\exists x)\varphi(x,y)\} \models_{\mathbb{K}} \psi \quad \text{iff} \quad T \cup \{(\forall y)\varphi(f_\varphi(y),y)\} \models_{\mathbb{K}} \psi.$$

4 Herbrand Theorems

In first-order classical logic, it can be assumed (using Skolemization and quantifier shifts) that only universal formulas appear on the left and existential formulas on the right of the consequence relation. Indeed we may even consider, using the deduction theorem, only existential formulas on the right, or, using also the double negation law, only universal formulas on the left. In general, for first-order fuzzy logics, formulas are not equivalent to prenex formulas and the deduction theorem and double negation law fail. Nevertheless, we can establish Herbrand theorems of the same scope using formulas that are *classically* equivalent to universal and existential formulas. Such formulas are defined using BNF as follows, denoting quantifier-free formulas (for a given language) by Δ_0:

g-universal formulas	$P ::= \Delta_0 \mid P \wedge P \mid P \vee P \mid P \,\&\, P \mid (\forall x)P \mid N \to P$
g-existential formulas	$N ::= \Delta_0 \mid N \wedge N \mid N \vee N \mid N \,\&\, N \mid (\exists x)N \mid P \to N.$

We refer to theories containing only (g-)universal and (g-)existential formulas as (g-)-universal and (g-)existential theories, respectively.

For any predicate language $\mathcal{P}$, the *Herbrand universe* $\mathcal{U}(\mathcal{P})$ is the set of closed $\mathcal{P}$-terms (assuming, for simplicity, that every predicate language contains at least one object constant and hence $\mathcal{U}(\mathcal{P}) \neq \emptyset$). The *$\mathcal{P}$-Herbrand expansion* $E(\varphi)$ of a $\mathcal{P}$-formula φ consists of all formulas obtained by applying the following two steps repeatedly, starting with φ, until no quantifiers remain:

I Replace $\psi[(\forall x)\chi(x,y)]$ where χ is quantifier-free with $\psi[\bigwedge_{t\in H}\chi(t,y)]$ for some finite $H \subseteq \mathcal{U}(\mathcal{P})$.

II Replace $\psi[(\exists x)\chi(x,y)]$ where χ is quantifier-free with $\psi[\bigvee_{t\in H}\chi(t,y)]$ for some finite $H \subseteq \mathcal{U}(\mathcal{P})$.

Notice that if φ is a sentence, then so are all formulas in $E(\varphi)$. We are now able to establish Herbrand theorems for the left and right sides of the consequence relation, obtaining an equivalence for the left side.

Theorem 3 (Herbrand Left). *The following are equivalent:*

(1) $\mathbb{K}$ *is finitary.*

(2) *For every g-universal theory* $T \cup \{\varphi\}$ *and g-existential* $\mathcal{P}$*-formula* χ*:*

$$T \cup \{\varphi\} \models_{\mathbb{K}} \chi \quad \textit{iff} \quad \textit{there exists } \varphi' \in E(\varphi) \textit{ such that } T \cup \{\varphi'\} \models_{\mathbb{K}} \chi.$$

Theorem 4 (Herbrand Right). *If* $\mathbb{K}$ *is finitary, then for every g-universal* $\mathcal{P}$*-theory* T *and g-existential* $\mathcal{P}$*-formula* ψ*:*

$$T \models_{\mathbb{K}} \psi \quad \textit{iff} \quad \textit{there exists } \psi' \in E(\psi) \textit{ such that } T \models_{\mathbb{K}} \psi'.$$

We show finally that finitarity and the Herbrand theorems fail for any logic admitting quantifier shifts that is defined by a class $\mathbb{K}$ with arbitrarily large chains; thus disproving e.g. for $\mathbb{K}$ consisting of the standard Łukasiewicz algebra $[0,1]_{Ł}$.

Proposition 1. *Suppose that:*

(a) $\{(\forall x)\varphi \to \psi\} \models_{\mathbb{K}} (\exists x)(\varphi \to \psi)$ *where* x *is not free in* ψ*.*

(b) *For each* $n \in \mathbb{N}$*, there is* $A \in \mathbb{K}$ *such that* $|A| \geq n$*.*

Then $\mathbb{K}$ *is not finitary and* $\models_{\mathbb{K}}$ *does not admit the left or right Herbrand theorem.*

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An approach to first-order quantum computational semantics

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Quantum computational logics are new forms of quantum logic, suggested by the theory of quantum computation. In these logics sentences are supposed to denote pieces of quantum information: *quregisters* or *mixtures of quregisters* that may by stored by quantum objects, while the logical connectives are interpreted as *quantum operations* that transform pieces of quantum information in a reversible way. In this paper we investigate the possibility of extending the semantic characterizations of sentential quantum computational logics to the case of first-order languages, discussing in particular the following questions:

1) How to interpret the logical quantifiers as special examples of quantum operations?
2) What might be the role of *universes of discourse* in the quantum case, where the concept of *individual* is highly problematic?

1 The mathematical environment

Any piece of quantum information is supposed to live in a Hilbert space $\mathcal{H}^{(n)} = \otimes^n \mathbb{C}^2$ (the n-fold tensor product of the space $\mathbb{C}^2$). *Quregisters* (representing possible pure states of quantum objects) are unit vectors $|\psi\rangle$ of a space $\mathcal{H}^{(n)}$, while mixtures of quregisters (briefly called *qumixes*) are density operators ρ of a space $\mathcal{H}^{(n)}$. Of course, any quregister $|\psi\rangle$ corresponds to a special case of a density operator (represented by the projection operator over the subspace determined by $|\psi\rangle$). In any space $\mathcal{H}^{(n)}$ the elements $|x_1,\ldots,x_n\rangle$ (with $x_i \in \{0,1\}$) of the canonical orthonormal basis represent the *classical registers*. A register $|x_1,\ldots,x_n\rangle$ is called *true* (*false*) iff $x_n = 1$ ($x_n = 0$). The *truth-property* (the *falsity-property*) of $\mathcal{H}^{(n)}$ is identified with the projection operator $P_1^{(n)}$ ($P_0^{(n)}$) that projects over the closed subspace spanned by the set of all true registers (false registers) of $\mathcal{H}^{(n)}$. Recalling the Born rule, one can define, for any qumix ρ of $\mathcal{H}^{(n)}$, the *probability that the information stored by* ρ *is true*: $\mathtt{p}(\rho) := \mathrm{Tr}(P_1^{(n)}\rho)$ (where

Tr is the trace-functional). On this basis, the set $\mathfrak{D}$ of all qumixes can be pre-ordered by the following relation:

$$\rho \preceq \sigma \text{ iff } \mathtt{p}(\rho) \leq \mathtt{p}(\sigma).$$

Quantum information is processed by *quantum logical gates*: unitary operators defined on a space $\mathcal{H}^{(n)}$. We consider three gates that have a special logical interest: the *negation*, the *Toffoli-gate* and the *Hadamard-gate*.

In any space $\mathcal{H}^{(n)}$ the negation $\mathtt{NOT}^{(n)}$ is the unitary operator such that:

$$\mathtt{NOT}^{(n)}(|x_1,\ldots,x_n\rangle) := |x_1,\ldots,x_{n-1},1-x_n\rangle.$$

The Hadamard-gate $\sqrt{\mathtt{I}}^{(n)}$ is the unitary operator such that:

$$\sqrt{\mathtt{I}}^{(n)}(|x_1,\ldots,x_n\rangle) := |x_1,\ldots,x_{n-1}\rangle \otimes \frac{1}{\sqrt{2}}((-1)^{x_n}|x_n\rangle + |1-x_n\rangle).$$

In any space $\mathcal{H}^{(m+n+p)}$ the Toffoli-gate $\mathtt{T}^{(m,n,p)}$ is the unitary operator such that:

$$\mathtt{T}^{(m,n,p)}(|x_1,\ldots,x_m,y_1,\ldots,y_n,z_1,\ldots z_p\rangle) :=$$
$$|x_1,\ldots,x_m,y_1,\ldots,y_n,x_m,z_1,\ldots z_{(p-1)},x_m \cdot y_n \oplus z_p\rangle$$

(where $\oplus$ is the sum modulo 2).

Any gate $G^{(n)}$ defined on $\mathcal{H}^{(n)}$ can be canonically extended to a *unitary quantum operation* ${}^{\mathfrak{D}}G^{(n)}$ defined on the set $\mathfrak{D}(\mathcal{H}^{(n)})$ of all qumixes of $\mathcal{H}^{(n)}$:

$${}^{\mathfrak{D}}G^{(n)}(\rho) := G^{(n)}\rho G^{(n)\dagger},$$

where $G^{(n)\dagger}$ is the adjoint of $G^{(n)}$.

The Toffoli-gate allows us to define a *holistic reversible conjunction* $\mathtt{AND}^{(m,n)}$:

$$\mathtt{AND}^{(m,n)}(\rho) := {}^{\mathfrak{D}}\mathtt{T}^{(m,n,1)}(\rho \otimes P_0^{(1)}),$$

for any qumix ρ of $\mathcal{H}^{(m+n)}$.

2 A first-order quantum computational language $\mathcal{L}$

The alphabet of $\mathcal{L}$ contains:

1) sentential constants, including two privileged sentences **t** and **f** that represent the truth-values *Truth* and *Falsity*, respectively;
2) individual names and individual variables;
3) *m*-ary predicates;
4) the following quantum computational connectives: the negation $\neg$ (which corresponds to the gate *Negation*), the square root of the identity $\sqrt{id}$ (which corresponds to the *Hadamard*-gate), a ternary connective $\intercal$ (which corresponds to the *Toffoli*-gate);
5) the universal quantifier $\forall$.

Recalling the definition of $\mathtt{AND}^{(m,n)}$ in terms of the Toffoli-gate, a binary conjunction $\wedge$ can be metalinguistically defined as follows: $\alpha \wedge \beta := \mathsf{T}(\alpha, \beta, \mathbf{f})$ (where $\mathbf{f}$ plays the role of an *ancilla*). The inclusive disjunction $\vee$ and the existential quantifier $\exists$ are supposed to be defined via de Morgan-law.

Any formula α can be naturally decomposed into its parts, giving rise to a special configuration called the *syntactical tree* of α (indicated by STree^{α}). Roughly, STree^{α} can be represented as a finite sequence of *levels*, where:

1) each $Level_i^{\alpha}$ is a sequence of subformulas of α;
2) the *bottom level* $Level_1^{\alpha}$ is (α);
3) the *top level* $Level_h^{\alpha}$ is the sequence of the atomic subformulas occurring in α;
4) for any i (with $1 \leq i < h$), $Level_{i+1}^{\alpha}$ is the sequence obtained by dropping the *principal connective* and the *principal quantifier* in all molecular formulas occurring at $Level_i^{\alpha}$, and by repeating all the atomic formulas that occur at $Level_i^{\alpha}$. For instance, the syntactical tree of the sentence $\alpha = P^1 a \wedge \neg P^2 ba = \mathsf{T}(P^1 a, \neg P^2 ba, \mathbf{f})$ is:

$$(\mathsf{T}(P^1 a, \neg P^2 ba, \mathbf{f})),\ (P^1 a, \neg P^2 ba, \mathbf{f}),\ (P^1 a, P^2 ba, \mathbf{f}).$$

3 A first-order holistic quantum computational semantics

The characteristic holistic features of the quantum-theoretic formalism (arising, for instance, in the case of entanglement phenomena) can be used as a "semantic resource". The basic intuitive idea can be sketched as follows. Any *model* $\mathtt{Hol}$ of $\mathcal{L}$ assigns to any formula α a global informational meaning $\mathtt{Hol}(\alpha)$, represented by a qumix ρ living in a Hilbert space $\mathcal{H}^{\alpha}$ (called the *semantic space* of α) that depends on the linguistic complexity of α. This meaning determines the *contextual meanings* of the subexpressions of α, and cannot be generally reconstructed as a function of the contextual meanings of the parts of α. Furthermore, a model $\mathtt{Hol}$ may assign different contextual meanings to different occurrences of α in different formulas.

Let us first refer to the quantifier-free sublanguage $\mathcal{L}^{-}$ of $\mathcal{L}$. Consider again the sentence $\alpha = \mathsf{T}(P^1 a, \neg P^2 ba, \mathbf{f})$. The choice of the semantic space $\mathcal{H}^{\alpha}$ depends on the non-logical constants of α. We assume that in order to store the information expressed by $P^1 a$ we need three qumixes of $\mathbb{C}^2$, representing respectively the meaning of P^1, the meaning of a and the truth-degree according to which the individual concept corresponding to a satisfies the property corresponding to P^1. In a similar way, in the case of $P^2 ba$, we need four qumixes, while for the sentential constant $\mathbf{f}$ one qumix will be sufficient. The number-sequence $(3,4,1)$ represents the *atomic complexity* of α. Accordingly $\mathcal{H}^{\alpha}$ is identified with the space $\mathcal{H}^{(3)} \otimes \mathcal{H}^{(4)} \otimes \mathcal{H}^{(1)}$.

The syntactical tree of any formula α uniquely determines a sequence of gates (all defined on $\mathcal{H}^{\alpha}$), called the *qumix tree* of α. For instance, in the case of $\alpha = \mathsf{T}(P^1 a, \neg P^2 ba, \mathbf{f})$, the qumix tree of α is the gate-sequence

$$(\mathtt{I}^{(3)} \otimes {}^{\mathfrak{D}}\mathtt{NOT}^{(4)} \otimes \mathtt{I}^{(1)},\ {}^{\mathfrak{D}}\mathtt{T}^{(8)})$$

(where $\mathtt{I}^{(3)}$ is the identity operator of $\mathcal{H}^{(3)}$).

By *holistic map* of $\mathcal{L}^-$ we mean a map that assigns to each level of the syntactical tree of any formula α a *global meaning* represented by a qumix of $\mathcal{H}^\alpha$. On this basis, any occurrence β_{i_j} of a subformula β (at the j-th position of the i-th level of the syntactical tree of α) receives a *contextual meaning*, indicated by $\mathtt{Hol}^\alpha(\beta_{i_j})$. We put:

$$\mathtt{Hol}(\alpha) := \mathtt{Hol}(Level_1^\alpha) = \mathtt{Hol}^\alpha(\alpha).$$

A *holistic* model of $\mathcal{L}^-$ is a holistic map that satisfies the following conditions:

1) for any α and for each $Level_i^\alpha$, different from the top level:

$$\mathtt{Hol}(Level_i^\alpha) = {}^{\mathfrak{D}}G_i(\mathtt{Hol}(Level_{i+1}^\alpha)),$$

   where ${}^{\mathfrak{D}}G_i$ is the i-th element of the qumix tree of α;
2) the contextual meanings of the sentences **t** and **f** are always the truth $P_1^{(1)}$ and the falsity $P_0^{(1)}$, respectively;
3) different occurrences of a subformula β in the syntactical tree of α receive the same contextual meaning (in the context $\mathtt{Hol}(\alpha)$).

The concepts of *truth* with respect to a model and of *logical consequence* are defined as follows:

$$\models_{\mathtt{Hol}} \alpha \text{ iff } \mathtt{p}(\mathtt{Hol}(\alpha)) = 1.$$

$\alpha \models \beta$ iff for any formula γ such that α and β are subformulas of γ and for any model $\mathtt{Hol}$, $\mathtt{Hol}^\gamma(\alpha) \preceq \mathtt{Hol}^\gamma(\beta)$.

The logic characterized by $\models$ is a weak form of *holistic quantum computational logic* where conjunction and disjunction violate idempotency, commutativity, associativity and distributivity.

How to extend this semantics to the full first-order language $\mathcal{L}$? Can $\forall$ correspond to a unitary quantum operation ${}^\forall\mathbf{Q}$? A reasonable condition that should be required seems to be the following: ${}^\forall\mathbf{Q}\rho \preceq \rho$. One is dealing with a condition that also characterizes *knowledge operations* $\mathbf{K}$ in a Hilbert-space environment, where $\mathbf{K}\rho$ is interpreted as "the information ρ is known". For convenient ρ (which belong to the so called *epistemic domain* of $\mathbf{K}$) we have: $\mathbf{K}\rho \preceq \rho$ (in other words, knowing ρ implies ρ). One can prove that non-trivial knowledge operations cannot be represented as unitary quantum operations. At the same time, they can be described as *quantum channels*, representing particular cases of quantum operations that are generally irreversible. It seems reasonable to assume that also the universal quantifier $\forall$ can be interpreted as a special example of a quantum channel. In fact, the use of $\forall$ seems to imply an irreversible step (a kind of theoretic "jump"), as happens in the case of quantum measurements. On this basis one can define a suitable notion of *holistic model* both for $\mathcal{L}$ and for an epistemic first-order language $\mathcal{L}^{Ep}$, whose alphabet contains epistemic operators (like *understanding* and *knowing*).

Unlike most first-order semantic approaches, the models of $\mathcal{L}$ and of $\mathcal{L}^{Ep}$ do not refer to any *domain of individuals* dealt with as a closed set (in a classical sense). The interpretation of a universal formula does not require "ideal tests" that should be performed on *all* elements of a hypothetical domain (which might be infinite). This way of thinking seems to be in agreement with a number of concrete semantic phenomena, where the individual-domain appears highly indeterminate and somehow evanescent; such situations, however, do not generally prevent a correct use of the universal quantifier.

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A fuzzy rough set model based on implicators and conjunctors

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In 1982, Pawlak [1] introduced rough set theory, where uncertain objects are approximated with respect to an equivalence relation representing indiscernibility. More formally, given a *Pawlak approximation space* (U,R), where U is a non-empty set (domain) and R is an equivalence relation, the *rough approximation* of a crisp subset A of U by R is the pair of sets $(R\downarrow A, R\uparrow A)$ in U defined by:

$$x \in R\downarrow A \Leftrightarrow (\forall y \in U)((y,x) \in R \Rightarrow y \in A), \quad (1)$$

$$x \in R\uparrow A \Leftrightarrow (\exists y \in U)((y,x) \in R \wedge y \in A). \quad (2)$$

A pair (A_1,A_2) of sets in U is called a *rough set* in (U,R) if there is a set A in U such that $A_1 = R\downarrow A$ and $A_2 = R\uparrow A$. The connection between rough sets and modal logic was already stated in [2].

In this talk, we discuss a general fuzzy rough set model, based on implicators and conjunctors. We show that this model covers many fuzzy rough set models studied in literature. Furthermore, we discuss an axiomatic approach to the model and explain how it is related to fuzzy modal logic.

The basic idea is to extend the rough set theory of Pawlak to the fuzzy setting, where we want to approximate a fuzzy set A with respect to a binary fuzzy relation R. This can be done by replacing the universal and existential quantifiers by the infimum and supremum operators and by using fuzzy implicators and conjunctors instead of the Boolean implication and conjunction. Namely, a *fuzzy approximation space* is just a pair (U,R) where U is a non-empty set as before but now R is a binary fuzzy relation R. Furthermore, let $\mathcal{I}$ be an implicator and $\mathcal{C}$ a conjunctor.

Definition 1. *The* $(\mathcal{I},\mathcal{C})$-fuzzy rough approximation *of a fuzzy set A in U by R is the pair of fuzzy sets* $(R\downarrow_{\mathcal{I}} A, R\uparrow_{\mathcal{C}} A)$ *defined by, for x in U,*

$$(R\downarrow_{\mathcal{I}} A)(x) = \inf_{y\in U} \mathcal{I}(R(y,x),A(y)), \quad (3)$$

$$(R\uparrow_{\mathcal{C}} A)(x) = \sup_{y\in U} \mathcal{C}(R(y,x),A(y)). \quad (4)$$

A pair (A_1, A_2) of fuzzy sets in U is called a *fuzzy rough set* in (U, R) if there is a fuzzy set A in U such that $A_1 = R \downarrow_{\mathcal{I}} A$ and $A_2 = R \uparrow_{\mathcal{C}} A$.

If the couple $(\mathcal{I}, \mathcal{C})$ consists of a left-continuous t-norm and its R-implicator, this definition coincides with the $\mathcal{T}$-modal operators $[R]_{\mathcal{T}}$ and $\langle R \rangle_{\mathcal{T}}$ defined in [3].

Table 1 succinctly describes the most important fuzzy rough set models proposed in the literature that can be seen as special cases of the above general model.

Model	Conjunctor	Implicator	Relation
[4, 5] Dubois & Prade, 1990	min	Kleene-Dienes	min-similarity
[6] Morsi & Yakout, 1998	left-cont. t-norm	R-implicator	$\mathcal{T}$-similarity
[7] Boixander et al., 2000	cont. t-norm	R-implicator	$\mathcal{T}$-similarity
[8] Radzikowska & Kerre, 2002	t-norm	border implicator	min-similarity
[9, 10] Wu et al., 2003	min	S-implicator	general
[11] Mi & Zhang, 2004	conjunctor	R-implicator	general
[12] Pei, 2005 and [13] Liu, 2008	min	S-implicator	general
[14] Wu et al., 2005	cont. t-norm	implicator	general
[15] Yeung et al., 2005	left-cont. t-norm	S-implicator	general
[15] Yeung et al., 2005	conjunctor	R-implicator	general
[16] De Cock et al., 2007	t-norm	border implicator	general
[17] Mi et al., 2008	cont. t-norm	S-implicator	general
[18, 19] Hu et al., 2010	left-cont. t-norm	S-implicator	$\mathcal{T}_{\cos}$-similarity
[18, 19] Hu et al., 2010	conjunctor	R-implicator	$\mathcal{T}_{\cos}$-similarity

Table 1. Overview of fuzzy rough set models in the literature.

A very interesting problem is to study these fuzzy rough set models from an axiomatic point of view, so as to get more insight in their logical structure. We will be working with unary operators on the set $\mathcal{F}(U)$ of fuzzy subsets of U. We use axioms on the operators to obtain a fuzzy relation R such that the operators behave as approximation operators with respect to R.

Our starting point is the axiomatic approach developed by Wu et al. [14]. Other papers that describe an axiomatic approach are [6, 9–12, 15, 13, 20].

In the following we denote by $\hat{\alpha}$ the constant fuzzy set of the value $\alpha \in [0, 1]$, i.e., $\forall x \in U$, $\hat{\alpha}(x) = \alpha$.

Definition 2. *Let H, L: $\mathcal{F}(U) \to \mathcal{F}(U)$, $\mathcal{C}$ a conjunctor and $\mathcal{I}$ an implicator. We call H a $\mathcal{C}$-*upper approximation *if it satisfies, for all $A, A_j \in \mathcal{F}(U)$, $\alpha \in [0, 1]$,*

$$(H1)\ \forall A \in \mathcal{F}(U), \forall \alpha \in I : H(\hat{\alpha} \cap_{\mathcal{C}} A) = \hat{\alpha} \cap_{\mathcal{C}} H(A),$$

$$(H2)\ \forall A_j \in \mathcal{F}(U), j \in J : H\left(\bigcup_{j \in J} A_j\right) = \bigcup_{j \in J} H(A_j).$$

*We call L an I-*lower approximation *if it satisfies, for all $A, A_j \in \mathcal{F}(U)$, $\alpha \in [0,1]$,*

$$(L1)\ \forall A \in \mathcal{F}(U), \forall \alpha \in I : L(\hat{\alpha} \Rightarrow_I A) = \hat{\alpha} \Rightarrow_I L(A),$$

$$(L2)\ \forall A_j \in \mathcal{F}(U), j \in J : L\left(\bigcap_{j\in J} A_j\right) = \bigcap_{j\in J} L(A_j).$$

Wu et al. [14] required $\mathcal{C}$ and I to be a continuous t-norm and implicator, resp., but these conditions can be slightly weakened. For this, we can use e.g. results from [3] obtained in the framework of fuzzy modal logics that can be easily adapted to approximation operators. For instance, one can show the following characterizations:

Proposition 1. *Let $H\colon \mathcal{F}(U) \to \mathcal{F}(U)$ and $\mathcal{T}$ a left-continuous t-norm. H is a $\mathcal{T}$-upper approximation if and only if for all $A \in \mathcal{F}(U)$, $H(A) = R \uparrow_{\mathcal{T}} A$, where $R(x,y) = H(\{x\})(y)$, for x, y in U. We denote H by $H^R_{\mathcal{T}}$.*

Recall that I is a *EP* implicator if it satisfies the exchange principle

$$\forall a,b,c \in [0,1]\colon\ I(a, I(b,c)) = I(b, I(a,c)).$$

Proposition 2. *Let $L\colon \mathcal{F}(U) \to \mathcal{F}(U)$ and I an EP implicator that is left-continuous in its first argument such that $\mathcal{N}_I$ is continuous. L is an I-lower approximation if and only if for all $A \in \mathcal{F}(U)$, $L(A) = R \downarrow_I A$, where $R(x,y) = L(U \setminus \{x\})(y)$, for x, y in U. We denote L by L^R_I.*

Adding more axioms to Definition 2, it will be shown that one can characterize specific properties of the fuzzy relation R (e.g. seriality, reflexivity, $\mathcal{T}$-transitivity, etc.), similarly as has been done in the realm of (fuzzy) modal logics.

The above propositions characterize lower and upper approximations separately. If these operators are dual, we can link them together.

Proposition 3. *Let $\mathcal{T}$ be a left-continuous t-norm, I an EP implicator that is left-continuous in its first argument such that $\mathcal{N}_I$ is involutive, H a $\mathcal{T}$-upper approximation and L an I-lower approximation. If H and L satisfy duality w.r.t. $\mathcal{N}_I$, then there exists a binary fuzzy relation R in U such that $H = H^R_{\mathcal{T}}$ and $L = L^R_I$.*

A drawback of the above approach is that it excludes some important operators. For instance, it can be verified that the R-implicator $I_{\min}$ does not satisfy the conditions of Proposition 2, because $\mathcal{N}_{I_{\min}}$ is not involutive. For this reason, below we introduce and characterize the alternative notion of a $\mathcal{T}$-coupled pair of approximations.

Definition 3. *Let $\mathcal{T}$ be a left-continuous t-norm, $H, L : \mathcal{F}(U) \to \mathcal{F}(U)$. We call (H,L) a $\mathcal{T}$-*coupled pair *of upper and lower approximations if the following conditions hold:*

(H1,H2) H is a $\mathcal{T}$-upper fuzzy approximation operator,

(L2) $L\left(\bigcap_{j\in J} A_j\right) = \bigcap_{j\in J} L(A_j)$,

(HL) $L(A \Rightarrow_{I_{\mathcal{T}}} \hat{\alpha}) = H(A) \Rightarrow_{I_{\mathcal{T}}} \hat{\alpha}$.

One can show that these properties actually characterize the fuzzy rough set model determined by a left-continuous t-norm and its residuum.

Proposition 4. *Let $\mathcal{T}$ be a left-continuous t-norm, $H, L : \mathcal{F}(U) \to \mathcal{F}(U)$. (H, L) is a $\mathcal{T}$-coupled pair of upper and lower approximations if and only if there exists a binary fuzzy relation R in U such that $H = H^R_{\mathcal{T}}$ and $L = L^R_{I_{\mathcal{T}}}$.*

Future work will include studying possible variations of the implicator-conjunctor model. For example, the model proposed in [21] by Inuiguchi: for x in U, the lower approximation of A by R is given by

$$(R \downarrow_I A)(x) = \min(A(x), \inf_{y \in U} I(R(y,x), A(y)))$$

and the upper approximation of A by R is given by

$$(R \uparrow_{\mathcal{C}} A)(x) = \max(A(x), \sup_{y \in U} \mathcal{C}(R(y,x), A(y))).$$

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Lattice-valued institutions

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1 Institutions: Introduction and Definition

Institutions as such were introduced by J. A. Goguen and R. M. Burstall in [1]; they had already introduced similar ideas in [2] under the term "language". As noted in [1], there were in the late 1970s and early 1980s many logical systems being introduced in computer science. In most cases, each system was designed, developed, and used independently of the others. Institutions were introduced so that these systems could be uniformly studied and so that results and applications of one system could, if appropriate, be used for or by other systems.

We show that the category of topological systems (more exactly, the dual category) may be interpreted as an institution. We define lattice-valued institutions and show that the category of lattice-valued topological systems, for a fixed L, may be interpreted as a lattice-valued institution.

Definition 1. *An* institution *is an 4-tuple* $(\mathbf{Sign}, sen, Mod, \models)$*, where*

- $\mathbf{Sign}$ *is a category;*
- *sen is a functor* $sen : \mathbf{Sign} \to \mathbf{Set}$*;*
- *Mod is a functor* $Mod : \mathbf{Sign}^{op} \to \mathbf{CAT}$*, where* $\mathbf{CAT}$ *is the quasicategory of "large" categories; and*
- *for each* $\Sigma \in |\mathbf{Sign}|$, $\models$ *determines a relation* $\models_\Sigma \subseteq |Mod(\Sigma)| \times sen(\Sigma)$*;*

such that the following "adjointness condition" is satisfied: for all $\sigma : \Sigma \to \Sigma'$ *in* $\mathbf{Sign}$*, and for each* $M' \in |Mod(\Sigma')|$ *and* $\varphi \in sen(\Sigma)$*:*

$$M' \models_{\Sigma'} sen(\sigma)(\varphi) \text{ iff } Mod(\sigma)(M') \models_\Sigma \varphi.$$

Terminology 2 *With notation in Definition 1,*

- $\mathbf{Sign}$ *is the category of signatures with signature morphisms;*

– *$sen(\Sigma)$ is the set of sentences for Σ, and $sen(\sigma)$ translates the sentences from $sen(\Sigma)$ to $sen(\Sigma')$;*
– *$Mod(\Sigma)$ is the category of models with signature Σ;*
– *$Mod(\sigma) : Mod(\Sigma') \to Mod(\Sigma)$ is a reduct functor; and*
– *for each Σ, $\models_\Sigma$ is a satisfaction relation.*

Given Goguen and Burstall's motivation, it is not surprising that first-order logic is an institution. However, the setting is sufficiently general to allow the construction of an institution based on the category **TopSys** of topological systems.

2 TopSys as an Institution

Example 1. We construct an institution based on **TopSys** [5] as follows:

– $\mathbf{Sign} = \mathbf{TopSys}^{op}$;
– $sen(X, A, \models) = A$;
– Given $\sigma^{op} = f : (X, A, \models) \to (X', A', \models)$; $sen(\sigma) = \Omega f$, where $f = (\mathrm{pt} f, \Omega f)$ with $\mathrm{pt} f : X \to X'$ a set function and $\Omega f : A' \to A$ a frame morphism;
– $Mod(X, A, \models) = X$ (considered as a discrete category);
– Given $\sigma^{op} = f$ in $\mathbf{Sign}^{op} = \mathbf{TopSys}$, $Mod(\sigma) = \mathrm{pt} f$; and
– $x \models a$ in the institution iff $x \models a$ in the appropriate topological system.

Since composition and identities in **TopSys** are taken componentwise, then *sen* and *Mod* are functors. That "everything goes in the right direction" requires only routine, but careful, checking.

The only thing left to check is the adjointness condition, which follows from the adjointness condition in the definition of a continuous function between topological systems. This is as follows:

$$x \models_\Sigma sen(\sigma)(a')$$
$$\textit{iff } x \models \Omega f(a') \text{ in the domain topological system}$$
$$\textit{iff } \mathrm{pt} f(x) \models a' \text{ in the codomain topological system}$$
$$\textit{iff } Mod(\sigma)(x) \models_{\Sigma'} a'.$$

3 Lattice-valued Institutions – Fixed-basis Case

We introduce lattice-valued institutions. Let L be a fixed complete lattice.

Definition 3. *An L-*institution *is an 4-tuple* $(\mathbf{Sign}, sen, Mod, \models)$*, where*

– **Sign** *is a category;*
– *sen is a functor* $sen : \mathbf{Sign} \to \mathbf{Set}$*;*
– *Mod is a functor* $Mod : \mathbf{Sign}^{op} \to \mathbf{CAT}$*, where* **CAT** *is the quasicategory of "large" categories; and*
– *for each $\Sigma \in |\mathbf{Sign}|$, $\models$ determines an L-valued relation $\models_\Sigma$: $|Mod(\Sigma)| \times sen(\Sigma) \to L$*

such that the following "adjointness condition" is satisfied: for all $\sigma : \Sigma \to \Sigma'$ in **Sign***, and for each $M' \in |Mod(\Sigma')|$ and $\varphi \in sen(\Sigma)$:*

$$\models_{\Sigma'} (M', sen(\sigma)(\varphi)) = \models_\Sigma (Mod(\sigma)(M'), \varphi).$$

4 L-TopSys as an L-institution

In this section, we assume that L is a fixed frame. For the definition of an L-**TopSys** and examples thereof, please see [3, 4].

Example 2. We construct an L-institution based on L-**TopSys** as follows:

– **Sign** $= (L\text{-}\mathbf{TopSys})^{op}$
– $sen(X, A, \models) = A$
– Given $\sigma^{op} = f : (X, A, \models) \to (X', A', \models)$, $sen(\sigma) = \Omega f$
– $Mod(X, A, \models) = X$ (considered as a discrete category)
– Given $\sigma^{op} = f$ in $\mathbf{Sign}^{op} = \mathbf{TopSys}$, $Mod(\sigma) = \mathrm{pt}f$
– $\models_\Sigma(x, a)$ in the institution is equal to $\models(x, a)$ in the appropriate L-topological system.

As in the crisp **TopSys** example, *sen* and *Mod* are functors since composition and identities in L-**TopSys** are taken componentwise. That "everything goes in the right direction" is essentially the same as for the crisp case.

Again, the "adjointness condition" follows from the definition of continuous function between L-topological systems. The proof is given by the following chain of equalities:

$$\models_\Sigma (x, sen(\sigma)(a'))$$

$$= \ \models (x, \Omega f(a')) \text{ (evaluated in the domain } L\text{-topological system)}$$

$$= \ \models (\mathrm{pt}f(x), a') \text{ (evaluated in the codomain } L\text{-topological system)}$$

$$= \ \models_{\Sigma'} (Mod(\sigma)(x), a').$$

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Nilpotent operator systems

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1 Nilpotent operator systems

Here, we show that a consistent logical system (the DeMorgan identity, the law of contradiction and the law of excluded middle all hold) represented by nilpotent operators is not necessarily isomorphic to Łukasiewicz-logic, which means that nilpotent logical systems are broader than we have thought earlier. Using more than one generator function, we examine three naturally derived negation operators in these systems. It is shown that the coincidence of the three negation operators leads back to a system that is isomorphic to Łukasiewicz-logic (which will be referred as a Łukasiewicz-system). Consistent nilpotent logical structures with three different negation operators are also provided (which will be referred as a bounded system). We will describe the structure of the bounded system and its properties, then give some examples.

2 Implications in Bounded Systems

Both R- and S-implications with respect to the three naturally derived negation operators of the bounded system are studied. It is shown that these implications never coincide in a bounded system. The condition of coincidence is equivalent to the three negation operators coinciding, which would lead to a Łukasiewicz system. The formulae and the basic properties of implications are given, where two of them fulfil all the basic properties generally required for implications.

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Toward a unified view of logics of incomplete and conflicting information

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We consider a simplified epistemic logic MEL, whose syntax is a fragment of the modal logic KD where an agent can express both beliefs and ignorance statements about propositional formulas. It is in fact a standard propositional language embedded into another one, whose role is to express beliefs about propositions of the former. Its semantics can be expressed in terms of subsets of interpretations of the inner propositional language, and does not explicitly use accessibility relations. A fragment of MEL is enough to capture three-valued logics of Łukasiewicz and Kleene as well as three-valued paraconsistent logics such as the Logic of Paradox by Priest, and also RM3. We also consider two extensions of MEL:

– The graded version of this epistemic logic generalizes possibilistic logic, and its semantics is in terms of sets of possibility distributions. It is a minimal logical setting for reasoning with Boolean formulas annotated with lower bounds of necessity or possibility degrees. We show the completeness of this logic w.r.t. this possibilistic semantics. This approach is general enough to capture answer-set programming.
– We may consider dropping axioms K and D and move to the MEL fragment of non-regular modal logics. This is the natural setting for encoding more general logics based on qualitative capacities viewed as imprecise possibilities, whereby the epistemic states of several agents are simultaneously handled. It has close connections to paraconsistent logics and to Belnap four-valued logics.

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Logics of graded consequence and a connection with decision support system

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1 Introduction

What do we understand by 'graded logical approaches'? Is it a logical system which endorses the idea that grades, other than the top and the least of a typical lattice structure, to be assigned to its sets of formulae, or to the formulae and the reasoning mechanism as well? The approaches, dealing with the former point of view, are usually known as many-valued logics, and/or fuzzy logics. There are some subtle distinctions between these two kinds of logics. Zadeh has differentiated fuzzy logic from many-valued logics in the following sense [10]. "*...fuzzy logic, FLn, is a logical system which aims at a formalization of approximate reasoning. In this sense, FLn is an extension of multivalued logic*" With the publication of the paper viz., 'The logic of inexact concepts' of Goguen [8] in the year 1968-69, fuzzy logic emerged as a discipline in logic. In 1975 based on the theory of fuzzy sets Zadeh proposed the idea of approximate reasoning [16] as a prototype of human reasoning. The word 'Fuzzy logic', then, was being used in a broad sense. Gradually, with the work in [6, 9–11, 13] the idea of fuzzy logic started to get a shape in a more strict sense where the use of fuzzy set theory alone does not determine the realm of fuzzy logic. Later, this branch of mathematical logic based on fuzzy set theory became familiar in the name of FLn, fuzzy logic in the narrow sense [10]. In Hájek's [10] term fuzzy logic is a system, endowed with the ability of *deriving partially true (graded) conclusion from partially true (graded) premises*. It is to be noted that *derivation* is not a graded concept here. So, what do fuzzy logicians mean by the term approximate reasoning or fuzzy/many-valued rule of inference? As pointed out by Pelta [14], there is no notion of multivalence in the concept of 'inferencing':"*Until now the construction of superficial many-valued logics, that is, logics with an arbitrary number (bigger than two) of truth values but always incorporating a binary consequence relation, has prevailed in investigations of logical many-valuedness.*"

It seems human brain does not always *derive* conclusions, certain to some degree, from a set of information, which are also certain to some degree, with full certainty. The prevalent prescriptions of logics do not handle that uncertainty of 'deriving' properly. The same concern was echoed in the lines of Parikh [12], where he mentioned "*...we seem to have come no closer to observationality by moving from two valued logic to real valued, fuzzy logic. A possible solution ... is to use continuous valued logic not only for the object language but also for the metalanguage*." And, perhaps, Zadeh's extended fuzzy logic [17] also could be counted as an account of the same concern. Let us present the formal theory of graded consequence (GCT), which is in existence from 1987 [1], as a general framework for the metatheory of a logic where *deriving partially true conclusion from a set of partially true premises is also a matter of grade*.

2 Theory of graded consequence

The idea behind the notion of GCT is, given a set X of premises, whose truth/credibility are of matter of grade, and a prospective conclusion α, which is also true/believable to some extent, the process of deriving α from X, denoted by $X \mid\sim \alpha$, could also be a matter of grade. That is, the derivation itself may have some strength. A graded consequence relation [2] is thus, a fuzzy relation ($\mid\sim$) from a set of all sets of formulae ($P(F)$) to the set of all formulae (F), satisfying the following basic conditions.

(GC1) If $\alpha \in X$ then $gr(X \mid\sim \alpha) = 1$ (reflexivity/overlap),

(GC2) if $X \subseteq Y$ then $gr(X \mid\sim \alpha) \leq gr(Y \mid\sim \alpha)$ (monotonicity/dilution),

(GC3) $\inf_{\beta \in Y} gr(X \mid\sim \beta) *_m gr(X \cup Y \mid\sim \alpha) \leq gr(X \mid\sim \alpha)$ (cut),

where $gr(X \mid\sim \alpha)$, the degree to which α follows from X, is an element of a complete residuated lattice $(L, \wedge, \vee, *_m, \rightarrow_m, 0, 1)$. The semantic counterpart of the notion starts with a collection of fuzzy sets of formulae, say $\{T_i\}_{i \in I}$, which may be regarded as the initial context formed by a set of experts assigning values to the object level formulae. So, the value of the metalinguistic sentence 'α is a semantic consequence of X', is obtained by literally computing the value of $\forall_{T_i}\{(X \subseteq T_i) \rightarrow \alpha \in T_i\}$. That is, computing the metalinguistic connective $\rightarrow$ by the operator $\rightarrow_m$, and quantifier $\forall$ by the operator for lattice 'infimum' of L, we have $gr(X \mid\approx_{\{T_i\}_{i \in I}} \alpha) = \inf_{i \in I}\{\inf_{\gamma \in X} T_i(\gamma) \rightarrow_m T_i(\alpha)\}$.

Then in [2] it has been shown that (i) given any $\{T_i\}_{i \in I}$, $\mid\approx_{\{T_i\}_{i \in I}}$ is a graded consequence relation (i.e. satisfies (GC1) to (GC3)), and (ii) given any graded consequence relation $\mid\sim$, there is a collection $\{T_i\}_{i \in I}$ such that $\mid\approx_{\{T_i\}_{i \in I}} = \mid\sim$. These two theorems are known as the representation theorems, which basically bridge a connection between the syntactic and semantic notion of graded consequence. The axiomatic notion of graded consequence also has been developed in [3]. Apart from the notion of consequence, the notion of inconsistency, consistency, and other metalogical notions also are introduced, and their interrelations are studied in [5, 3].

The discussion above gives an idea about the metatheory of GCT. Let us now concentrate on the logic building part based on the metatheory of GCT. As usually a logic does have, a logic of graded consequence too has a language, called object language, containing some or all of the connectives $\neg$, $\supset$, &, $\vee$, and, perhaps, a few more. For the time being the focus is only on the propositional fragment of a language. Once the object language is specified, correspondingly the object level algebraic structure is formed; the set L endowed with the respective operators $\neg_o$, $\rightarrow_o$, $*_o$, $\oplus_o$ for the connectives forms the object level algebraic structure. The availability of rules (of inference) corresponding to each connective is determined by the interrelation between the object and metalevel algebraic structures, may be called L_o and L_m respectively. Thus the scheme for generating different logics with graded notion of consequence is as follows.

A collection $\{T_i\}_{i \in I}$, may be called a set of experts assigning values to the atomic formulae, is considered. Depending on user's choice of object language, presence of different connectives in the object level are assumed. Hence based on the meanings of the connectives, according to the users, the object level algebraic structure L_o is formed. A metalevel algebra $(L, *_m, \rightarrow_m, 0, 1)$ is fixed so that for any set of formulae X and formula α, $gr(X \mid\approx_{\{T_i\}_{i \in I}} \alpha)$ can be calculated. The properties of the object level algebraic structure as well as the metalevel algebraic structure along with their interrelations give shape to a particular logic with graded notion of consequence. This leads towards generating logics based on GCT, and the following table is an initial outcome of this study.

	DT	MP DT_c	Tran	&-I &-R	&-E &-L	$\underline{\vee}$-I $\underline{\vee}$-R	$\underline{\vee}$-E $\underline{\vee}$-L	$\neg$-I $\neg$-R	GC^M5	GC4
$(O_{\text{Gödel}}, M_{\text{Łukasiewicz}})$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	–	$\surd$ $(k = 1)$
$(O_{\text{Łukasiewicz}}, M_{\text{Gödel}})$	$\surd$	$\times$	$\times$	$\times$	$\surd$	$\surd$	$\times$	$\surd$	$\surd$ $(c = \frac{1}{2}$ in $[0, 1])$	$\times$
$(O_{\text{Goguen}}, M_{\text{Gödel}})$	$\surd$	$\times$	$\times$	$\times$	$\surd$	$\surd$	$\times$	$\surd$	–	$\surd$ $(k = 1)$
$(O_{\text{Gödel}}, M_{\text{Goguen}})$	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	–	$\surd$ $(k = 1)$

For any connective #, #-I, #-E, #-R, #-L respectively denote the graded counterparts of the introduction, elimination, right and left rule of the connective. DT, DT_c, MP, Trans are the abbreviations for the graded version of the deduction theorem, its converse, modus ponens and transitivity respectively. It is to be noted that in graded context a general structure of a classical rule, like $X, \alpha \vdash \beta$ implies $Y \vdash \gamma$ would be translated as $gr(X \cup \{\alpha\} \mid\sim \beta) \leq gr(Y \mid\sim \gamma)$. GC4 and GC^M5 [4, 7] are the graded counterpart of the law of explosion and reasoning by cases respectively. GC4 ensures, there is a $k(>0) \in L$ such that $\inf_{\alpha,\beta} gr(\{\alpha, \neg\alpha\} \mid\sim \beta) = k$, and GC^M5 states, there is a $c(>0) \in L$ such that $gr(X \cup \{\alpha\} \mid\sim \beta) *_m gr(X \cup \{\neg\alpha\} \mid\sim \beta) *_m c \leq gr(X \mid\sim \beta)$. The pair of structures, given by $(O_S, M_{S'})$, indicates that the logical base of the object language is the system S, whereas that of the metalevel is S'. It can be shown that the t-norm based many-valued logics can be obtained as a special case of this scheme.

3 Key ideas of GCT vis-à-vis that of a decision support system

In order to give an overview of the suitability of this approach pertaining to real life decision making, let us start with an example. It is often observed that in order to come to a decision in a complex real life situation, the decision maker needs to rely on an initial set of data gathered from a set of experts/users/daily-stake-holders of the subject of concern. These information may be of imprecise, conflicting nature. These experts may put forward their opinion based on their everyday experiences and reasoning. While taking decision, based on this data, the decision maker may incorporate her subjective knowledge and reasoning in a particular context. So, these two levels' of reasoning may not be the same; rather they need to have a meaningful coordination and interaction in between so that both the real life factors, i.e. the users' data, and subjective knowledge base are taken into account while taking decision. Theory of graded consequence (GCT) takes care of these two levels', may be called object and metalevel, of reasoning in its mathematical formalism. This seems to be lacking in most of the existing approaches, including many-valued/fuzzy logics, for dealing with uncertainty [7].

In this presentation we first present the idea of generating different logics of graded consequence, and show that the many-valued logics can be rediscovered following this scheme. Then we would try to exploit this general framework of GCT, which allows to have the flexibility of choosing different logical bases for different layers of decision making, in order to show a good connection with the key ideas of a decision support system [15], typically an interactive system between two agents, an human user and a decision making machine.

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Fuzzy quantifiers of type $\langle 1,1\rangle$ determined by fuzzy measures and integrals – recent results

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We wrote a series of papers [3–5] investigating fuzzy quantifiers of type $\langle 1,1\rangle$ (that is, with two arguments) based on fuzzy measures and integrals analyzed in [2]. These fuzzy quantifiers are intended as models of natural language quantifiers which involve *vagueness*, such as *many*, *few*, *almost all*, etc. We investigated various semantic properties of these fuzzy quantifiers, for example permutation invariance, conservativity, property of extension, etc. These properties are analyzed in detail for the classical case in [10]. Semantic properties of (general) fuzzy quantifiers were studied in [6, 7].

We previously studied fuzzy quantifiers of type $\langle 1\rangle$ (with one argument) [1],[1] which denote important noun phrases of natural language such as something in "Something is broken", everyone in "Everyone likes Bob", and nobody in "Nobody knows everything". Classical logical quantifiers such as "for all" and "there exists" also belong to this type. A natural extension of this research is to study quantifiers of type $\langle 1,1\rangle$ (e.g. every in "Every book has leaves", most in "Most birds fly") that take two arguments. It has been suggested that these type $\langle 1,1\rangle$ quantifiers are the most important from the point of view of natural language semantics [10]. The reason is that two-argument quantifiers are most common in natural language usage. Moreover, they can often be used to express or decompose quantifiers of other types.

In the case of quantifiers with two arguments, it is advantageous to work with a slightly different definition of fuzzy measures and integrals. The first argument of a $\langle 1,1\rangle$ quantifier is called *restriction* and the second is *scope*; for example, in "Every book has leaves", to be a book is the restriction and to have leaves is the scope. It is natural to think of the restriction as a *new universe* for the quantifier (in our example, to determine the truth value, only objects fulfilling the restriction condition are important, i.e., books). Because we are working with fuzzy subsets of some universe M, we should be able to define quantifiers on *fuzzy universes*. Therefore, we introduced a new type of fuzzy measure space defined on algebras of subsets of a *fuzzy* set A and a corresponding fuzzy integral, the so-called $\odot$-fuzzy integral [2].

Why do we think that our $\odot$-fuzzy integral is an appropriate tool for modeling of natural language quantifiers? We previously argued that a possible logical analysis of a sentence such as "Many sportsmen are tall" is as follows [2]: we search for a fuzzy subset of the fuzzy set of sportsmen that is large (i.e., its measure is as great as possible)

[1] The notation type $\langle 1\rangle$ and type $\langle 1,1\rangle$ originated in [9], where quantifiers are understood to be classes of relational structures of a certain type (representing a number of arguments and variable binding). It is widely used in the literature on generalized quantifiers [8, 10].

and for all elements x from its support it holds that if x is a sportsman, then x is tall. This leads to the second-order formula

$$\mathsf{many}(Sp,Ta) := (\exists Y \in \mathcal{F}^{-}_{Sp})(\forall x \in \mathrm{Supp}(Y)(\mu(Y)\&(Sp(x) \Rightarrow Ta(x)))), \qquad (1)$$

where Sp and Ta denote fuzzy sets of sportsmen and tall people, respectively, $\mathcal{F}^{-}_{Sp}$ is the set of all non-empty fuzzy subsets of fuzzy set Sp, $\mathrm{Supp}(Y)$ is the support of fuzzy set Y, and μ denotes a fuzzy measure. The semantic counterpart of this formula is exactly the $\odot$-integral of the fuzzy set $Sp \to Ta$, where $\to$ is the operation of residuum that models the implication. Based on this idea, fuzzy quantifiers are represented using functionals assigning *fuzzy measure spaces* to crisp universal sets and their fuzzy subsets. Using these fuzzy measure spaces, the truth value is assigned (using the $\odot$-fuzzy integral) to a pair of fuzzy sets.

To introduce fuzzy quantifiers of type $\langle 1,1 \rangle$ for modeling of various natural language quantifiers, we need a general means for combining arguments of a fuzzy quantifier. Therefore, we define *residuated lattice operations*. These operations allow us to establish induced operations on fuzzy sets from operations $\{\wedge,\vee,\otimes,\to\}$ of a given residuated lattice. Then we define a fuzzy quantifier of type $\langle 1,1 \rangle$ as a fuzzy integral of combinations of the restriction and scope arguments using a residuated lattice operation (e.g., the residuum $\to$ in the previous paragraph).

Semantic properties under consideration are:

- *Permutation and isomorphism invariances* - these properties hold if quantifiers are invariant with respect to permutations (bijective mappings) on the universe of discourse (permutation invariance) and with respect to bijections between different universes of discourse (isomorphism invariance).
- *Property of extension* - this property expresses the invariance of quantifier values with respect to possible extensions of the universe of discourse.
- *Conservativity* says that quantifiers are in their second argument sensitive only to these objects which lie in the intersection of their arguments.
- *Extensionality* represents a form of smoothness of fuzzy quantifiers.

In the investigation of semantic properties discussed above for fuzzy quantifiers determined by fuzzy measures and integrals, we adhere to the following strategy. Fuzzy quantifiers are defined by means of the pair of functionals $(\mathcal{S},\varphi)$ assigning a fuzzy measure space and an rl-operation for a combination of quantifier arguments, respectively, to any universe M. Hence, we provide a characterization of a semantic property using a corresponding characterization of these functionals. For example, the property of conservativity of fuzzy quantifiers is characterized by the property of *cons-closedness* of the functional and the *conservativity* of φ.

In this contribution we overview and summarize our results and point out these which can be of interest from the point of view of fuzzy logic and/or theory of generalized quantifiers.

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CP- and OCF-networks – a comparison

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Abstract. In modelling reasoning, network approaches are quite common, the best known probably being Bayesian networks in probabilistics. These networks have been transferred to the qualitative framework of ranking functions, resulting in so-called OCF-networks. Also in decision making, network approaches are used. Ceteris parius (CP) networks model a preference between instances of single variables, assuming that all other variables are kept equal. Here, we discuss under which conditions OCF-networks can be used to model the information of CP-networks and vice versa, and whether and how one representation can be transferred to the other.

1 Preliminaries

Let $\mathcal{V} = \{V_1, \ldots, V_n\}$ be a set of propositional atoms and a *literal* a positive or negative atom representing variables in their positive resp. negated form; for a specific, nevertheless undetermined, outcome of V_i, we write $\dot{v}_i \in \{v_i, \overline{v}_i\}$. The set of formulas $\mathfrak{L}$ over $\mathcal{V}$ joined with the symbols for tautology ($\top$) and contradiction ($\bot$), with the connectives $\wedge$ (*and*), $\vee$ (*or*) and $\neg$ (*not*) shall be defined in the usual way. For $A, B \in \mathfrak{L}$, we usually omit the connective $\wedge$ and write AB instead of $A \wedge B$ as well as indicate negation by overlining, that is, $\overline{A}$ means $\neg A$. *Interpretations*, or *possible worlds*, a syntactical representation of interpretations, are also defined in the usual way; the set of all possible worlds is denoted by Ω. For $\mathbf{A} \subseteq \mathcal{V}$ we denote the assignments or instantiations of this subset in $Asst(\mathbf{A})$ and interpret their elements as complete conjunctions over $\mathbf{A}$.

Let $\Gamma = \langle \mathcal{V}, \mathcal{E} \rangle$ be a directed, acyclic graph (DAG) with the propositional variables $\mathcal{V}$ as set of vertices and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We define the *parents* of a vertex V, $\mathrm{pa}(V)$, as the direct predecessors of V, the *descendants* of V, $\mathrm{desc}(V)$, as the set of vertices for which there is a path in Γ from V to this vertex, and the set of *non-descendants* of V is the set of all vertices that are neither the parents nor the descendants of V, nor V itself. For each $\omega \in \Omega$, we indicate by $V(\omega)$ respectively $\mathrm{pa}(V)(\omega)$ the outcome $\dot{v}$ of V with $\omega \models \dot{v}$ resp. the configuration $\dot{p}$ of the variables in $\mathrm{pa}(V)$ with $\omega \models \dot{p}$.

2 Ranking functions and OCF-networks

An *ordinal conditional function* (OCF), also known as *ranking function* is a function that assigns to each world a rank of *disbelief* or *implausibility*, that is, the higher the rank of a world is, the less plausible this world is.

Definition 1 (Ranking function (OCF, [5])). *A ranking function* κ *is a function* $\kappa : \Omega \to \mathbb{N}_0^\infty$ *such that the set* $\{\omega \,|\, \kappa(\omega) = 0\}$ *is not empty, that is, there have to be worlds that are maximally plausible. The rank of a formula* $A \in \mathfrak{L}$ *is defined to be the minimal rank of all worlds that satisfy* A*,* $\kappa(A) = \min\{\kappa(\omega) \,|\, \omega \models A\}$*, which implies* $\kappa(\bot) = \infty$ *and* $\kappa(\top) = 0$*. The rank of a conditional* $(B|A) \in (\mathfrak{L}|\mathfrak{L})$ *is the rank of the conjunction of the conditional's premise and conclusion reduced by the rank of the conditional's premise, formally* $\kappa(B|A) = \kappa(AB) - \kappa(A)$*.*

There is a notion of independence for ranking functions that resembles probabilistic independence and is defined as follows:

Definition 2 (Conditional κ-independence [5]). *Let* $\mathbf{A},\mathbf{B},\mathbf{C} \subseteq \mathcal{V}$ *be disjoint subsets of variables.* $\mathbf{A}$ *is* (conditionally) κ-independent of $\mathbf{B}$ given $\mathbf{C}$*, written* $\mathbf{A} \perp\!\!\!\perp_\kappa \mathbf{B} \mid \mathbf{C}$ *if and only if* $\kappa(\dot{a}\dot{b}|\dot{c}) = \kappa(\dot{a}|\dot{c}) + \kappa(\dot{b}|\dot{c})$ *for all* $\dot{a} \in Asst(\mathbf{A})$*,* $\dot{b} \in Asst(\mathbf{B})$ *and* $\dot{c} \in Asst(\mathbf{C})$*.*

κ-independence can be characterized equivalently as in the probabilistic case, that is, for all disjoint sets of variables $\mathbf{A},\mathbf{B},\mathbf{C} \subseteq \mathcal{V}$, $\mathbf{A}$ is κ-independent of $\mathbf{B}$ given $\mathbf{C}$ ($\mathbf{A} \perp\!\!\!\perp_\kappa \mathbf{B} \mid \mathbf{C}$) if and only if $\kappa(\dot{a}|\dot{b}\dot{c}) = \kappa(\dot{a}|\dot{c})$ for all $\dot{a},\dot{b},\dot{c}$ in $Asst(\mathbf{A})$, $Asst(\mathbf{B})$, $Asst(\mathbf{C})$, respectively [4].

OCF-networks, a network approach that resembles Bayesian networks using local conditional ranks instead of local conditional probabilites at the vertices, have been proposed in [2] and recalled in [3].

Definition 3 (OCF-network). *A DAG* $\Gamma = \langle \mathcal{V}, \mathcal{E} \rangle$ *over a set of propositional atoms* $\mathcal{V}$ *is an* OCF-network *if each vertex* $V \in \mathcal{V}$ *is annotated with a table of local rankings* $\kappa_V(V|pa(V))$ *with (local) ranking values specified for every configuration of* V *and* $pa(V)$*. The local rankings must be normalised, i.e.,* $\min_{\dot{v}}\{\kappa(\dot{v}|pa(V))\} = 0$ *for every configuration of* $pa(V)$*.*

The local ranking information in Γ can be used to define a global ranking function κ over $\mathcal{V}$ by applying the idea of stratification [2]: A function κ is *stratified* relative to an OCF-network Γ if and only if

$$\kappa(\omega) = \sum \kappa_V(V(\omega)|\,\mathrm{pa}(V)(\omega)) \tag{1}$$

for every world ω. It has been shown that (1) is indeed an OCF [4].

Conversely, given a DAG Γ with vertices $\mathcal{V}$ and an OCF κ over $\mathcal{V}$ such that each vertex $V \in \mathcal{V}$ is κ-independent of its non-descendants given its parents, we obtain a stratification of κ relative to Γ.

It has been shown that for an OCF-network $\Gamma = \langle \mathcal{V}, \mathcal{E}, \{\kappa_V\}_{V \in \mathcal{V}} \rangle$, for the gobal ranking function κ we have $\kappa(V_1, \ldots, V_n) = \sum_{i=1}^{n} \kappa(V_i|\,\mathrm{pa}(V_i))$ [4].

3 CP-networks

Most of our everyday preferences seem to be of the type *ceteris paribus*, that is, our preferences are represented keeping "*everything else equal*", meaning that if, for example, asked whether we prefer one thing to another we answer in the context of the actual situation, mentally keeping all other variables constant.

A *preference ranking* $\prec$ over a set of assignments $Asst(\mathbf{A})$, $\mathbf{A} \subseteq \mathcal{V}$, is a transitive, irreflexive, asymmetric and total relation with meaning: $\dot{a}_1 \prec \dot{a}_2$ iff $\dot{a}_1$ is strictly more preferred than $\dot{a}_2$.

For CP-Networks, we determine for each variable $V \in \mathcal{V}$ the set of parent variables $pa(V) \subseteq \mathcal{V}$, which can affect the preference of V irrespectively of all further variables.

Definition 4 (CP-network [1]). *A DAG $\Gamma = (\mathcal{V}, \mathcal{E}, \{CPT(V)\}_{V \in \mathcal{V}})$ is a* CP-network*, if its nodes are annotated with conditional preference tables $CPT(V)$, which associates a preference ranking $\prec_{\dot{p}}$ over $\{V\}$ for every instantiation $\dot{p}$ of $pa(V)$.*

While CP-networks define a preference on variables given their parents it has to be examined whether this preference can be transferred to the set of possible worlds, which is done by the notion of *satisfiability*.

Definition 5 (Satisfiability of CP-networks [1]). *Let $V \in \mathcal{V}$ and $\dot{v}_1, \dot{v}_2 \in Asst(V)$, let $\dot{p} \in Asst(pa(V))$, let $X \in \mathcal{V} \setminus (pa(V) \cup \{V\})$. A CP-network is* satisfiable *if and only if there is a preference relation $<_{cp} \subseteq \Omega \times \Omega$ such that for every $\dot{p}$ and every $\dot{x} \in Asst(X)$ with $\omega \models \dot{v}_1 \dot{p} \dot{x}$ and $\omega \models \dot{v}_2 \dot{p} \dot{x}$ we have $\dot{v}_1 \prec_{\dot{p}} \dot{v}_2$ if and only if $\omega <_{cp} \omega'$.*

[1] show that every CP-network is satisfiable. In a CP-network, each $V \in \mathcal{V}$ is *conditionally preferentially independent* of $\mathcal{V} \setminus (\mathrm{pa}(V) \cup \{V\})$ given $\mathrm{pa}(V)$ [1], formally defined as follows:

Definition 6 (Conditionally preferential independence [1]). *Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C} \subseteq \mathcal{V}$ be nonempty partitions of $\mathcal{V}$ such that $\mathcal{V} = \mathbf{A} \dot{\cup} \mathbf{B} \dot{\cup} \mathbf{C}$. $\mathbf{A}$ is* conditionally preferentially independent *from $\mathbf{B}$ given $\mathbf{C}$, written $\mathbf{A} \perp\!\!\!\perp_{\prec} \mathbf{B} \mid \mathbf{C}$, iff, for all $\dot{a}_1, \dot{a}_2 \in Asst(\mathbf{A})$, $\dot{b}_1, \dot{b}_2 \in Asst(\mathbf{B})$ and $\dot{c} \in Asst(\mathbf{C})$, we have*

$$\dot{a}_1 \dot{b}_1 \dot{c} \prec \dot{a}_2 \dot{b}_1 \dot{c} \textit{ iff } \dot{a}_1 \dot{b}_2 \dot{c} \prec \dot{a}_2 \dot{b}_2 \dot{c}.$$

4 Comparison of both approaches

Ranking function induce a preference relation $<_\kappa$ on worlds, and CP-networks are directly connected with a preference relation $<_{cp}$ by the networks satisfiability. So both OCF- and CP-networks are techniques of expressing a global preference relation on the set of possible worlds on DAGs, what brings up the question whether the approaches are related, or could be used to express similar information, respectively, if they could be tranferred into another.

In this work we show that each CP-network can be transferred into an OCF network and that this OCF-network expresses the same preferences and independences as the CP-network: Let $\langle \mathcal{V}, \mathcal{E}, \{CPT(V)\}_{V \in \mathcal{V}} \rangle$ be a CP-network and let $\langle \mathcal{V}, \mathcal{E}, \{\kappa_V\}_{V \in \mathcal{V}} \rangle$ be an OCF-network such that $\{CPT(V)\}_{V \in \mathcal{V}}$ induces $\{\kappa_V\}_{V \in \mathcal{V}}$, then $\omega <_{cp} \omega'$ implies $\omega <_\kappa \omega'$ for all $\omega \in \Omega$. Additionally the resulting OCF-network can be retransfered into the CP-network used to set it up.

This does not hold for ranking functions in general: The ceteris paribus preference relation $\prec$ is a strict relation which does not allow for two instances of a variable being equally preferred, whereas for ranking functions $\kappa(A) = \kappa(\overline{A}) = 0$, $A \in \mathfrak{L}$, is possible. Therefore there are ranking functions and hereby local ranking tables in an OCF-network that cannot be represented by a ceteris paribus preference relation, which directly implies that not all OCF-networks can be transferred to CP-networks.

An additional challenge is the rank itself. In OCF-networks, the value of a literal given its parents can be any natural number (with respect to the normalisation condition), implying a strength or firmness of the rejection of this option, which cannot be found in CP-networks. We show that for OCF-networks with an extended normalisation condition, namely $\kappa(V | \mathrm{pa}(V)) = 0$ for exactly one $\dot{v}$ for each $\dot{p} \in Asst(\mathrm{pa}(V))$, CP-networks expressing the same local preference can be constructed, with the drawback that the exact firmness is lost in this process.

Additionally we present an approach that allows to generate an OCF-network from a CP-network that inherits the CP-network's stronger indepence condition by bottom-up setting firmness values combining the ranks of a vertex's childrens.

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Inductive generation of OCF-LEG networks using System Z

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Abstract. Network approaches for probabilistic environments are well known and have proven to be slim, lightweight and efficient, just to mention Bayesian networks. For qualitative environments, ranking functions prove to be a proper alternative of probabilities, although network approaches to reduce storage space and boost the calculation are still few. Here, we recall a recent transferal of OCFs to LEG networks (called OCF-LEG networks) and show how the inductive reasoning approach System Z can be used to generate the ranking component of an OCF-LEG network. We demonstrate for an exemplary knowledge base that the global ranking function obtained for the System Z generated OCF-LEG network does not coincide with a global OCF calculated by System Z on the knowledge base, directly, and that this is a structural challenge for System Z.

1 Preliminaries

Let $\mathfrak{L}$ be a propositional language with an underlying finite alphabet of variables $\Sigma = \{V_1, \dots, V_n\}$, logical connectives $\wedge$ (and), $\vee$ (or) and $\neg$ (negation) and the symbols $\top$ and $\bot$ for tautology respectively contradiction. Let ϕ, ψ be formulas in $\mathfrak{L}$. We abbreviate conjunction by juxtaposition (that is, $\phi \wedge \psi$ is abbreviated as $\phi\psi$) and negation by overlining (that is, $\neg\phi$ is abbreviated as $\overline{\phi}$).

A *literal* is a variable interpreted to *true* or *false*; we write v_i to denote the interpretation of V_i to *true*, $\overline{v}_i$ to denote the interpretation of V_i to *false* and $\dot{v}_i$ to denote a fixed interpretation of V_i. We write $V_i \sqsubseteq \phi$ if $\dot{v}_i$ appears in ϕ.

The set of *possible worlds* Ω is defined as the set of all complete conjunctions of literals in Σ. For a set $\Sigma_i \subseteq \Sigma$ we denote by Ω_i the *local possible worlds* of Σ_i. With $\Sigma_i(\omega)$ we indicate the configuration $\omega^i \in \Omega_i$ with $\omega \models \omega^i$. We denote by $[\![\phi]\!]_\omega$ the evaluation of ϕ under ω with respect to the junctors, as usual.

A *conditional* $(\psi|\phi)$ represents the defeasible rule "if ϕ then *usually / normally* ψ" with the trivalent evaluation $[\![(\psi|\phi)]\!]_\omega = \mathit{true}$ if and only if $\omega \models \phi\psi$ (verification / acceptance), $[\![(\psi|\phi)]\!]_\omega = \mathit{false}$ if and only if $\omega \models \phi\overline{\psi}$ (falsification/refutation) and $[\![(\psi|\phi)]\!]_\omega = \mathit{undefined}$ iff $\omega \models \overline{\phi}$ (non-applicability) [1, 3].

The language of all conditionals over $\mathfrak{L}$ is denoted by $(\mathfrak{L} \mid \mathfrak{L})$. A finite set of conditionals $\Delta = \{(\psi_1|\phi_1), \dots, (\psi_n|\phi_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$ is called a *knowledge base*. A conditional $(\psi|\phi)$ is *tolerated* by Δ if and only if there is a world $\omega \in \Omega$ such that $\omega \models \phi\psi$ and $\omega \models \phi_i \Rightarrow \psi_i$ for every $1 \leq i \leq n$.

2 OCF and System Z

An *ordinal conditional function* (OCF) is a function that assigns to each world a rank of *disbelief* or *implausibility*, that is, the higher the rank of a world is, the less plausible this world is.

Definition 1 (Ordinal conditional function (OCF, [8])). *An* ordinal conditional function *(*OCF*, also called* ranking function*)* κ *is a function* $\kappa : \Omega \rightarrow \mathbb{N}_0^\infty$ *such that the set* $\{\omega \,|\, \kappa(\omega) = 0\}$ *is not empty, that is, there have to be worlds that are maximally plausible. The rank of a formula* $\phi \in \mathfrak{L}$ *is defined to be the minimal rank of all worlds that satisfy* ϕ*,* $\kappa(\phi) = \min\{\kappa(\omega)|\omega \models \phi\}$*, which implies* $\kappa(\bot) = \infty$ *and* $\kappa(\top) = 0$*. The rank of a conditional* $(\psi|\phi) \in (\mathfrak{L}|\mathfrak{L})$ *is the rank of the conjunction of the conditional's premise and conclusion reduced by the rank of the conditional's premise, formally* $\kappa(\psi|\phi) = \kappa(\phi\psi) - \kappa(\phi)$*.*

A ranking function κ *accepts* a conditional $(\psi|\phi)$, written $\kappa \models (\psi|\phi)$ if and only if $\kappa(\phi\psi) < \kappa(\phi\overline{\psi})$. κ *accepts/is admissible to* a conditional knowledge base

$$\Delta = \{(\psi_1|\phi_1), \dots, (\psi_n|\phi_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$$

(written $\kappa \models \Delta$) if and only if κ accepts all conditionals in Δ.

System Z [7] is an approach to generate a ranking function κ_Δ^Z which is admissible to a consistent knowledge base $\Delta = \{(\psi_1|\phi_1), \dots, (\psi_n|\phi_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$, realising the most plausible rankings among all such ranking functions. This system is set up by an algorithm which partitions the knowledge base Δ in maximal disjoint sets of tolerated conditionals $\Delta = \Delta_0 \uplus \dots \uplus \Delta_k$, starting with Δ_0 containing all conditionals that are tolerated by all other conditionals in Δ and applying this recursively. The function $Z : \Delta \rightarrow \mathbb{N}_0$ is defined to be $Z(B|A) = z$ iff $(B|A) \in \Delta_z$, and by this the ranking function κ_Δ^Z is given as

$$\kappa_\Delta^Z(\omega) = \begin{cases} 0 \text{ iff } \omega \models (A_i \Rightarrow B_i) \text{ for all } 1 \leq i \leq n \\ \max\limits_{1 \leq i \leq n} \{Z(\omega)|\omega \models A_i\overline{B_i}\} + 1 \quad \text{otherwise.} \end{cases} \tag{1}$$

3 OCF-LEG networks

An OCF-LEG network, a qualitative variant of the probabilistic LEG networks [4], is a hypergraph on the alphabet Σ with a local ranking function κ_i on each hyperedge Σ_i which is the marginal of a global OCF.

Definition 2 (OCF-LEG network [2]). *Let* Σ *be a propositional alphabet and let* Σ_1*,* $\dots, \Sigma_m$ *be a set of covering subsets such that* $\Sigma_i \subseteq \Sigma$*,* $1 \leq i \leq m$ *and* $\Sigma = \bigcup_{i=1}^m \Sigma_i$*. Let* $\kappa_1, \dots, \kappa_m$ *be ranking functions* $\kappa_i : \Omega_i \rightarrow \mathbb{N}_0^\infty$*,* $1 \leq i \leq m$*. The system* $\langle(\Sigma_1, \kappa_1), \dots,$ $(\Sigma_m, \kappa_m)\rangle$*, abbreviated as* $\langle(\Sigma_i, \kappa_i)\rangle_{i=1}^m$*, is a* ranking network of local event groups (OCF-LEG network) *iff there is a global OCF* κ *on* Ω *with the property that* $\kappa(\omega^i) = \kappa_i(\omega^i)$ *for all* $\omega^i \in \Omega_i$ *and all* $1 \leq i \leq n$*.*

For a system $\langle(\Sigma_i, \kappa_i)\rangle_{i=1}^m$ to have a global ranking function as defined above the following consistency condition has to be fulfilled.
(Consistency condition) There is a global ranking function $\kappa : \Omega \to \mathbb{N}_0^\infty$ to the system $\langle(\Sigma_i, \kappa_i)\rangle_{i=1}^m$ only if $\kappa_i((\Sigma_i \cap \Sigma_j)(\omega)) = \kappa_j((\Sigma_i \cap \Sigma_j)(\omega))$ for all pairs $1 \leq i, j \leq m$ and all worlds $\omega \in \Omega$.

We define by $\mathbb{S}_i = \Sigma_i \cap \left(\bigcup_{j=1}^{i-1} \Sigma_j\right)$ the *separators* of Σ_i (cf. [6]). It has been shown that if $\Sigma_1, \dots, \Sigma_m$ is a hypertree with local OCFs $\kappa_1, \dots, \kappa_m$ that satisfy the consistency condition, a global OCF of the system $\langle(\Sigma_i, \kappa_i)\rangle_{i=1}^m$ can be computed as

$$\kappa(\omega) = \sum_{i=1}^m \kappa_i(\Sigma_i(\omega)) - \sum_{i=1}^m \kappa_i(\mathbb{S}_i(\omega)) \tag{2}$$

and henceforth, $\langle(\Sigma_i, \kappa_i)\rangle_{i=1}^m$ is an OCF-LEG network [2].

4 Inductively generating OCF-LEG networks

We recall an algorithm to set up an OCF-LEG network from a finite conditional knowledge base $\Delta = \{(\psi_1|\phi_1), \dots, (\psi_n|\phi_n)\} \subseteq (\mathfrak{L} \mid \mathfrak{L})$: The hypergraph-component is generated by assigning to each conditional $(\psi_i|\phi_i) \in \Delta$ a set $\Sigma_i = \{V | V \sqsubseteq \psi\phi\} \subseteq \Sigma$ and joining up these sets to hypercliques $\mathcal{C} = \{\mathbb{C}_1, \dots, \mathbb{C}_m\}$ such that $\mathcal{C}$ is a hypertree [5, 6]. This partitions Δ into sets Δ_j such that $\Delta_j = \{(\psi_i|\phi_i) | \Sigma_i \subseteq \mathbb{C}_j\}$, on which we compute an admissible ranking function κ_j using methods of inductive reasoning (cf. [2]), namely System Z. It has been shown that the global OCF κ calculated with Equation (2) is admissible to Δ [2]. We demonstrate the generation of an OCF-LEG network using System Z with a more complex example than in [2]. Thereby, we show that even if $\kappa_i \models \Delta_i$ for all $1 \leq i \leq m$ and $\kappa \models \Delta$ holds, the global ranking function κ is usually not a System Z-representation of Δ. It is confirmed formally that this is a structural incompatibility between System Z and the OCF-LEG approach in general.

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The fundamentals of lative logic

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Traditional logic is informal about the production of terms and sentences, and even worse, often avoids to clearly describe how terms latively[4] appear in sentences, i.e., how sentences proceed from terms, and are in fact constructed using terms. Continuing that lativity towards entailment and provability, it is clear that sentences appear in provability, but provability as a statement should not be seen as a sentence. This creates self-referentiality which often leads to peculiar situations.

Existential quantification '$\exists x$' in expressions like $\exists x.P(x)$ is obviously not an operator, as little as 'λx' would be an operator producing the lambda expression $\lambda x.\omega(x)$. In [2] we showed how we can use *levels of signatures* to provide a precise term functor based definition of λ-terms. The key point is that 'λ' should not be seen as a universal abstractor, as already pointed out in [1], but indeed that "ω owns its abstraction". For existential quantification, the expression $\exists x.P(x)$ must be seen as sentence, not a term, but P is then an operator symbol in some underlying signature, so that $P(x)$ is a term. A negation operator $\neg$ can be applied to the term $P(x)$ so that $\neg P(x)$ and $P(x)$ are of the same sort, as terms. However, as $\exists x.P(x)$ is not a term, but a sentence, and it is very questionable whether $\neg$ in $\neg\exists x.P(x)$ and $\exists x.\neg P(x)$ really is the same symbol. In $\exists x.\neg P(x)$, it acts an operator, changing a term to term, but in $\neg\exists x.P(x)$ it changes a sentence to a sentence, so it is strictly speaking not an 'operator'.

Gödel's incompleteness is seen as a theorem, but could also be seen as a paradox arising from inlative logic. Gödel's numbering [5] creates sentences based on provability. This is clearly seen using notations in [6], where a predicate symbol A and a predicate $A(x)$ is used when speaking about "$A(x)$ is provable", using the notation "$\vdash A(x)$". Then, a "metamathematical proposition" $\mathfrak{R}(x,Y)$ is created to represent "Y is a proof of $A(x)$", followed by $(\exists Y)\mathfrak{R}(x,Y) \equiv \vdash A(x)$. Kleene [6] then wonders "What is the nature of the predicate $\mathfrak{R}(x,Y)$?", and continues to say that requires an "effectively decidable" metamathematical predicate, and that "there must be a decision procedure or algorithm for the question whether $\mathfrak{R}(x,Y)$ holds". Mathematical propositions and metamathematical propositions are thus allowed to be in the same bag, and in [5] there is frequent

[4] 'Lative' is "motion", motion 'to' and 'from', so when terms appear in sentences, terms 'move into' sentence, and sentences 'move away from' terms. In comparison, 'ablative' is "motion away", and nominative is static. The lative locative case (casus) indeed represents "motion", whereas e.g. a vocative case is identification with address.

use of that degree of freedom to mix bags, not just between term and sentence, but indeed between truth and provability.

A further subtlety concerns the underlying signature which is a basis for producing terms. Type constructors such as the one producing function types are frequently treated as if they are outside signatures, i.e., that they are not operators in some signature. In order to see it more precisely, let $\mathtt{s}_1$ and $\mathtt{s}_2$ be two sorts in some set of sorts S. The function sort involving $\mathtt{s}_1$ and $\mathtt{s}_2$ can be denoted $\mathtt{s}_1 \Rrightarrow \mathtt{s}_2$. Even if we want to view $\mathtt{s}_1 \Rrightarrow \mathtt{s}_2$ as a (constructed) sort, it is not part of S. This creates an awkward meta-level of constructors, and the formalism for treating these constructors is rather lose. This is also the reason why the definition of λ-terms is correspondingly lose, even if it is seen as compact and elegant. However, the traditional so called 'set' of λ-terms is not well-defined even if it is 'well understood'. In other words, there hasn't been any strict term functor, or term monad for that matter, for producing λ-terms. This situation has been made more clear in [2], once type constructors are accommodate properly into suitable signatures. The main question is how to expand a signature $\Sigma = (S, \Omega)$, often called the *basic signature*, to a signature $\Sigma' = (S', \Omega')$ so that $\mathtt{s}_1 \Rrightarrow \mathtt{s}_2 \in S'$ whenever $\mathtt{s}_1, \mathtt{s}_2 \in S$, and where Σ' is the underlying signature for the λ-terms. Such an arrangement also enables to keep λ-abstractions, as members of Ω', clearly apart from λ-terms, residing in the set of λ-terms as defined by the λ-term functor.

The three-level arrangement of signatures starts from the basic signature Σ on level one. On level two we have the (Σ-)*superseding type signature* as a one-sorted signature $\mathtt{S}_\Sigma = (\{\mathtt{type}\}, \mathtt{Q})$, where $\mathtt{Q}$ is a set of *type constructors* satisfying

(i) $\mathtt{s} :\to \mathtt{type}$ is in $\mathtt{Q}$ for all $\mathtt{s} \in S$

(ii) there is a $\Rrightarrow : \mathtt{type} \times \mathtt{type} \to \mathtt{type}$ in $\mathtt{Q}$

If $\mathtt{Q}$ does not contain any other type constructors, apart from those given by (i) and (ii), we say that $\mathtt{S}_\Sigma$ is a (Σ-)*superseding simple type signature*.

The term monad construction can then obviously be used also for any Σ-superseding type signature $\mathtt{S}_\Sigma$. We may write $\mathtt{s} \Rrightarrow \mathtt{t}$ for the type term $\Rrightarrow (\mathtt{s}, \mathtt{t})$.

The signature $\Sigma' = (S', \Omega')$ on level three then is based on $S' = \mathsf{T}_{\mathtt{S}_\Sigma} \varnothing$, i.e., the sorts on level three are those from level one together with the constructed sorts, on level two appearing as terms (the type terms), added to those basic sorts coming from level one.

For the operators in Ω' it may sound natural to include all operators from Ω into Ω' so that $\Omega \subseteq \Omega'$, but it is not always desirable. If we consider the NAT signature[5] on level one we obviously may have both $0 :\to \mathtt{nat}$ and $\mathtt{succ} : \mathtt{nat} \to \mathtt{nat}$ included in the operators for $\mathtt{NAT}'$. However, the unary operator succ, i.e., unary both on level one and level three, can alternatively be (λ-)abstracted to be become a constant (0-ary) operator $\lambda_1^{\mathtt{succ}} :\to (\mathtt{nat} \Rrightarrow \mathtt{nat})$ on level three. Clearly, the constant $0 :\to \mathtt{nat}$ converts to $\lambda_0^0 :\to \mathtt{nat}$, i.e., a constant on level one remains as a constant on level three. Note also that nat on level one is not the same as nat on level three. If we need to be strict, we should use e.g. $\mathtt{nat}'$ for the corresponding sort on level three.

Church's type constructor [1] is in effect our $\Rrightarrow$, so that $(\beta \Rrightarrow \alpha)$ is Church's $(\beta\alpha)$. An interpretation of Church's ι to be our type is clearly less controversial, but for the

[5] The signature for natural numbers is usually given by $\mathtt{NAT} = (\{\mathtt{nat}\}, \{0 :\to \mathtt{nat}, \mathtt{succ} : \mathtt{nat} \to \mathtt{nat}\})$.

interpretation of o there are a number of alternative intuitions, a formalism for which modern type theory has been incapable of producing.

In summary, the three signature levels underlying the production of λ-terms are then following.

1. the level of primitive underlying operations, with a usual many-sorted signature $\Sigma = (S, \Omega)$
2. the level of type constructors, with a single-sorted signature $\mathtt{S}_\Sigma = (\{\mathtt{type}\}, \{\mathtt{s} :\to \mathtt{type} \mid \mathtt{s} \in S\} \cup \{\Rrightarrow : \mathtt{type} \times \mathtt{type} \to \mathtt{type}\})$
3. the level including λ-terms based on the signature $\Sigma' = (S', \Omega')$ where $S' = \mathsf{T}_{\mathtt{S}_\Sigma}\varnothing$, $\Omega' = \{\lambda^{\omega}_{i_1,\ldots,i_n} :\to (\mathtt{s}_{i_1} \Rrightarrow \cdots \Rrightarrow (\mathtt{s}_{i_{n-1}} \Rrightarrow (\mathtt{s}_{i_n} \Rrightarrow \mathtt{s})) \mid \omega : \mathtt{s}_1 \times \ldots \times \mathtt{s}_n \to \mathtt{s} \in \Omega\} \cup \{\mathtt{app}_{\mathtt{s},\mathtt{t}} : (\mathtt{s} \Rrightarrow \mathtt{t}) \times \mathtt{s} \to \mathtt{t}\}$

Here $(i_1, \ldots, i_n)$ is a permutation of $(1, \ldots, n)$. Note also that level one operators are always transformed to constants on level three. In traditional notation in λ-calculus, substituting x by $\mathtt{succ}(y)$ in $\lambda y.\mathtt{succ}(x)$ requires a renaming of the bound variable y, e.g., $\lambda z.\mathtt{succ}(\mathtt{succ}(y))$. In our approach we avoid the need for renaming. On level one, and in the case of NAT, we have the substitution (Kleisli morphism) $\sigma_{\mathtt{nat}} : X_{\mathtt{nat}} \to \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}(X_{\mathtt{t}})_{\mathtt{t}\in\{\mathtt{nat}\}}$, where $\sigma_{\mathtt{nat}}(x) = \mathtt{succ}(y)$, x being a variable on level one, and the extension of $\sigma_{\mathtt{nat}}$ is

$$\mu_{X_{\mathtt{nat}}} \circ \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}\,(\sigma_{\mathtt{t}})_{\mathtt{t}\in\{\mathtt{nat}\}} : \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}(X_{\mathtt{t}})_{\mathtt{t}\in\{\mathtt{nat}\}} \to \mathsf{T}_{\mathtt{NAT},\mathtt{nat}}(X_{\mathtt{t}})_{\mathtt{t}\in\{\mathtt{nat}\}}.$$

On level three we have

$$\sigma_{\mathtt{nat}'} : X_{\mathtt{nat}'} \to \mathsf{T}_{\mathtt{NAT}',\mathtt{nat}'}(X_{\mathtt{t}})_{\mathtt{t}\in S'},$$

with $\sigma_{\mathtt{nat}'}(x) = \mathtt{app}_{\mathtt{nat}',\mathtt{nat}'}(\lambda_1^{\mathtt{succ}}, x)$, x a variable on level three, and no renaming needed in $\mu_{\mathtt{nat}'} \circ \mathsf{T}_{\mathtt{NAT}',\mathtt{nat}'}\,\sigma_{\mathtt{nat}'}(\mathtt{app}_{\mathtt{nat}',\mathtt{nat}'}(\lambda_1^{\mathtt{succ}}, x))$.

At this point we have the crisp set of λ-terms, given the term functor $\mathsf{T}_{\Sigma'} : \mathtt{Set}_{S'} \to \mathtt{Set}_{S'}$. The sets of λ-terms with respect to each end sort $\mathtt{s}' \in S'$ are then represented by respective sets $\mathsf{T}_{\Sigma',\mathtt{s}'}(X_{\mathtt{s}})_{\mathtt{s}\in S'}$.

Note indeed that the λ-term monad may be considered to be over other monoidal biclosed categories [2, 4], and recent constructions show more clearly how monoidal closed categories come into play [4].

Semantically, we come to an interesting question concerning $\mathfrak{A}_{\mathtt{S}_\Sigma}(\mathtt{type})$. Obviously, we could use free algebras, but this would, in the case of natural numbers, make $\mathfrak{A}_{\mathtt{S}_\Sigma}(\mathtt{nat})$ into a point at level two, given that here nat is a 0-ary operator. We would rather prefer to have $\mathfrak{A}_{\mathtt{S}_\Sigma}(\mathtt{nat})$ as a set, and this then means we would have $\mathfrak{A}_{\mathtt{S}_\Sigma}(\mathtt{nat} \Rrightarrow \mathtt{nat})$ as a homset, i.e., we then need $\mathfrak{A}_{\mathtt{S}_\Sigma}(\mathtt{type})$ to be $\mathrm{Ob}(\mathtt{Set})$. Obviously, other categories can be considered for these purposes, and the use of monoidal biclosed categories would enable to use internal homsets as semantics for function types.

Concerning sentences, a fundamental difference as compared with terms is that term functors are extendable to term monads, so that substitutions are composable (composition of Kleisli morphisms), whereas sentence functors are not necessarily monads, in fact should not be monads, since we do not substitute with sentences. If a sentence functor is a monad, then sentences produced by such a sentence functor are in fact terms. Lambda calculus is a special case where lambda terms appear simultaneously as sentences, so that rewriting is a proof sequence given β-reduction as an inference rule. Informally, the lativity between terms and sentence, i.e., that terms appear "inside" sentences, is a situation where a sentence functor 'Sen' is composed with a 'T' to produce the "set of sentences" '$(\mathsf{Sen} \circ \mathsf{T})X$' over some (many-sorted) "set" X of variables. In the case of lambda calculus, $\mathsf{Sen} = \mathsf{id}$, and in the case of equational logic, $\mathsf{Sen} = \mathsf{id} \times \mathsf{id}$. For Horn clause logic, i.e., without existential quantification, we produced a many-sorted sentence functor [3], where implication is not seen as an operator using terms to produce terms, but rather as a pair, not in form of a "sentential equation" but a "sentential implication".

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Term monad in monoidal biclosed categories

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Abstract. A *prequantale* is pair $(X, *)$ where X is a complete lattice and $*$ is a binary operation on X preserving arbitrary joins in each variable separately.

Question. Does there exist a free prequantale for every complete lattice?

We make the previous question categorically precise. For this purpose let Sup be the category of complete lattices and join preserving maps. It is well kown that there exists a tensor product on Sup turning Sup into a monoidal closed category (cf. [3]). In particular, for any bimorphism (cf. [1]) —- i.e. for any map $X \times Y \xrightarrow{b} Z$ which preserves arbitrary joins in each variable separately — there exists a unique join preserving map $X \otimes Y \xrightarrow{\ulcorner b \urcorner} Z$ making the following diagram commutative:

$$\begin{array}{ccc} X \times X & \xrightarrow{\otimes} & X \otimes Y \\ & \searrow_{b} & \vdots \, {\ulcorner b \urcorner} \\ & & Z \end{array}$$

Hence prequantales and *magmas in* Sup are the same thing. Therefore we can reformulate the question as follows:

Question. Does the forgetful functor from the category of magmas in Sup to Sup have a left adjoint?

In the category Set of sets the previous question means the construction of free groupoids and has a positive answer. It is well known that this construction is based on the *term construction* w.r.t. a signature consisting of a single binary operator symbol. Therefore we are motivated to ask the more general question:

Question. Does there exist a term construction in Sup?

In this talk I prove the following result:

Theorem. The term monad exists in any monoidal biclosed category.

Comment. In this context a *one-sorted signature* Ω is a sequence $(\Omega_n)_{n \in \mathbb{N}_0}$ of objects Ω_n of the underlying category.

This theorem has various consequences. First of all our previous question has a positive answer. Further, the term monad exists also in Goguen's category. Viewing Goguen's category as a category for fuzzy set theory we have therefore *fuzzy terms* (cf. [2]) — a concept which sems to be completely new in *fuzzy logic*.

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Local finiteness in t-norm bimonoids: overlap cases

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1 Introduction

An algebraic structure $\mathfrak{A}$ is *locally finite* iff each of its finite subsets G generates a finite subalgebra $\langle G\rangle_{\mathfrak{A}}$ only.

An element $a \in \mathfrak{A}$ has *order n* iff $\langle\{a\}\rangle_{\mathfrak{A}}$ has n elements; it has *finite order* iff it has order n for some $n \in \mathbb{N}$, and it has *infinite order* otherwise.

Continuous t-norms can be represented as ordinal sums of isomorphic copies of T_{L} and T_{P}.

An ordinal sum representation for a t-norm T is based upon a disjoint family $(]l_i, r_i[)_{i\in I}$ of open subintervals of $[0,1]$, and a family $(T_i)_{i\in I}$ of t-norms, written as

$$T = \bigoplus_{i\in I}([l_i, r_i], T_i)\,. \tag{1}$$

By the *summand domain* for the k-th summand of such an ordinal sum we will understand the closed interval $[l_k, r_k]$ as well as its square $[l_k, r_k]^2$, depending on the context.

2 The t-norm bimonoids

Now we are interested in the t-norm based bimonoids $([0,1], T, S_T, 1, 0)$. In general, a bimonoid is an algebraic structure $\mathfrak{A} = (A, *_1, *_2, e_1, e_2)$ such that both $(A, *_1, e_1)$ and $(A, *_2, e_2)$ are monoids.

Theorem 1. *Suppose that the continuous t-norm T has an ordinal sum representation with only Łukasiewicz isomorphic summands, let for those summands their determining order automorphisms of $[0,1]$ be rational based, and let all T-idempotents be rationals. Then the t-norm bimonoid $\mathfrak{A}_T$ which is determined by T is rationally locally finite.*

3 T-S-overlap

The problem will now be to discuss what happens for bimonoids *with* T-S-overlap. Then the domains of a T-summand and an S-summand have an interval in common.

Only this type of binary overlap is possible. It is impossible that three summand-domains have a common overlap because no two T-summand and no two S-summands can overlap. It is, however, possible that a T-summand overlaps with more than one S-summand, and vice versa.

Distinction shall be made between *partial* and *total* overlap. One has four possibilities for $\{T_i, S_k\}$-overlap with the following characterizations:

$$\begin{array}{ll} \text{partial } \langle T_i, S_k\rangle\text{-overlap:} & l_i < 1 - r_k < r_i < 1 - l_k\,, \\ \text{total } \langle T_i, S_k\rangle\text{-overlap:} & l_i \le 1 - r_k < 1 - l_k \le r_i\,, \\ \text{partial } \langle S_k, T_i\rangle\text{-overlap:} & 1 - r_k < l_i < 1 - l_k < r_i\,, \\ \text{total } \langle S_k, T_i\rangle\text{-overlap:} & 1 - r_k \le l_i < r_i \le 1 - l_k\,. \end{array}$$

Corollary 1. *Both types of total overlap coincide in the sense that total $\langle T_i, S_k\rangle$-overlap is equivalent to total $\langle S_i, T_k\rangle$-overlap.*

4 Local finiteness: negative results

The considerations on self-overlap offer immediately a negative result on local finiteness which is a straightforward generalization of the fact that the Łukasiewicz bimonoid is not locally finite.

Proposition 1. *If in a t-norm bimonoid $\mathfrak{A}$ its t-norm T has a summand $([l_i, r_i], T_{\mathsf{L}}, h_i)$ with a rational-based order automorphism h_i and with full self-overlap, then $\mathfrak{A}$ is not locally finite.*

Example 1. For each $0 < a < \frac{1}{2}$ the t-norm $T = ([a, 1-a], T_{\mathsf{L}}, id)$ has full self-overlap and determines, thus, a bimonoid which is not locally finite.

Corollary 2. *Suppose to have partial $\langle T_i, S_k\rangle$-overlap.*

(i) If some $\mathfrak{A}$-iteration of a $c \in [a, b]$ reaches a then it reaches l_i after finitely many further steps and stops there, provided there is no overlap of $[l_i, a]$ with some other S_j-domain, $j \neq k$.

(ii) If some $\mathfrak{A}$-iteration of a $c \in [a, b]$ reaches b then it reaches $1 - l_k$ after finitely many further steps and stops there, provided there is no overlap of $[b, 1 - l_k]$ with some other T_j-domain, $j \neq i$.

Proposition 2. *Suppose to have partial $\langle T_i, S_k\rangle$-overlap with rational borders of the overlap interval $[a, b]$, and that T_i, T_k are zoomed versions of T_{L}. Then each irrational $c \in [a, b]$ is of infinite $\mathfrak{A}$-order.*

5 Local finiteness: partly positive results

Proposition 3. *Suppose to have partial $\langle S_k, T_i\rangle$-overlap in the t-norm bimonoid $\mathfrak{A}$ together with $T_i(b, b) \le a$, then each $c \in [a, b]$ is of finite $\mathfrak{A}$-order.*

Proposition 4. *Suppose to have total $\langle T_i, S_k\rangle$-overlap in the t-norm bimonoid $\mathfrak{A}$. If $T_i(b, b) \le a$ and the T_i-domain does not overlap with another S_j-domain, $j \neq k$, then each $c \in [a, b]$ is of finite $\mathfrak{A}$-order.*

Proposition 5. *Suppose to have total* $\langle T_i, S_k \rangle$*-overlap in the t-norm bimonoid* $\mathfrak{A}$*. Let the* T_i*-range totally overlap with just the* S_j*-ranges for* $j \in J$*, and let be* b *the supremum of all* $1 - l_j$ *for* $j \in J$*. If each one of these* S_j*-ranges is covered by one of the intervals* $[T_i(b,b), b]$*,* $[T_i(b,b,b), T_i(b,b)], \ldots$*, then each* $c \in [l_i, r_i]$ *is of finite* $\mathfrak{A}$*-order.*

Example 2. The assumptions of this Proposition 5 are satisfied by the following t-norm

$$T = ([\frac{1}{6}, \frac{2}{6}], T_{\mathsf{L}}, id) \oplus ([\frac{2}{6}, \frac{1}{2}], T_{\mathsf{L}}, id) \oplus ([\frac{1}{2}1], T_{\mathsf{L}}, id)\,.$$

Proposition 6. *Suppose to have total* $\langle S_k, T_i \rangle$*-overlap in the t-norm bimonoid* $\mathfrak{A}$*. If* $S_k(a,a) \geq b$ *and the* S_k*-domain does not overlap with another* T_j*-domain,* $j \neq i$*, then each* $c \in [a,b]$ *is of finite* $\mathfrak{A}$*-order.*

Graded equipollence: a functional approach to cardinality of finite fuzzy sets

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1 Introduction

In the classical set theory, we can recognize two approaches to the cardinality of sets. One of them is a functional approach that uses one-to-one correspondences between sets to compare their sizes. More precisely, we say that two sets a and b are equipollent (equipotent, equivalent, bijective or have the same cardinality) and write $a \sim b$ if there exists a one-to-one mapping of a onto b. The relation "being equipollent" is an equivalence on the class of all sets and is called equipollence (or equipotence, equinumerosity etc.). The equipollence of (finite) fuzzy sets has been investigated primarily by S. Gottwald [1, 2] and M. Wygralak [6–9] (see also [5]). S. Gottwald proposed a graded approach to the equipollence of fuzzy sets defined using the uniqueness of fuzzy mappings in his set theory for fuzzy sets of higher level. Additionally, a graded generalization of equipollence suggesting that fuzzy sets have approximately the same number of elements has been noted by M. Wygralak in [7], but substantial development of cardinal theory based on this type of equipollence has not been realized yet.

In this contribution, we propose the concept of graded equipollence of finite fuzzy sets which in a special case can collapse to the equipollence noted by M. Wygralak. We provide a functional approach to the cardinality of finite fuzzy sets based on this type of equipollence.

2 Preliminaries

We assume that the truth values are interpreted in a *residuated - dually residuated lattice* (rdr-lattice for short), i.e., in an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, \oplus, \ominus, \bot, \top \rangle$ with six binary operations and two constants satisfying the following conditions:

(i) $\langle L, \wedge, \vee, \bot, \top \rangle$ is a bounded lattice, where $\bot$ is the least element and $\top$ is the greatest element of L,
(ii) $\langle L, \otimes, \bot \rangle$ and $\langle L, \oplus, \top \rangle$ are commutative monoids,
(iii) the pairs $\langle \otimes, \rightarrow \rangle$ and $\langle \oplus, \ominus \rangle$ form adjoint pairs, i.e.,

$$\alpha \leq \beta \rightarrow \gamma \quad \text{if and only if} \quad \alpha \otimes \beta \leq \gamma,$$
$$\alpha \leq \beta \oplus \gamma \quad \text{if and only if} \quad \alpha \ominus \beta \leq \gamma$$

hold for each $\alpha, \beta, \gamma \in L$ ($\leq$ denotes the corresponding lattice ordering).

In order to integrate some alternative constructions based on the operations of $\wedge$ and $\otimes$, in the sequel, we will use the common symbol $\odot$.

To express the cardinality of finite fuzzy set, one may imagine something like fuzzy sets defined on the set $\mathbb{N}$ of natural numbers (with 0) as suitable candidates for this purpose. Because $\mathbb{N}$ is denumerable, it is advantageous to use the proper class $\mathfrak{Count}$ of all countable sets as a framework of our theory.

Definition 1. *A mapping $A : x \to L$ is called a fuzzy set in $\mathfrak{Count}$ if x is a set in $\mathfrak{Count}$.*

Let us denote by $\mathfrak{FCount}$ the class of all countable fuzzy sets and by $\mathrm{Supp}(A)$ the support of fuzzy set A. We say that two fuzzy sets are the same if they coincide as mappings. An essential predicate in our theory is a binary relation that extends the concept of being the same fuzzy sets and states that two fuzzy sets are the same except for elements having the zero membership degree.

Definition 2. *We say that fuzzy sets A and B are* equivalent *(symbolically, $A \equiv B$) if* $\mathrm{Supp}(A) = \mathrm{Supp}(B)$ *and $A(x) = B(x)$ for any $x \in \mathrm{Supp}(A)$. The class of all equivalent fuzzy sets with A is denoted by* $\mathrm{cls}(A)$.

Bellow we demonstrate on the definition of the union and difference the principle how to introduce operations on fuzzy sets. Similarly, one may introduce the intersection ($\cap$), product ($\times$) or disjoint union ($\sqcup$) of fuzzy sets.

Definition 3. *Let $A, B \in \mathfrak{FCount}$, $x = \mathrm{Dom}(A) \cup \mathrm{Dom}(B)$ and $A' \equiv A$, $B' \equiv B$ such that* $\mathrm{Dom}(A') = \mathrm{Dom}(B') = x$. *Then,*

- *the* union *of A and B is a mapping $A \cup B : x \to L$ defined by*

$$(A \cup B)(a) = A'(a) \vee B'(a)$$

- the difference *of A and B is a mapping $A \setminus B : x \to L$ defined by*

$$A \setminus B(a) = A'(a) \otimes (B'(a) \to \top)$$

for any $a \in x$.

Further, we propose the following definitions of fuzzy power set and exponentiation.

Definition 4. *Let $A \in \mathfrak{FCount}$ and $x = \{y \mid y \subseteq \mathrm{Dom}(A)\}$. A fuzzy set $\mathbf{P}(A) : x \to L$ defined by*

$$\mathbf{P}(A)(y) = \bigwedge_{z \in \mathrm{Dom}(A)} (\chi_y(z) \to A(z))$$

is called a fuzzy power set of A.

Definition 5. *Let $A, B \in \mathfrak{FCount}$ and put $x = \mathrm{Dom}(A)$ and $y = \mathrm{Dom}(B)$. A fuzzy set $B^A : y^x \to L$ defined by*

$$B^A(f) = \bigwedge_{z \in x} (A(z) \to B(f(z)))$$

is called an exponentiation *of A to B.*

Definition 6. *We say that a fuzzy set A from $\mathfrak{FCount}$ is* finite *if there exists $A' \in \mathrm{cls}(A)$ such that* $\mathrm{Dom}(A')$ *is a finite set. The class of all finite fuzzy sets in $\mathfrak{Count}$ is denoted by $\mathfrak{Ffin}$.*

3 Graded equipollence

Let us start with the concept of one-to-one mapping between fuzzy sets in a degree.

Definition 7. *Let* $A, B \in \mathfrak{Fin}$*,* $x, y \in \mathrm{Count}$ *and* $f : x \to y$ *be a one-to-one mapping of* x *onto* y *in* Count*. We shall say that* f *is a* one-to-one mapping of A onto B in the degree α with respect to $\odot$ *if* $\mathrm{Supp}(A) \subseteq x \subseteq \mathrm{Dom}(A)$ *and* $\mathrm{Supp}(B) \subseteq y \subseteq \mathrm{Dom}(B)$ *and*

$$\alpha = \bigodot_{z \in x} (A(z) \leftrightarrow B(f(z))).$$

We write $[A \sim_f^\odot B] = \alpha$ *if* f *is a one-to-one mapping of* A *onto* B *in the degree* α *with respect to* $\odot$*.*

As could be seen above not all one-to-one mappings are considered to specify the degree in which a mapping is a one-to-one mappings between fuzzy sets. The following establishes the set of all important one-to-one mappings between fuzzy sets.

Definition 8. *Let* $A, B \in \mathfrak{Fin}$*. A mapping* $f : x \to y$ *belongs to the set* $\mathrm{Bij}(A, B)$ *if* f *is a one-to-one mapping of* x *onto* y*,* $\mathrm{Supp}(A) \subseteq x \subseteq \mathrm{Dom}(A)$*, and* $\mathrm{Supp}(B) \subseteq y \subseteq \mathrm{Dom}(B)$*.*

Now we can proceed to the definition of graded equipollence.

Definition 9. *Let* $A, B \in \mathfrak{Fin}$*. We shall say that* A is equipollent with B *(or* A has the same cardinality as B*)* in the degree α with respect to $\odot$ *if there exist fuzzy sets* $C \in \mathrm{cls}(A)$ *and* $D \in \mathrm{cls}(B)$ *such that*

$$\alpha = \bigvee_{f \in \mathrm{Bij}(C,D)} [C \sim_f^\odot D]$$

and, for each $A' \in \mathrm{cls}(A)$*,* $B' \in \mathrm{cls}(B)$ *and* $f \in \mathrm{Bij}(A', B')$*, there is* $[A' \sim_f B'] \leq \alpha$*.*

Similarly to the equipollence of sets (or fuzzy sets), the graded equipolence of fuzzy set is a $\otimes$-similarity relation (i.e., reflexive, symmetric and $\otimes$-transitive) on the class of all finite fuzzy sets as the following theorem shows.

Theorem 1. *The fuzzy class relation* $\sim^\odot : \mathfrak{Fin}^2 \to L$ *is a* $\otimes$*-similarity relation on* $\mathfrak{Fin}$*.*

4 Graded versions of selected fundamental results in set theory

The most familiar theorems in set theory is the Cantor-Bernstein theorem (CBT). One of its forms states that if a, b, c, d are sets such that $b \subseteq a$ and $d \subseteq b$ and $a \sim d$ and $b \sim c$, then $a \sim c$. Unfortunately, we cannot prove its graded form in a full generality. However, restricting to the case of $\odot = \wedge$ and the linearity of rdr-lattice, we obtain the following theorem - a graded version of CBT.

Theorem 2. *Let* $\mathbf{L}$ *be a linearly ordered rdr-lattice and* $A, B, C \in \mathfrak{Fin}$ *such that* $A \subseteq B \subseteq C$*. Then,*

$$[A \sim^\wedge C] \leq [A \sim^\wedge B] \wedge [B \sim^\wedge C].$$

Let a,b,c,d be sets such that $a \sim c$ and $b \sim d$. Then, it is well-known that $a \cup b \sim c \cup d$, whenever $a \cap b = \emptyset$ and $c \cap d = \emptyset$, $a \times b \sim c \times d$, $a \sqcup b \sim c \sqcup d$. The following theorem shows the graded versions of these and two further statements.

Theorem 3. *Let $A,B,C,D \in \mathfrak{Ffin}$. Then,*

(i) $[A \sim^{\circ} B] \leq [\overline{A} \sim^{\circ} \overline{B}]$
(ii) $[A \sim^{\circ} B] \otimes [C \sim^{\circ} D] \leq [A \otimes C \sim^{\circ} B \otimes D]$,
(iii) $[A \sim^{\circ} B] \otimes [C \sim^{\circ} D] \leq [A \times C \sim^{\circ} B \times D]$,
(iv) *if* $\mathrm{Supp}(A) \cap \mathrm{Supp}(B) = \mathrm{Supp}(C) \cap \mathrm{Supp}(D) = \emptyset$, *then*

$$[A \sim^{\circ} C] \otimes [B \sim^{\circ} D] \leq [A \cup B \sim^{\circ} C \cup D],$$

(v) $[A \sim^{\circ} B] \otimes [C \sim^{\circ} D] \leq [A \sqcup C \sim^{\circ} B \sqcup D]$.

If a,b are sets and $a \sim b$, then $\mathbf{P}(a) \sim \mathbf{P}(b)$ and $a \not\sim \mathbf{P}(a)$. The following theorem provides a graded version of these two classical statements (with a restriction on the operation $\wedge$ in the first case).

Theorem 4. *Let $A,B \in \mathfrak{Ffin}$. Then, $[A \sim^{\circ} B] \leq [\mathbf{P}(A) \sim^{\wedge} \mathbf{P}(B)]$ and $[A \sim^{\circ} \mathbf{P}(A)] < \top$.*

If a,b,c,d are sets such that $a \sim c$ and $b \sim d$, then $b^a \sim d^c$. The following theorem is a graded version of this statement.

Theorem 5. *Let $A,B,C,D \in \mathfrak{Ffin}$ such that $|\mathrm{Dom}(A)| = |\mathrm{Dom}(C)| = m$ and $|\mathrm{Dom}(B)| = |\mathrm{Dom}(D)| = n$. Then, $[A \sim^{\circ} C] \otimes [B \sim^{\circ} D] \leq [B^A \sim^{\wedge} C^D]$.*

Theorem 6. *Let $A,B,C \in \mathfrak{Ffin}$ such that their universes are finite. Then,*

$$[C^{A \otimes B} \sim^{\circ} (C^B)^A] = \top.$$

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Rethinking uncertainty: some key questions

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1 Prologue or *The uncertain reasoner's nightmare*

As economist G.L.S. Shackle once put it, decision is the *human predicament*. For when it really matters, we just don't know. We don't know what to do because we don't know what to believe and sometimes we don't even know what we want, or aspire to. And yet, in spite of all this, we must decide.

Classical decision theorists, a set to whom Shackle did not belong, have nurtured over the last five decades or so a consensus on the recommendation that we should weigh probabilistically our beliefs before making decisions. This suggests that the first step in order to tackle the practical aspects of the human predicament involves answering the following problem

how should we choose our probabilities?

Probabilists disagree. Some insist that the theory of probability should not give us probability values for the events we are interested in. Rather, probability theory can and should only guard us against having incoherent beliefs for our relevant uncertainties, thereby leaving to "practice" the choice of specific values. Others agree that rational beliefs should not be incoherent, but contend that not all coherent beliefs are born equal. Some versions of this line of reasoning ends with the mathematically most rewarding answer, namely one (unique) probability value. Many take issues with the appeal to symmetry which these arguments use.

Whilst subjective and objective Bayesians argue about the meaning of probability and how this may or may not justify further logical constraints on the choice of probability functions, non-Bayesians take the opportunity to ask

why probability in the first place?

Some believe that probability is in fact far from being a good model of our uncertainty, in things that matter. A recurrent case in point is the unsuitability of additive measures –of which probability is an example– to represent ignorance. Likewise, for the unreasonable demand that rational agents should have a complete representation of their uncertainty, i.e. that a probability should be available to quantify one's uncertainty for every event of interest. Alternative models are put forward in reaction to that. Whilst supporters of probability argue with its detractors about how should we measure all uncertainty, epistemologists cannot help but asking do we really need to measure uncertainty?

Objectivists –those who see uncertainty as a property or feature of the "world"– are likely to answer that the quantitative approach to uncertainty is necessary to let statistical analysis bear on complex decision problems. Subjectivists about uncertainty, i.e. those who focus primarily on the epistemic aspect of uncertainty, see the measurement of uncertainty as necessary to communicate their individual appraisal of uncertainty Those favouring the probabilistic representation of epistemic uncertainty also add that under suitable conditions, probability guarantees that individuals have no interest in communicating degrees of belief which depart from their own epistemic state Suppose, however, that there was no real need to *quantify* uncertainty after all. Then, one could gather all the above discussants and suggest them to re-orientate their efforts towards practically relevant models of uncertainty. The provisions of the Maastricht Treaty (1992) did not contemplate the event that any of the member States might cease to fulfil the conditions that allowed them in the Euro. Back then, exiting the Euro was not considered a possibility at all. Things changed dramatically, in 2011 when GREXIT did become a serious possibility. In (partial) hindsight, all that really seemed to have mattered about GREXIT was the policy-makers' ability to distinguish between logical and practically-relevant *possibility*. For when GREXIT became a serious possibility, the CEB did "whatever it took" to *not* make it happen.[1]

The qualitative approach to uncertain reasoning in artificial intelligence is one prominent research area which has largely been motivated by this similar sort of considerations. Qualitative uncertain reasoning naturally raises the question as to whether all uncertainties are born equal or:

are there many kinds of uncertainty?

Much recent work in uncertain reasoning and decision theory aims at unfolding the consequences of assuming that uncertainty does come in various forms. Keynes and Knight were among the first to put forward suggestions to this effect, albeit from rather distinct points of view. Knight, in particular, argued for a basic distinction between uncertainty which is quantifiable, and uncertainty which is not. Some contemporary decision theorists pay homage to the economist by referring to the latter as *Knightian uncertainty*. Bayesians (of all sorts) have ever since challenged the well-foundedness of this distinction. Ongoing work in economic theory shows that the debate is far from begin settled.

What this leaves us with is the rather discomforting feeling that our grasp of the very concept of uncertainty is a lot less firm than we like to think. The purpose of this paper is to put forward some preliminary suggestions as to how the uncertain reasoner's nightmare can be turned into a basis for a robust framework for uncertainty quantification and decision-making.

[1] A position which awarded the head of the CBE the Financial Time's person of the Year 2012 (FT, 13 December 2012).

Multiple iterated belief revision for ranking functions

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Belief revision has been shaped predominantly by the so-called AGM theory, named after a seminal paper by Alchourron, Gärdenfors, and Makinson in 1985 that set up a framework of rationality postulates for reasonable belief change when the prior knowledge is a deductively closed set of propositions, and the new information comes in also as a proposition. This theory prepared the grounds on which the field of modern belief revision grew. However, limitations of the AGM theory became apparent soon. First, AGM theory deals with just one step of revision, not caring about further revisions in the future. So, the need for an extended framework also dealing with "iterated revision" became apparent soon and has been a topic of intense research since the nineties of the last century. Further problems which are caused not by the AGM approach itself but by the chosen framework of classical propositional logic have been discussed in the broad community only quite recently: How to change beliefs rationally if both prior knowledge and new information need richer semantical frameworks than propositional logic? What to do if multiple pieces of new information ("multiple revision") have to be integrated? In particular, this last problem has been ignored for a long time because in propositional logic, a set of propositions is equivalent to the conjunction of the propositions, i.e., in classical logic, one proposition can replace a set of propositions, so this case seemed to have been covered by AGM theory as well. However, counterintuitive examples showed that unsatisfactory belief sets result from this simplification.

In this talk, I will present an approach to belief revision from a broader point of view that offers quite natural methods for iterated revision and tackles the problem of multiple revision right from the beginning. This approach also takes the ideas of AGM as a starting point but investigates belief revision in richer epistemic structures like probabilities, or qualitative rankings. Therefore, it is compatible to AGM theory (and proposed extensions for iterated revision) in propositional logic but is not trapped by its limitations that are caused by the classical propositional view. I will explain how this approach unifies belief revision in different semantical frameworks and offers powerful approaches for belief revision even for very advanced scenarios, i.e., when an epistemic state has to be revised by a set of conditional beliefs. For the framework of Spohn's ranking functions, the talk will present a constructive and concise schema for multiple iterated revision that is evaluated with respect to well-known and also recently proposed postulates. Furthermore, some novel postulates for multiple iterated revision are proposed and discussed.

A functional equation involving (quasi-)copulas and their duals

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Quasi-copulas [2, 7] and copulas [17, 16] are special binary aggregation functions. They play a significant role in probability theory [10, 14], generalized integration theory [13], preference modeling [3], but also in fuzzy logics and the theory of fuzzy sets [4, 9, 15].

Recall that a (*binary*) *quasi-copula* $Q\colon [0,1]^2 \to [0,1]$ is a binary aggregation function on the unit interval $[0,1]$ which is 1-Lipschitz and has neutral element 1.

A (*binary*) *copula* $C\colon [0,1]^2 \to [0,1]$ is a supermodular quasi-copula, i.e., for all $\mathbf{x}, \mathbf{y}[0,1]^2$

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y}).$$

Given a binary 1-Lipschitz aggregation function $A\colon [0,1]^2 \to [0,1]$, its *dual* $A^*\colon [0,1]^2 \to [0,1]$ is defined [8] by

$$A^*(x,y) = x + y - A(x,y).$$

Observe that, for each quasi-copula Q, 1 is not a neutral element od Q^*, so the dual of a quasi-copula is never a quasi-copula (nor is the dual of a copula a copula). For a copula C, also the *co-copula* [1, 16] $\overline{C}\colon [0,1]^2 \to [0,1]$ given by

$$\overline{C}(x,y) = 1 - C(1-x, 1-y)$$

is considered. Note that $\overline{C}$ is never a copula.

A copula C is said to be *ultramodular* [11, 12] on the upper left triangle $\Delta = \{(x,y) \in [0,1]^2 \mid x \leq y\}$ if for all $x, y, \alpha, \beta, \gamma, \delta \in [0,1]$ which satisfy $\{(x,y), (x+\alpha, y+\gamma), (x+\beta, y+\delta\} \subseteq \Delta$ and $(x+\alpha+\beta, y+\gamma+\delta) \in [0,1]^2$:

$$C(x+\alpha+\beta, y+\gamma+\delta) - C(x+\alpha, y+\gamma) \geq C(x+\beta, y+\delta) - C(x,y).$$

We also will consider copulas C which are *Schur concave* [18, 5] on the upper left triangle Δ, i.e., for each $x]0,1[$ and for all $(\alpha,\beta) \in \Delta \cap \left[0, \min(x, 1-x^2)\right]$ we have

$$C(x-\alpha, x+\alpha) \leq C(x-\beta, x+\beta).$$

The well-known *Frank functional equation* [6] can be formulated as follows: find all associative copulas F and G such that $F = (\overline{G})^*$.

We are interested in functional equations of the type

$$S = R(Q, Q^*),$$

i.e., $S(x,y) = R(Q(x,y), Q^*(x,y))$ for all $(x,y) \in [0,1]^2$, where Q, R and S are (quasi-) copulas.

Proposition 1. *Let Q and R be two binary quasi-copulas. Then $R(Q,Q^*)$ is a quasi-copula.*

Looking at the lower and upper Fréchet-Hoeffding bounds W and M, and at the product copula Π given by, respectively

$$W(x,y) = \max(x+y-1, 0), \qquad M(x,y) = \min(x,y), \qquad \Pi(x,y) = x \cdot y,$$

it is not difficult to show:

Proposition 2. *For all binary quasi-copulas Q, R and for all binary copulas C, D we have:*

(i) *$R(Q,Q^*) \leq Q$ and $D(C,C^*) \leq C$;*
(ii) *$M(Q,Q^*) = Q$;*
(iii) *$R(W,W^*) = W$;*
(iv) *if D is a symmetric copula then $D(M,M^*) = D$;*
(v) *$\Pi(Q,Q^*)$ is a quasi-copula and $\Pi(C,C^*)$ is a copula;*
(vi) *if we put $Q_1 = \Pi(Q,Q^*)$ and $Q_{n+1} = \Pi(Q_n, Q_n^*)$ for each $n \in \mathbb{N}$ then*

$$\lim_{n\to\infty} Q_n = W.$$

Proposition 3. *Let C be a binary copula and let D be a binary copula which is ultramodular and Schur concave on the upper left triangle Δ. Then $D(C,C^*)$ is a copula.*

We also provide examples which illustrate the importance of the hypotheses in Proposition 2(iv) and in Proposition 3:

(a) an asymmetric copula D and a copula C such that $D(C,C^*)$ is not a copula;
(b) a copula D which is ultramodular on Δ (but not Schur concave on Δ) and a copula C such that $D(C,C^*)$ is not a copula;
(c) a copula D which is Schur concave on Δ (but not ultramodular on Δ) and a copula C such that $D(C,C^*)$ is not a copula.

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Probabilities in Łukasiewicz logic

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The history of many-valued logics begins with the 3-valued system defined by Łukasiewicz, which was extended to a ∞-valued system $\mathcal{L}_\infty$ by Łukasiewicz and Tarski in 1930 [11]. The evolution of Łukasiewicz logic is strongly connected with its algebraic counterpart: the theory of MV-algebras.

MV-algebras were defined by Chang [2] and we refer to [3] for a comprehensive study of their general theory. One can see [17] for advanced topics. MV-algebras are structures $(A, \oplus, \neg, 0)$ of type $(2,1,0)$ and they stand to Łukasiewicz propositional logic as boolean algebras stand to classical logic. The standard MV-algebra is the real unit interval $[0,1]$ equipped with the following operations: $\neg x = 1 - x$ and $x \oplus y = (x+y) \wedge 1$ for any x, $y \in [0,1]$. Chang's completeness theorem states that an equation is satisfied in all MV-algebras if and only if it is satisfied in the MV-algebra $[0,1]$. The theory of MV-algebras was highlighted by Mundici's categorical equivalence between MV-algebras and abelian lattice-ordered groups with strong unit [13].

MV-algebras are twofold structures, generalizations of boolean algebras and unit intervals of lattice-ordered groups with strong unit. *The theory of states* is the MV-algebraic correspondent of the boolean probability theory, being also intimately related with the theory of states defined on lattice-ordered groups.

If A is an MV-algebra, a *state* [14] is a function $s : A \to [0,1]$ such that $s(\neg 0) = 1$ and $s(x \oplus y) = s(x) + s(y)$ whenever $x \leq \neg y$ for any x, $y \in A$. When A is a free algebra, this notion captures the average truth-degree of a formula in Łukasiewicz logic. A probability MV-algebra is a pair (A, s), where A is a σ-complete MV-algebra and s is a σ-continuous faithful state. Probability MV-algebras were introduced and studied by Mundici and Riečan in [19]. Further major results on states are the Kroupa-Panti representation theorem [9, 18] and the generalization of de Finetti's coherence criterion [15].

We present an overview of the MV-algebraic theory of states, focusing on the following issues.

– *The notion of conditional probability.*
  We are aiming to summarize the present approaches of the conditional probabilities in Łukasiewicz logic [1, 8, 6, 16, 12] and to explore the notion of conditional expectation. An open problem raised by Mundici and Riečan [19] was to generalize the theory of "stochastically independent" algebras to probability MV-algebras. We proposed a solution to this problem using a categorical duality between a subclass of MV-algebras and a particular class of topological measure spaces [10]. The con-

cept of stochastic independence is closely related to the definition of conditional probability and we intend to further analyze the consequences of this approach.

- *The study of hyperreal states.*
  Any MV-algebra is, up to isomorphism, the Lindenbaum algebra $L(\Theta)$ determined by a theory Θ of $Ł_\infty$. As a consequence of Di Nola's representation theorem [5], $L(\Theta)$ is an algebra of ${}^*[0,1]$-valued functions, where ${}^*[0,1]$ is the non-standard real unit interval. If Θ satisfies the strong completeness theorem, then $L(\Theta)$ is an algebra of $[0,1]$-valued functions, so it contains no infinitesimals. For arbitrary Θ, if s is a state on $L(\Theta)$, then $s(\tau) = 0$ whenever τ is an infinitesimal. Hence, states on MV-algebras "ignore" the infinitesimals. In order to overcome this problem a notion of *hyperreal state* is proposed in [7]. An important class of non-semisimple structures are the *lexicographic MV-algebras*, i.e. those MV-algebras that correspond to lexicographic products $H \otimes_{lex} G$, where (H,u) is an abelian totally-ordered group with strong unit and G is an arbitrary abelian lattice-ordered group. These structures are introduced in [4], where an appropriate notion of ${}^*[0,1]$-valued state is also proposed under the name of *lexicographic state*. We present the main representation theorems, both for lexicographic MV-algebras and lexicographic states.

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On fuzzy power algebras and compatible fuzzy relations

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The subject of our investigation is the powering of binary fuzzy relations and their relationship with fuzzy power algebras. In this paper, we assume that the structure of truth values is a complete residuated lattice.

The notion of a fuzzy power algebra (see [3, 4]) can be defined using the Zadeh's extension principle ([6]) in two ways. If the induced operations are defined by the cartesian product, we get fuzzy power algebras $\mathcal{F}^+(\mathcal{A})$, studied in [4]. If the definition of a fuzzy power algebra rely on the tensor product, we obtain fuzzy power algebras $\mathcal{F}^*(\mathcal{A})$, investigated in [5]. In [3] it is demonstrated that these two kinds of power algebras behave differently with respect to homomorphisms and direct products.

In [4] fuzzy power algebras $\mathcal{F}^+(\mathcal{A})$ are studied in the framework of fuzzy set theory based on a continuous t-norm. Three ways of lifting binary fuzzy relations from a set X to the set $\mathcal{F}(X)$ of all fuzzy subsets of X are defined and the properties of these constructions are studied. The notions of good, Smyth good, Hoare good and very good fuzzy relations are introduced and some connections between them are established, generalizing some results from [1, 2]. In [5] these fuzzy power constructions are studied in the case when the structure of truth values is a complete residuated lattice and the fuzzy power algebras are defined by the tensor product.

The main aim of the present paper is to answer some questions posed in [4] and to clarify the difference between the two kinds of fuzzy power algebras. For example we prove the following:

Theorem 1. *Let $\mathcal{L}$ be a complete residuated lattice. The following conditions are equivalent:*

(a) $\mathcal{L}$ is a Heyting algebra.
(b) $(R \circ Q)^+ = R^+ \circ Q^+$ for all binary $\mathcal{L}$-relations R and Q.

Theorem 2. *Let $\mathcal{L}$ be a complete residuated lattice. The following conditions are equivalent:*

(a) $\mathcal{L}$ is a Heyting algebra.
(b) For any algebra $\mathcal{A}$ and any $\mathcal{L}$-relation R on A, if R is a congruence on $\mathcal{A}$, then R^+ is a congruence on the fuzzy power algebra $\mathcal{F}^+(\mathcal{A})$.

Theorem 3. *Let $\mathcal{L}$ be a complete residuated lattice, $\mathcal{A}$ an algebra and R a binary $\mathcal{L}$-relation on A.*

(a) *If R is a fuzzy preorder and $R^{\rightarrow}$ is a good relation on $\mathcal{F}^+(\mathcal{A})$, then R is compatible on $\mathcal{A}$.*
(b) *If R is a fuzzy preorder and $R^{\leftarrow}$ is a good relation on $\mathcal{F}^+(\mathcal{A})$, then R is compatible on $\mathcal{A}$.*

Theorem 4. *Let $\mathcal{L}$ be a complete residuated lattice. The following conditions are equivalent:*

(a) *$\mathcal{L}$ is a Heyting algebra.*
(b) *For any algebra $\mathcal{A}$, every $\wedge$-compatible fuzzy preorder on $\mathcal{A}$ is $\wedge$-Hoare good.*
(c) *For any algebra $\mathcal{A}$, every $\wedge$-compatible fuzzy preorder on $\mathcal{A}$ is $\wedge$-Smyth good.*

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On the roots of fuzzy logic with evaluated syntax

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In this contribution, we will return to the roots of fuzzy logic with evaluated syntax ($Ev_Ł$). We will remember the foundational role of Jan Pavelka ([7]). First we recall the main concepts in the metamathematics of this logic. The basic results in the propositional $Ev_Ł$ including the completeness theorem were obtained by J. Pavelka. Its extension to the predicate version was done by V. Novák in [4] and the whole theory was elaborated in detail in the book [6].

The starting point for the development of $Ev_Ł$ is to accept that axioms need not be fully convincing that is, we cannot take them as fully true. The canonical example are axioms of the sorites theory (cf. [2, 6]) where we cannot take as fully true that, for any n, "if n stones do not form a heap then $n+1$ stones do not form it as well". This naturally leads to the concept of *evaluated formula* a/A where A is a formula and $a \in L$ is its evaluation and L is the set of truth values. Let us emphasize that the evaluation is given from outside and does not belong to the syntax.

An important principle applied in $Ev_Ł$ is the *principle of maximality*. Namely, we always want to obtain as high evaluation of formulas as possible. Hence, if we find a formula A evaluated by more values, i.e., we have more evaluated formulas a_i/A, $i \in J$ at disposal, then we consider only the highest evaluation of A which means that the resulting evaluated formula is $(\bigvee_{i\in J} a_i)/A$.

Further step is to introduce special many-valued inference rules that manipulate with evaluated formulas. An n-ary many-valued inference rule r is a scheme

$$r : \frac{a_1/A_1, \dots, a_n/A_n}{r^{evl}(a_1,\dots,a_n)/r^{syn}(A_1,\dots,A_n)}, \tag{1}$$

where r^{syn} is a syntactic and r^{evl} an evaluation part of the rule r. Finally, we consider the concept of *fuzzy theory* — a fuzzy set of formulas obtained from the fuzzy set of axioms using many-valued inference rules.

A fuzzy set $V \underset{\sim}{\subset} F_J$ (F_J is the set of all formulas of the language J) is *closed* with respect to r if

$$V(r^{syn}(A_1,\dots,A_n)) \geq r^{evl}(V(A_1),\dots,V(A_n)) \tag{2}$$

holds for all formulas $A_1,\dots,A_n \in \mathrm{dom}(r^{syn})$.

Let R be a set of inference rules. Then the fuzzy set of *syntactic consequences* of a fuzzy set X of formulas is a fuzzy set of formulas with the membership function

$$\mathscr{C}^{syn}(X)(A) = \bigwedge\{V(A) \mid V \underset{\sim}{\subset} F_J, X \leq V \text{ and } V \text{ is closed w.r.t. to all } r \in R\}. \tag{3}$$

An *evaluated formal proof* of a formula A from the fuzzy set $X \underset{\sim}{\subset} F_J$ is a finite sequence of evaluated formulas

$$w := {}^{a_0}\!/A_0,\ {}^{a_1}\!/A_1, \ldots,\ {}^{a_n}\!/A_n \tag{4}$$

such that $A_n := A$ and for each $i \leq n$, ${}^{a_i}\!/A_i$ is either an axiom or it was derived from some previous formulas using an inference rule. The evaluation a_n is *value* of the proof w and is denoted by $\mathrm{Val}(w)$.

Theorem 1 (J. Pavelka).

$$\mathscr{C}^{syn}(X)(A) = \bigvee\{\mathrm{Val}(w_A) \mid w_A \textit{ is a proof of } A \textit{ from } X\}. \tag{5}$$

The proof of this theorem uses the principle of maximality. With respect to this theorem, the degree (3) is called the *provability degree* of the formula A in a fuzzy theory T determined by the fuzzy set of axioms X. In correspondence with the classical notation, we write $T \vdash_a A$ where $a = \mathscr{C}^{syn}(X)(A)$.

Semantics of $\mathrm{Ev}_{Ł}$ is many-valued and defined in a standard way. The fuzzy set of *semantic consequences* of $X \underset{\sim}{\subset} F_J$ is given by the membership function

$$\mathscr{C}^{sem}(X)(A) = \bigwedge\{\mathscr{M}(A) \mid \text{for all truth valuations } \mathscr{M} : F_J \longrightarrow L, X \leq \mathscr{M}\}. \tag{6}$$

In correspondence with the classical notation, we write $T \models_a A$ where $a = \mathscr{C}^{sem}(X)(A)$ and call a the *truth degree* of A in T.

The $\mathrm{Ev}_{Ł}$ is *complete* if $\mathscr{C}^{syn}(X) = \mathscr{C}^{sem}(X)$ for all $X \underset{\sim}{\subset} F_J$. The following theorem holds true for any formal logical system with evaluated syntax based on a complete residuated lattice.

Theorem 2 (J. Pavelka). *If interpretation $\rightarrow$ of the logical implication $\Rightarrow \in J$ does not fulfil the equations*

$$\bigvee_{b\in I}(a \rightarrow b) = a \rightarrow (\bigvee_{b\in I} b), \qquad a \in L, \tag{7}$$

$$\bigvee_{a\in I}(a \rightarrow b) = (\bigwedge_{a\in I} a) \rightarrow b \qquad b \in L, \tag{8}$$

$$\bigwedge_{b\in I}(a \rightarrow b) = a \rightarrow (\bigwedge_{b\in I} b) \qquad a \in L, \tag{9}$$

$$\bigwedge_{a\in I}(a \rightarrow b) = (\bigvee_{a\in I} a) \rightarrow b \qquad b \in L, \tag{10}$$

for arbitrary subset of $I \subseteq L$ then such a system cannot be complete.

It follows from this theorem that $\mathrm{Ev}_{Ł}$ based on $L = [0, 1]$ is limited to Łukasiewicz implication (or its isomorphs) and so, the Łukasiewicz logic lays in the core of $\mathrm{Ev}_{Ł}$. Namely, it can be represented inside $\mathrm{Ev}_{Ł}$and by [3], $\mathrm{Ev}_{Ł}$ is a conservative extension of the Łukasiewicz logic.

The following two theorems written as generalization of the Gödel completeness theorems for classical first-order logic have algebraic proof extending the original Pavelka's completeness proof for the propositional $\text{Ev}_{Ł}$ given by Novák and also two syntactical proofs given by Hájek and independently by Novák.

A fuzzy theory T is *contradictory* if there is a formula A and proofs w_A of A and $w_{\neg A}$ of $\neg A$ such that

$$\text{Val}(w_A) \otimes \text{Val}(w_{\neg A}) > 0.$$

Otherwise it is *consistent*.

Theorem 3 (Completeness theorem II). *A fuzzy theory T is consistent iff it has a model.*

Theorem 4 (Completeness theorem I).

$$T \vdash_a A \quad \textit{iff} \quad T \models_a A$$

holds for every formula $A \in F_J$ and every consistent fuzzy theory T.

Besides completeness, we will in this contribution overview the main results obtained in $\text{Ev}_{Ł}$. Among them, we will also address one of the discussed problems of $\text{Ev}_{Ł}$, namely the necessity to introduce in the language J truth constants (i.e. constants representing truth values) for all elements of L. This means that in case of $L = [0,1]$, the language J is uncountable. Novák and also Hájek showed that this is unnecessary and that only countable number of logical constants is sufficient (cf. [1, 5, 6]).

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Fuzzy plane geometry modeled by linear fuzzy space and applications

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Identification of basic geometric features, such as lines and other objects (triangles, circles), from digital raster image is one of fundamental processes in image analysis. Motivated with this problem we have introduced the notion of linear fuzzy space [5–7] based on the model of fuzzy imprecise point, see also [1, 8, 9].

Definition 1. *Fuzzy point $P \in \mathbb{R}^2$, denoted by $\tilde{P}$ is defined by its membership function $\mu_{\tilde{P}} \in \mathcal{F}^2$, where the set $\mathcal{F}^2$ contains all membership functions $\mu : \mathbb{R}^2 \to [0,1]$ satisfying following conditions:*

(i) $(\forall \mu \in \mathcal{F}^2)(\exists_1 P \in \mathbb{R}^2) \quad \mu(P) = 1$,
(ii) $(\forall X_1, X_2 \in \mathbb{R}^2)(\lambda \in [0,1]) \quad \mu(\lambda X_1 + (1-\lambda)X_2) \geqslant \min(\mu(X_1), \mu(X_2))$,
(iii) *function μ is upper semi continuous,*
(iv) $[\mu]^\alpha = \{X \mid X \in \mathbb{R}^2, \mu(X) \geqslant \alpha\}$ *α-cut of function μ is convex.*

A point from $\mathbb{R}^2$, with membership function $\mu_{\tilde{P}}(P) = 1$, will be denoted by P (P is the core of the fuzzy point $\tilde{P}$), and the membership function of the point $\tilde{P}$ will be denoted by $\mu_{\tilde{P}}$. By $[P]^\alpha$ we denote the α-cut of the fuzzy point.

Definition 2. *Linear fuzzy space is the set $\mathcal{H}^2 \subset \mathcal{F}^2$ of all functions which, in addition to the properties given in Definition 1, are:*

(i) *Symmetric with respect to the core $S \in \mathbb{R}^2$ ($\mu(S) = 1$),*

$$\mu(V) = \mu(M) \wedge \mu(M) \neq 0 \Rightarrow d(S,V) = d(S,M),$$

where $d(S,M)$ is the distance in $\mathbb{R}^2$.
(ii) *Inverse-linear decreasing w.r.t. pointsÂŠ distance from the core according to:*

$$\mu_{\tilde{S}}(V) = \max(0, 1 - \frac{d(S,V)}{|r|}), \quad \text{if } r \neq 0,$$

$$\mu_{\tilde{S}}(V) = \begin{cases} 1 & \text{if } S = V \\ 0 & \text{if } S \neq V, \end{cases} \quad \text{if } r = 0,$$

where $d(S,V)$ is the distance between the point V and the core S ($V,S \in \mathbb{R}^2$) and $r \in \mathbb{R}$ is a constant.

Elements of that space are represented as ordered pairs $\tilde{S} = (S, r_S)$ where $S \in \mathbb{R}^2$ is the core of $\tilde{S}$, and $r_S \in \mathbb{R}$ is the distance from the core for which the function value becomes 0; in the sequel parameter r_S will be denoted as *fuzzy support radius*.

We have introduced basic operations over linear fuzzy space $\mathcal{H}^2$ defined on $\mathbb{R}^2$, and we proved their properties which are used in definitions of basic fuzzy plane geometry objects, see [5, 6]. Based on our results [5] we have introduced a mathematical model of fuzzy line, fuzzy triangle and fuzzy circle. We give here only the definition of the fuzzy line.

Definition 3. *Let $\mathcal{H}^2$ be a linear fuzzy space and function $f : \mathcal{H}^2 \times \mathcal{H}^2 \times [0,1] \to \mathcal{H}^2$ is a linear combination of the fuzzy points $\tilde{A}$ and $\tilde{B}$, i.e.,*

$$f(\tilde{A}, \tilde{B}, u) = \tilde{A} + u \cdot (\tilde{B} - \tilde{A}),$$

where $u \in [0,1]$. Then a fuzzy set $\widetilde{AB}$ given by

$$\widetilde{AB} = \bigcup_{u \in [0,1]} f(\tilde{A}, \tilde{B}, u)$$

is called fuzzy line.

We have defined basic spatial relations: coincidence, between and collinear, see [6].

Definition 4. *Let λ be the Lebesgue measure on the set $[0,1]$ and $\mathcal{H}^2$ is a linear fuzzy space. A fuzzy relation $coin : \mathcal{H}^2 \times \mathcal{H}^2 \to [0,1]$ is called fuzzy coincidence represented by the following membership function*

$$\mu_{coin}(\tilde{A}, \tilde{B}) = \lambda(\{\alpha \mid [\tilde{A}]^\alpha \cap [\tilde{B}]^\alpha \neq \emptyset\}).$$

Since the lowest α is 0, a membership function of the *fuzzy coincidence* is given by

$$\mu_{coin}(\tilde{A}, \tilde{B}) = \max\{\alpha \mid [\tilde{A}]^\alpha \cap [\tilde{B}]^\alpha \neq \emptyset\}.$$

Theorem 1. *The membership function of the fuzzy relation fuzzy coincidence is determined according to the following formula*

$$\mu_{coin}(\tilde{A}, \tilde{B}) = \begin{cases} 0 & \text{if } |r_A| + |r_B| = 0 \wedge d(A,B) \neq 0, \\ \max(0, 1 - \frac{d(A,B)}{|r_A| + |r_B|}) & \text{if } |r_A| + |r_B| \neq 0, \\ 1 & \text{if } |r_A| + |r_B| = 0 \wedge d(A,B) = 0. \end{cases}$$

Definition 5. *Let λ be Lebesgue measure on the set $[0,1]$, $\mathcal{H}^2$ linear fuzzy space and $\mathcal{L}^2$ be set of all fuzzy lines defined on $\mathcal{H}^2$. Then fuzzy relation $contain : \mathcal{H}^2 \times \mathcal{L}^2 \to [0,1]$ is fuzzy contain represented by following membership function*

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \lambda(\{\alpha \mid [\tilde{A}]^\alpha \cap [\widetilde{BC}]^\alpha \neq \emptyset\}).$$

Definition 6. *Let $\tilde{A}, \tilde{B} \in \mathcal{H}^2$ and $\tilde{A} \neq \tilde{B}$. Then a point $T_{AB} \in \mathbb{R}^2$ is called internal homothetic center if the following holds*

$$T_{AB} = A + \frac{r_A}{r_A + r_B}(B - A),$$

where $\tilde{A} = (A, r_A)$ and $\tilde{B} = (B, r_B)$.

Theorem 2. *Let $\tilde{A}, \tilde{B}, \tilde{C} \in \mathcal{H}^2$, $u \in [0,1]$ and $\widetilde{A'}$ be fuzzy image of point A on fuzzy line $\widetilde{BC}$. If points T_{AB} and T_{AC} are internal homothetic center fuzzy points for fuzzy points $\tilde{A}$ and $\tilde{B}$ and $\tilde{A}$ and $\tilde{C}$ respectively. Then the membership function of the fuzzy relation fuzzy contain is determined according to the following formula*

$$\mu_{contain}(\tilde{A}, \widetilde{BC}) = \begin{cases} \mu_{coin}(\tilde{A}, \widetilde{A'}) & \text{if } u \in \{0,1\} \\ \mu_{\tilde{A}}(A^*) & \text{if } u \in (0,1), \end{cases}$$

where point A^* is a projection of the core of $\tilde{A}$ on the line passing through the points T_{AB} and T_{AC}.

Fuzzy points are used to describe the position of a real object when there is some uncertainty to the measured position. Most often this uncertainty in practical applications is ignored. If the points that represent the path are imprecise, then the whole line should be described in way similar to imprecise points description. Real-world objects are mapped to the digital raster image through a variety of sensors, making the image only an approximation to the real-world object. Due to imperfections in either the image data or the edge detector, there may be missing points or pixels on lines as well as spatial deviations between ideal line and the set of imprecise points obtained from the edge detector. The overall effect is an image that has some distortion in its geometry. The proposed models of imprecise line objects could be used in various applications, such as image analysis (imprecise feature extraction), GIS (imprecise spatial object modeling) [4, 10] and robotics (environment models). In [6] we have used fuzzy line as model of the road lane. The algorithm for lane detection is primarily based on fuzzy spatial relations introduced by this work, and it is characterized by reduced computational complexity versus the standard Hough transformation [3]. Further applications are obtained in medicine in the interpretation of DICOM medical images [2].

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Tense operators on Q-effect algebras

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Abstract. For effect algebras, the so-called tense operators were already introduced by Chajda and Paseka. They presented also a canonical construction of them using the notion of a frame.
Tense operators express the quantifiers "it is always going to be the case that" and "it has always been the case that" and hence enable us to express the dimension of time in the logic of quantum mechanics.
A crucial problem concerning tense operators is their representation. Having an effect algebra with tense operators, we can ask if there exists a frame such that each of these operators can be obtained by the canonical construction. Introducing the notion of a q-effect algebra we solve this problem for E-tense operators on E-representable E-Jauch-Piron q-effect algebras.

1 Preliminaries and basic facts

Effect algebras were introduced by Foulis and Bennett [7] as an abstraction of the Hilbert space effects which play an important role in the logic of quantum mechanics. However, this notion does not corporate the dimension of time.

This means that effect algebras can serve to describe the states of effects in a given time but they cannot reveal what these effects expressed in the past or what they will reveal in the next time.

By an *effect algebra* is meant a structure $\mathcal{E} = (E; +, 0, 1)$ where 0 and 1 are distinguished elements of E, $0 \neq 1$, and $+$ is a partial binary operation on E satisfying the following axioms for $x, y, z \in E$:

(E1) if $x + y$ is defined then $y + x$ is defined and $x + y = y + x$
(E2) if $y + z$ is defined and $x + (y + z)$ is defined then $x + y$ and $(x + y) + z$ are defined and $(x + y) + z = x + (y + z)$
(E3) for each $x \in E$ there exists a unique $x' \in E$ such that $x + x' = 1$; x' is called a *supplement* of x
(E4) if $x + 1$ is defined then $x = 0$.

Having an effect algebra $\mathcal{E} = (E; +, 0, 1)$, we can introduce the *induced order* $\leq$ on E and the partial operatin $-$ as follows

$$\begin{array}{l} x \leq y \quad \text{if for some } z \in E \quad x + z = y, \\ \text{and in this case } z = y - x \end{array}$$

(see e.g. [5] for details). Then $(E; \leq)$ is an ordered set and $0 \leq x \leq 1$ for each $x \in E$.

It is worth noticing that $a+b$ exists in an effect algebra $\mathcal{E}$ if and only if $a \leq b'$ (or equivalently, $b \leq a'$). This condition is usually expressed by the notation $a \perp b$ (we say that a, b are orthogonal). Dually, we have a partial operation $\cdot$ on E such that $a \cdot b$ exists in an effect algebra $\mathcal{E}$ if and only if $a' \leq b$ in which case $a \cdot b = (a' + b')'$. This allows us to equip E with a dual effect algebraic operation such that $\mathcal{E}^{op} = (E; \cdot, 1, 0)$ is again an effect algebra, ${}'^{\mathcal{E}^{op}} = {}'^{\mathcal{E}} = {}'$ and $\leq_{\mathcal{E}^{op}} = \leq^{op}$.

Let $d, q : E \to E$ be maps such that, for all $x, y, z \in E$,

(Q1) $d(x') = q(x)'$,
(Q2) $d(0) = 0 = q(0)$,
(Q3) d is order-preserving,
(Q4) $x' \leq x$ implies $x \cdot x = d(x)$,
(Q5) $z \leq x, z \leq y \leq x'$ imply $d(z) \leq x \cdot y$,
(Q6) $d(x) \leq x$.

We then say that $\mathcal{E} = (E; +, q, d, 0, 1)$ is a *q-effect algebra*. Note that a dual of $\mathcal{E} = (E; +, q, d, 0, 1)$ is a q-effect algebra $\mathcal{E}^{op} = (E; \cdot, d, q, 1, 0)$.

A *morphism of effect algebras* (*morphism of q-effect algebras*) is a map between them such that it preserves the partial operation $+$, (and the unary operations q and d),the bottom and the top elements. In particular, $' : \mathcal{E} \to \mathcal{E}^{op}$ is a morphism of effect algebras (morphism of q-effect algebras).

A map $s : E \to [0,1]$ is called a *state* (*an E-state*) on $\mathcal{E}$ if $s(0) = 0$, $s(1) = 1$, $(s(q(x)) = s(x) + \min(s(x'), s(x)$, $s(d(x)) = 1 - s(x') - \min(s(x'), s(x))$ and $s(x+y) = s(x) + s(y)$ whenever $x+y$ exists in $\mathcal{E}$.

A *morphism* $f : P_1 \to P_2$ *of bounded posets* is an order, top element and bottom element preserving map. Any morphism of effect algebras is a morphism of corresponding bounded posets. A morphism $f : P_1 \to P_2$ of bounded posets is *order reflecting* if $(f(a) \leq f(b)$ if and only if $a \leq b)$ for all $a, b \in P_1$.

If, moreover $(E; \leq)$ is a lattice (with respect to the induced order), then $\mathcal{E}$ is called a *lattice effect algebra*. On any lattice effect algebra $\mathcal{E}$ we may introduce total operations $\oplus$ and $\odot$ as follows: $x \oplus y = x + (y \wedge x')$ and $x \odot y = (x' \oplus y')'$. Note that a lattice effect algebra $\mathcal{E}$ is an MV-algebra (see [3]) with respect to the operations $\oplus$ and $'$ if and only if $x \wedge y = 0$ implies $x \leq y'$. In this case the unary operations $q(x) = x \oplus x$ and $d(x) = x \odot x$ satisfy the conditions (Q1)-(Q5) and $\mathcal{E} = (E; +, q, d, 0, 1)$ is a q-effect algebra. Moreover, any morhism of MV-algebras is a morphism of q-effect algebras.

In what follows, motivated by the above situation, we will always use for q-effect algebras the notation $\mathcal{E} = (E; +, \oplus, \odot, 0, 1)$ such that $\oplus(x) = x \oplus x$ and $\odot(x) = x \odot x$.

Tense operators on q-effect algebras

Let $\mathcal{E} = (E; +, 0, 1)$ be an effect algebra. Unary operators G and H on $\mathcal{E}$ are called *partial tense operators* if they are are partial mappings of E into itself satisfying the following axioms:

(T1) $G(0) = H(0) = 0$, $G(1) = H(1) = 1$,
(T2) $x \leq y$ implies $G(x) \leq G(y)$ whenever $G(x), G(y)$ exist and $H(x) \leq H(y)$ whenever $H(x), H(y)$ exist

(T3) if $x+y$ and $G(x), G(y), G(x+y)$ exist then $G(x)+G(y)$ exists and $G(x)+G(y) \leq G(x+y)$ and if $x+y$ and $H(x), H(y), H(x+y)$ exist then $H(x)+H(y)$ exists and $H(x)+H(y) \leq H(x+y)$

(T4) $x \leq GP(x)$ if $H(x')$ exists, $P(x) = H(x')'$ and $GP(x)$ exists, $x \leq HF(x)$ if $G(x')$ exists, $F(x) = G(x')'$ and $HF(x)$ exists.

If both G and H are total (i.e., G and H are mappings of E into itself defined for each $x \in E$) then G and H are called *tense operators* and P (or F) is a left adjoint to G (or H, respectively) (see [2]).

It is quite natural to ask that our (total) tense operators on q-effect algebras preserve unary operations $\oplus$ and $\odot$ (see [4]). This can be accomplished by the following axioms:

(T5) $G(x \oplus x) = G(x) \oplus G(x)$, $H(x \oplus x) = H(x) \oplus H(x)$,

(T6) $G(x \odot x) = G(x) \odot G(x)$, $H(x \odot x) = H(x) \odot H(x)$.

We call such tense operators G and H *tense E-operators*. The main aim of our paper is to establish a representation theorem for tense E-operators.

E-semi-states on q-effect algebras

Definition 1. *Let $\mathcal{E} = (E; +, \oplus, \odot, 0, 1)$ be a q-effect algebra. A map $s : E \to [0,1]$ is called*

1. an E-semi-state on $\mathcal{E}$ *if*
   (i) $s(0) = 0, s(1) = 1$,
   (ii) $s(x) + s(y) \leq s(x+y)$ *whenever $x+y$ is defined,*
   (iii) $s(x) \odot s(x) = s(x \odot x)$,
   (iv) $s(x) \oplus s(x) = s(x \oplus x)$,
2. a Jauch-Piron E-semi-state on $\mathcal{E}$ *if s is an E-semi-state and*
   (v) $s(x) = 1 = s(y)$ *implies* $s(x \wedge y) = 1$*;*

Definition 2. *Let $\mathcal{E} = (E; +, \oplus, \odot, 0, 1)$ be a q-effect algebra.*

(a) If S is an order reflecting set of E-states on $\mathcal{E}$ then $\mathcal{E}$ is said to be E-representable.
(b) If S is an order reflecting set of Jauch-Piron E-states on $\mathcal{E}$ then $\mathcal{E}$ is said to be E-Jauch-Piron representable.
(c) If any E-state is E-Jauch-Piron then $\mathcal{E}$ is called an E-Jauch-Piron q-effect algebra.

2 The representation of E-tense operators

In this section we outline the problem of a representation of E-tense operators H and H and we solve it for E-representable E-Jauch-Piron q-effect algebras. This means that we get a procedure how to construct a corresponding time frame (it will be the set of all Jauch-Piron E-states equipped with an induced relation ρ_G) to be in accordance with the canonical construction from [2].

By a *frame* is meant a couple (S,R) where S is a non-void sets and $R \subseteq S \times S$. For our sake, we will assume that for all $x \in S$ there are $y,z \in S$ such that xRy and zRx. Having a q-effect algebra $\mathcal{E} = (E;+,\oplus,\odot,0,1)$ and a non-void set T, we can produce the direct power $\mathcal{E}^T = (E^T;+,\oplus,\odot,o,j)$ where the operation $+$ and the induced operations $\vee$, $\wedge$, $\oplus$, $\odot$ are defined and evaluated on $p,q \in E^T$ componentwise. Moreover, o,j are such elements of E^T that $o(t) = 0$ and $j(t) = 1$ for all $t \in T$. The direct power $\mathcal{E}^T$ is again a q-effect algebra.

The notion of frame allows us to construct E-tense operators on q-effect algebras.

Theorem 1. *Let $\mathcal{M}$ be a linearly ordered complete MV-algebra, (S,R) be a frame, G^* and H^* be maps from M^S into M^S defined by*

$$\begin{aligned} G^*(p)(s) &= \bigwedge\{p(t) \mid t \in S, sRt\}, \\ H^*(p)(s) &= \bigwedge\{p(t) \mid t \in S, tRs\} \end{aligned}$$

for all $p \in M^S$ and $s \in S$. Then G^* (H^*) is an E-tense operator on M^S which has a left adjoint P^* (F^*). In this case, for all $q \in M^S$ and $t \in S$,

$$\begin{aligned} P^*(q)(t) &= \bigvee\{q(s) \mid s \in S, sRt\} \\ F^*(q)(t) &= \bigvee\{q(s) \mid s \in S, tRs\}. \end{aligned}$$

Now we are able to establish our main result which is a generalization of the main results from [1,8].

Theorem 2. *Let $\mathcal{E}$ be an E-representable E-Jauch-Piron q-effect algebra with an order reflecting set S of E-states and with tense E-operators G and H. Then $(\mathcal{E},G,H)$ can be embedded into the tense MV-algebra $([0,1]^S,G^*,H^*)$ induced by the frame (S,ρ_G), where S is the set of all Jauch-Piron E-states from $\mathcal{E}$ to $[0,1]$ and the relation ρ_G is defined by*

$$s\rho_G t \text{ if and only if } s(G(x)) \leq t(x) \text{ for any } x \in E.$$

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Automata-based reasoning in Fuzzy Description Logics

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Description logics (DLs) are a family of well-studied knowledge representation formalisms designed to express and reason with the conceptual knowledge of application domains in a clear and well-understood manner. They have been successfully applied for representing large application domains, most prominently from the biological and medical fields. In their classical form, DLs are not adequate for handling vague or imprecise knowledge, which is a common staple in bio-medical knowlege. To alleviate this problem, fuzzy extensions of DLs have been introduced. As a prototypical example, we consider here the smallest propositionally closed fuzzy DL, which we call $\otimes$-$\mathcal{ALC}$.[1]

The fuzzy DL $\otimes$-$\mathcal{ALC}$ is based on *concepts* and *roles*, which are interpreted as (fuzzy) unary and binary relations, respectively. Given the disjoint sets $\mathsf{N_R}$, and $\mathsf{N_C}$ of *role*, and *concept names*, respectively, $\otimes$-$\mathcal{ALC}$ *concepts* are built through the grammar rule

$$C ::= A \mid \bot \ \mid C \sqcap C \mid C \to C \mid \exists r.C \mid \forall r.C,$$

where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. The concept $\top$ is often used as an abbreviation of $\bot \to \bot$. The terminological knowledge of a domain is represented through a *TBox*: a finite set of *general concept inclusions* (GCIs) of the form $\langle C \sqsubseteq D \geq q \rangle$, where C, D are $\otimes$-$\mathcal{ALC}$-concepts, and $q \in [0,1]$.

The semantics of this logic is given through *interpretations*, which are pairs $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ where $\Delta^{\mathcal{I}}$ is a non-empty set called the *domain*, and $\cdot^{\mathcal{I}}$ is a function that maps every $A \in \mathsf{N_C}$ to a function $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$, and every $r \in \mathsf{N_R}$ to a function $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. Intuitively, for every domain element $x \in \Delta^{\mathcal{I}}$ the value $A^{\mathcal{I}}(x)$ represents the degree to which x is a member of A. The interpretation function is extended to arbitrary concepts using the continuous t-norm $\otimes$ and its (unique) residuum $\Rightarrow$. In the case of G-$\mathcal{ALC}$, where the semantics is based on the Gödel t-norm, complex concepts are interpreted as shown in Table 1.

The interpretation $\mathcal{I}$ is a *model* of the TBox $\mathcal{T}$ if for every GCI of the form $\langle C \sqsubseteq D \geq q \rangle \in \mathcal{T}$ and every $x \in \Delta^{\mathcal{I}}$, $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geq q$ holds. Reasoning tasks in fuzzy DLs are based on the class of models of a TBox. However, it is customary to further restrict this class allow only so-called *witnessed* models, where the suprema and infima stated by the semantics of the existential and value restrictions, respectively, are in fact maxima and minima. We keep this restriction, and for the rest of this paper call *witnessed models* simply *models* for brevity.

[1] Unfortunately, there is no agreed naming for fuzzy DLs. We use this name to emphasize the relationship with $\mathcal{ALC}$, the smallest propositionally closed classical DL.

Table 1: Semantics of G-$\mathcal{ALC}$

constructor	**syntax**	**semantics**
bottom concept	$\bot$	0
conjunction	$C \sqcap D$	$\min(C^I(x), D^I(x))$
implication	$C \to D$	$C^I(x) \Rightarrow D^I(x)$
existential restriction	$\exists r.C$	$\sup_{y \in \Delta^I} \min(r^I(x,y), C^I(y))$
value restriction	$\forall r.C$	$\inf_{y \in \Delta^I} r^I(x,y) \Rightarrow C^I(y)$

Most reasoning tasks in fuzzy DLs can be reduced to deciding the existence of a model that satisfies an additional set of restrictions, or *restricted consistency*. A *restriction* is an expression of the form $\langle C \rhd q \rangle$, where C is a concept, $q \in [0,1]$, and $\rhd \in \{\leq, \geq\}$. A finite set of restrictions $\mathcal{R}$ is *consistent* w.r.t. the TBox $\mathcal{T}$ if there is a model I of $\mathcal{T}$ and an element $x \in \Delta^I$ such that $C^I(x) \rhd q$ holds for every restriction $\langle C \rhd q \rangle \in \mathcal{R}$.

Restricted consistency and other associated reasoning tasks have been recently shown to be hard (even undecidable) for non-idempotent t-norms; i.e., any continuous t-norm that is not Gödel [5]. One culprit for this hardness is the fact that, for those t-norms, $\otimes$-$\mathcal{ALC}$ does not have the finite model property nor the *finitely-valued model property*. That is, there exist consistent restrictions that are only satisfied by infinite models that use infinitely many different membership degrees [3]. This fact is used to prove that the existence of such a model cannot be decided in finite time. Given the simplicity of the operators associated to the Gödel t-norm, it was generally believed that G-$\mathcal{ALC}$ has the finite model property. Moreover, it is often claimed that all reasoning tasks in this logic can be restricted to only a finite set of truth degrees, which can be computed *a priori*, depending only on the values explicitly provided in the input. This belief seems to arise from the results in [6] which, however, depend on different semantics.

Consider the set of restrictions $\mathcal{R} = \{\langle A \leq 0.6 \rangle\}$ and the TBox

$$\mathcal{T} = \{\langle \forall r.A \sqsubseteq A \geq 1 \rangle, \langle \exists r.\top \sqsubseteq A \geq 1 \rangle\}.$$

It is easy to see that $\mathcal{R}$ is consistent w.r.t. $\mathcal{T}$. For any model I of $\mathcal{T}$ that satisfies $\mathcal{R}$ there must exist an element $x_1 \in \Delta^I$ such that $A^I(x_1) < 0.6$. As I is witnessed, there exists a $x_2 \in \Delta^I$ with $(\forall r.A)^I(x_1) = r^I(x_1,x_2) \Rightarrow A^I(x_2)$. The first axiom of $\mathcal{T}$ entails $r^I(x_1,x_2) \Rightarrow A^I(x_2) \leq A^I(x_1) < 1$, and in particular $r^I(x_1,x_2) > A^I(x_2)$. The second axiom of the TBox $\mathcal{T}$ implies that

$$r^I(x_1,x_2) = \min(r^I(x_1,x_2),1) \leq (\exists r.\top)^I(x_1) \leq A^I(x_1),$$

and thus $A^I(x_1) > A^I(x_2)$. Repeating the same argument, there must exist elements $x_3, x_4, \ldots \in \Delta^I$ such that $A^I(x_i) > A^I(x_{i+1})$ for all $i \geq 1$. This means that any model of $\mathcal{T}$ satisfying the restriction $\mathcal{R}$ must have infinitely many elements that belong to the concept A to a different degree.

While it is not possible to explicitly construct a model that uses infinitely many membership degrees in finite time, we can still decide its existence by considering the

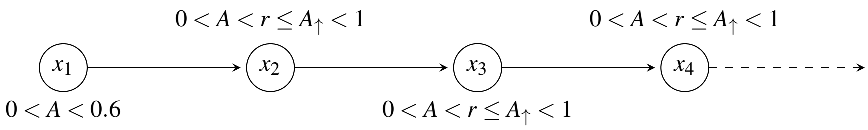

Fig. 1: An abstract description of models of $\mathcal{T}$ satisfying $\mathcal{R}$

local ordering relations between the membership degrees of all relevant concepts, at every node and its parents. As seen in the example above, it is possible to provide an abstract description of the models of interest through a preorder over all subconcepts and membership degrees explicitly appearing in the input TBox and set of restrictions. Figure 1 provides an abstract representation of all models of $\mathcal{T}$ that satisfy the restriction $\mathcal{R}$. In the figure, $A_{\uparrow}$ represents the membership degree of the parent node to A. As it can be seen, although the models of this TBox satisfying the restriction can be arbitrarily complex, they can all be represented using a very simple recurrent structure. In general, the existence of a model satisfying a set of restrictions can be characterised through *Hintikka trees*.

Consider the set $\mathcal{U} := \mathsf{V}_{\mathcal{T},\mathcal{R}} \cup \mathsf{sub}(\mathcal{T},\mathcal{R}) \cup \mathsf{sub}_{\uparrow}(\mathcal{T},\mathcal{R}) \cup \{\lambda\}$, where $\mathsf{V}_{\mathcal{T},\mathcal{R}}$ represents the set of all constants appearing in the input extended with $0, 1$, $\mathsf{sub}(\mathcal{T},\mathcal{R})$ is the set of all subconcepts from $\mathcal{T},\mathcal{R}$, and λ is an arbitrary new symbol. A *Hintikka order* is a total preorder $\lesssim$ over $\mathcal{U}$ that preserves the standard ordering of real numbers over $\mathsf{V}_{\mathcal{T},\mathcal{R}}$ and is consistent with the semantics of the propositional constructors. For example, if $X, C \sqcap D \in \mathcal{U}$ and $X \lesssim C \sqcap D$, then it must also hold that $X \lesssim C$ and $X \lesssim D$. All other cases can be treated similarly. Intuitively, a Hintikka ordering represents the relation between the membership degrees at a specific element of the domain of an interpretation. To ensure that it is a model, this ordering must also be *compatible* with the GCIs in the TBox; that is, for every $\langle C \sqsubseteq D \geq q \rangle \in \mathcal{T}$, either $C \lesssim D$ or $q \lesssim D$.

Existential and value restrictions are verified producing a sequence of successors that witness them. For each existential restriction $E = \exists r.C$ in the input, every node in the Hintikka tree has a distinguished successor $\phi(E)$. The Hintikka ordering associated with this node is required to satisfy $(\exists r.C)_{\uparrow} \equiv \min(\lambda, C)$, thus serving as a witness for the concept at the parent node. Moreover, for all other successors associated to a concept quantified over the same role r, the ordering must satisfy $\min(\lambda, C) \lesssim (\exists r.C)_{\uparrow}$. These conditions ensure that the semantics of existential restrictions are satisfied. Similar conditions guarantee the satisfaction of value restrictions $\forall r.C$.

A *Hintikka tree* for $\mathcal{T},\mathcal{R}$ is an infinite tree of constant arity where every node is labelled with a Hintikka ordering compatible with the TBox $\mathcal{T}$, the successors satisfy the transition conditions, and the root node satisfies the restrictions in $\mathcal{R}$. It can be shown that $\mathcal{R}$ is consistent w.r.t. $\mathcal{T}$ if and only if there is a Hintikka tree for $\mathcal{T},\mathcal{R}$. Notice moreover that there are only finitely many partial orderings over the set $\mathcal{U}$, and hence also finitely many Hintikka orderings. In fact, the number of Hintikka orderings is bounded exponentially by the size of the input.

To decide the existence of a Hintikka tree, we construct a simple looping automaton on (unlabeled) infinite trees. The set of Hintikka orderings defines the states of the automaton; the transition relation is determined by the transition conditions for quantified concepts; and the initial states are those that satisfy the input restrictions. Essentially, the successful runs of this automaton correspond to the Hintikka trees sought. Thus, the automaton has a successful run iff a Hintikka tree for $\mathcal{T}, \mathcal{R}$ exists iff $\mathcal{R}$ is consistent w.r.t. $\mathcal{T}$. For further details see [4].

This automata-based decision procedure not only provides a tight complexity bound for reasoning in the fuzzy DL G-$\mathcal{ALC}$. It also opens the door to the application of other automata-based techniques, originally developed for classical DLs (e.g. [1, 2]), to this and other fuzzy DLs based on the Gödel t-norm.

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Similarity-based reasoning and interpolating fuzzy function

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One of general schemes of approximate reasoning can be expressed in the following form:

$$\begin{array}{lll} \textit{given a set of IF-THEN rules:} & \text{IF } X = A_i \text{ THEN } Y = B_i, & \\ \textit{and the fact} & X \text{ is } A, & \\ \textit{infer a conclusion} & & Y \text{ is } B, \end{array} \tag{1}$$

on the basis of the *meta-inference rule*

$$\text{the } \textit{closer} \text{ the input } A \text{ is to } A_i, \text{ the } \textit{closer} \text{ the output } B \text{ is to } B_i. \tag{2}$$

If A, B, A_i, B_i, $i = 1, \ldots, n$, are fuzzy predicates/sets then scheme (1) is widely known as the *Generalized Modus Ponens* [3, 9]. If in addition, it is assumed that in the case $A = A_i$, the conclusion is equal to B_i, then scheme (1) characterizes the problem of *fuzzy interpolation*. The latter was intensively investigated in e.g., [5, 1, 4, 6–8].

It is important to stress that two different closeness relations, say $close_X$ on $\mathscr{F}(X)$ (set of fuzzy subsets of X) and $close_Y$ on $\mathscr{F}(Y)$, are involved into interpretation of (2). This meta-rule can be schematically expressed as

$$close_X(A, A_j) \le close_Y(B, B_j),\ j = 1, \ldots, n. \tag{3}$$

Assuming that scheme (1) is interpolating, i.e. valid for facts A_i and conclusions B_i, the meta-rule (3) should be valid for all pairs (A_i, B_i), $i = 1, \ldots, n$. This means that relations $close_X$ and $close_Y$ should be chosen in such a way that

$$close_X(A_i, A_j) \le close_Y(B_i, B_j),\ i, j = 1, \ldots, n. \tag{4}$$

Summarizing, we can formulate the *Problem of interpolative approximate reasoning* with fuzzy sets as follows:

> *given fuzzy sets $A, A_1, \ldots, A_n$ on X and $B_1, \ldots, B_n$ on Y and two closeness relations: $close_X$ on $\mathscr{F}(X)$ and $close_Y$ on $\mathscr{F}(Y)$ such that (4) is fulfilled, find a fuzzy set B on Y such that (3) is fulfilled.*

In fuzzy literature, a closeness between fuzzy sets is used to be specified with the help of a *similarity*, where the latter is any binary fuzzy relation that fulfills three axioms: reflexivity, symmetry and transitivity. If this is accepted and moreover, $A, A_1, \ldots, A_n$ and $B, B_1, \ldots, B_n$ are expressions of a formal language, and B is a result of an approximate entailment, then the above formulated problem is known [4, 6] as fuzzy similarity-based reasoning.

In the proposed contribution, we will be looking at the above formulated Problem from the *functional* point of view. Our purpose is to show that if

- a complete residuated lattice $\mathscr{L} = \langle L, \vee, \wedge, *, \rightarrow, 0, 1\rangle$ is chosen as an underlying algebraic structure,
- similarities S on X and Q on Y are $*$-transitive,
- closeness relation $close_X$ on L^X (similarly, $close_Y$ on L^Y) is chosen (see [1, 8]) in accordance with

$$close_X(E,D) = (E \lesssim S \circ D), \text{ or}$$

$$close_X(E,D) = \min(E \lesssim S \circ D, D \lesssim S \circ E),$$

then a natural interpretation of (3) leads to the conclusion that a solution B of the Problem of interpolative approximate reasoning is the value of a certain *fuzzy function* whose argument is A. This fuzzy function is represented by the following fuzzy relation

$$\hat{R} = \bigwedge_{i=1}^{n} (S \circ A_i \rightarrow Q \circ B_i), \tag{5}$$

and B is computed on the basis of (3) as follows:

$$B = \bigwedge_{x \in X} (S \circ A) \rightarrow \hat{R}. \tag{6}$$

We will be discussing whether fuzzy relation $\hat{R}$ in (5) is a representation of a fuzzy function in the sense of [2][1]. We will show that the latter can be represented by any of two specially constructed fuzzy relations. Each representation uses an ordinary (core) function that is extensional with respect to given similarities. On the other side, if we are given two similarities S on X and Q on Y and a correspondence, say g, between some classes, then we can easily construct two fuzzy relations that represent two fuzzy functions in the sense of [2], provided that a corresponding to g core function is extensional with respect to S and Q.

Therefore, from the semantical point of view, fuzzy similarity based reasoning, that is expressed by scheme (1) and rule (2), produces conclusions that are values of a certain fuzzy function. In the contribution, the issue of computational complexity will be discussed as well.

[1] It is a fuzzy relation on $X \times Y$ that is double extensional with respect to chosen similarities on X and Y and that fulfills a generalized property of uniqueness

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Finite negative commutative tomonoids and the level-set approach

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Abstract. Following an idea due to the first author and P. Sarkoci, we propose the representation of finite negative, commutative totally ordered monoids by means of level sets. For the Archimedean case, we show that the elementary extensions of such totally ordered monoids can be constructed in a quite efficient way.

1 Introduction

Triangular norms, or t-norms for short, are binary operations on the real unit interval used in fuzzy logic for the interpretation of the conjunction. Just as should reasonably be expected about a conjunction, these operations are defined to be associative, commutative, neutral w.r.t. 1, and in each argument isotone.

Quite a number of algebraic approaches exists to examine t-norms. Under the assumption of left continuity, a t-norm gives alternatively rise to an MTL-algebra, to a quantale, or simply to a totally ordered monoid. We choose in this contribution the presumably simplest framework, the last mentioned one.

Definition 1. A structure $(L;\leq,\odot,1)$ is called a *totally ordered monoid*, or *tomonoid* for short, if (i) $(L;\leq)$ is a totally ordered set, (ii) $(L;\odot,1)$ is a monoid, and (iii) $\leq$ is compatible with $\odot$, that is, for any $a,b,c\in L$, $a\leq b$ implies $a\odot c\leq b\odot c$ and $c\odot a\leq c\odot b$.

L is called *commutative* if $\odot$ is commutative. L is called *negative* if 1 is the top element.

The real unit interval endowed with the natural order, a t-norm, and the constant 1 is a negative, commutative tomonoid. The general aim that we follow in our work is to classify this type of algebras. A classification is difficult for the general case, and the restriction imposed in this contribution – finiteness – hardly makes the situation easier.

The crucial property that we have to cope with is associativity. This property is fundamental in mathematics and numerous approach exist to shed light on it. Let us enumerate some ideas that are in our context applicable.

– We can lead back the associativity of a commutative monoid to the probably most common situation where this property arises: the addition of natural numbers. In fact, any commutative monoid, provided it is finitely generated, is a quotient of an $\mathbb{N}^n$, where n is the number of generators. How to deal in this framework with a compatible total order is, e.g., the topic of [5].
– There is another situation in which associativity arises naturally: the composition of functions. In fact, we may represent any monoid as a monoid of mappings under composition. This is the regular representation; see, e.g., [1]. To include a compatible total order poses no difficulty; we are then led to a monoid of pairwise commuting, order-preserving mappings [4]. Here, the fact that any two mappings commute corresponds to both associativity and commutativity.
– A third and totally different approach, which is inspired by the field of web geometry, is due to the first author and Peter Sarkoci [3]. Here, a tomonoid is represented by its contour lines. Associativity corresponds to the so-called Reidemeister condition.

Each of these three approaches has its benefits and drawbacks. However, when it comes to the systematic construction of finite tomonoids, it seems to us that the last approach, to which we devote this contribution, is particularly useful.

2 Tomonoids as partitions

Let us first review how tomonoids are represented by means of their contour lines. The idea is developed in [3] for the case of t-norms, but can be adapted to the present setting without any difficulty.

Definition 2. Let $(L;\leq,\odot,1)$ be a negative, commutative tomonoid. For two pairs (a,b), $(c,d) \in L \times L$, we define

$$(a,b) \sim (c,d) \quad \text{if} \quad a \odot b = c \odot d,$$

and we say in this case that (a,b) and (c,d) are *level equivalent*. We call $\sim$ itself the *level equivalence* associated with L.

In this way, each negative, commutative tomonoid L induces a partition of $L \times L$, which is clearly characteristic for L. The following theorem is easily proved.

Theorem 1. *Let $\sim$ be the level equivalence associated with the negative, commutative tomonoid $(L;\leq,\odot,1)$. Then the following properties hold:*

(L1) *For any $a,b \in L$, there is exactly one $c \in L$ such that $(a,b) \sim (c,1)$.*
(L2) *For any $a,b,c,d,e \in L$, $(a,b) \sim (d,1)$ and $(b,c) \sim (1,e)$ implies $(d,c) \sim (a,e)$.*
(L3) *For any $a,b \in L$, we have $(b,a) \sim (a,b)$.*
(L4) *For any $a,a',b,c,c' \in L$, $a \leq a'$, $(a,b) \sim (c,1)$, and $(a',b) \sim (c',1)$ implies $c \leq c'$.*

Conversely, let $(L;\leq,1)$ a totally ordered set with the top element 1. Let $\sim$ be an equivalence relation on $L \times L$ such that (L1)–(L4) *hold. Then $\sim$ is the level equivalence associated with the negative, commutative tomonoid $(L;\leq,\odot,1)$, where, for $a,b \in L$,*

$$a \odot b = \text{ the unique } c \text{ such that } (a,b) \sim (c,1).$$

The crucial property – the one corresponding to associativity – is evidently (L2). In web geometry it is referred to as the *Reidemeister condition* and has a quite appealing geometric interpretation. Namely, given two rectangles hitting the upper respectively right edge of the square $L \times L$, the equivalence of all corresponding vertices but the lower left ones implies the equivalence of the lower left vertices as well. See Fig. 1.

It is furthermore clear that (L3) corresponds to the commutativity of the monoid and (L4) ensures the compatibility of the total order.

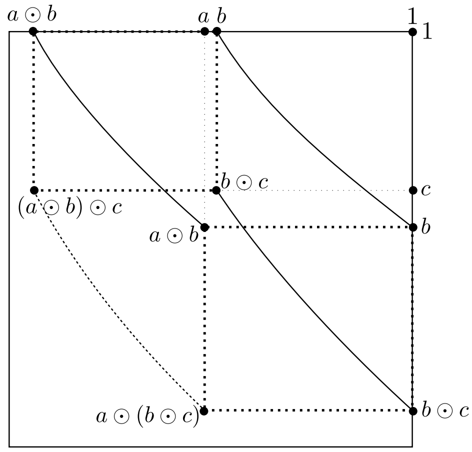

Fig. 1. The condition (L2). The two axes depict a negative, commutative tomonoid L, and the level equivalence of two elements of $L \times L$ is indicated by a connecting line. By (L2), the equivalences of the pairs connected by a solid line implies the equivalence of the two points connected by a broken line.

3 Rees quotients and elementary extensions

It seems appropriate to say that a negative, commutative tomonoid has many quotients. In fact, we can associate a quotient with each of its elements. Let $(L; \leq, \odot, 1)$ be a negative, commutative tomonoid and let $q \in L$. For $a, b \in L$, let $a \approx_q b$ if $a = b$ or $a, b \leq q$. Then $\approx_q$ is a tomonoid congruence; see, e.g., [2]. The quotient, which we denote by L/q, is called the *Rees quotient* of L w.r.t. q. In the sequel, we will identify L/q with the subset $[q, 1] = \{a \in L \colon q \geq a\}$ of L. Clearly, the Rees quotient is again negative and commutative.

Given a finite negative, commutative tomonoid L, let e be the atom of L, that is, the smallest element distinct from the bottom element of L. Let us then call L/e the *elementary quotient* of L and, taking the opposite viewpoint, let us call L an *elementary extension* of L/e. Assuming that L consists of the elements $1 = a_0 > a_1 > \ldots > a_n$, we have a chain

$$L/a_0, \quad \ldots, \quad L/a_n;$$

here, L/a_0 is the trivial (one-element) tomonoid; for each $i = 0, ..., n-1$, L/a_{i+1} is an elementary extension of L/a_i; and $L/a_n = L$.

The construction of finite negative, commutative tomonoids can thus be understood as the problem of determining all the elementary extensions of a given such tomonoid. We shall demonstrate how the latter problem can be solved in the present framework.

In the level-set picture, the formation of Rees quotients is straightforward.

Proposition 1. *Let $(L; \leq, \odot, 1)$ be a negative, commutative tomomoid, and let $\sim$ be its level equivalence. Let $q \in L$ and let $\sim_q$ be the equivalence relation on $[q,1] \times [q,1]$ defined as follows:* (i) *for each $a > q$, the $\sim_q$-class of $(a,1)$ coincides with its $\sim$-class;* (ii) *the $\sim_q$-class of $(q,1)$ comprises all remaining elements. Then $\sim_q$ is the level equivalence associated with the Rees quotient of L w.r.t. q.*

In other words, a Rees quotient arises from the partition on $L \times L$ by "cutting off" all columns and rows indexed by elements $< q$, and pairs belonging to equivalence classes of elements $(a,1)$ such that $a < q$ are joined to the class of $(q,1)$. In the special case that q is the atom of a finite negative, commutative tomonoid, we "cut off" just one column and row, and we join pairs belonging to the class of the former bottom element to the class of the new bottom element q.

We now turn to the reverse procedure: How can we determine the elementary extensions of a negative, commutative tomonoid?

We restrict to the Archimedean case. Recall that a negative, commutative tomonoid L is called *Archimedean* if, for any $a < b < 1$, there is an $n \geq 1$ such that $b^n \leq a$, where b^n is the n-fold product of b with itself.

Theorem 2. *Let $(L; \leq, \odot, 1)$ be a negative, commutative tomonoid, and let $\sim$ be its level equivalence. Assume that L is finite and Archimedean. Let the totally ordered set $(\bar{L}, \leq)$ arise from the totally ordered set $(L; \leq)$ by replacing the bottom element 0 of L by two new elements $\bar{0}$ and e and by requiring $\bar{0} < e < a$ for any $a \in L \backslash \{0\}$. We shall construct an equivalence relation $\bar{\sim}$ on $\bar{L} \times \bar{L}$ in two steps.*

As our first step, let $\bar{\sim}_0$ be the smallest equivalence relation on $\bar{L} \times \bar{L}$ such that conditions (L2)–(L4) *are fulfilled as well as the following ones:*

(E1) *Let Q be the subset of $\bar{L} \times \bar{L}$ consisting of those pairs (a,b) such that $a, b \in L \backslash \{0\}$ and $(a,b) \sim (f,1)$ for some $f \in L \backslash \{0\}$. For any $(a,b), (c,d) \in Q$, let then $(a,b) \bar{\sim}_0 (c,d)$ if $(a,b) \sim (c,d)$.*

(E2) *For all $a \in \bar{L}$, let $(0,a) \bar{\sim}_0 (0,1)$. For all $a \in \bar{L} \backslash \{1\}$, let $(e,a) \bar{\sim}_0 (0,1)$.*

(E3) *For any $(a,b) \notin Q$, let $(a,b) \bar{\sim}_0 (0,1)$ if there are $c > a$ and $d < 1$ such that $(c,b) \notin Q$ and $(c,d) \sim (a,1)$.*

As our second step, let $\bar{\sim}$ be an equivalence relation extending $\bar{\sim}_0$ and fulfilling (L1) *and* (L4). *Then $\bar{\sim}$ is a level equivalence on $\bar{L}$ and the corresponding tomonoid is an Archimedean elementary extension of L.*

Moreover, all Archimedean elementary extensions of L arise in this way.

4 Conclusion

The structure of finite MTL-algebras, or with reference to the present framework: of finite negative, commutative tomonoids, has been a research issue for quite a time. In the present contribution, we have explored ways of constructing such algebras in an efficient way. Namely, starting from the one-element tomonoid, we construct those Archimedean tomonoids whose Rees quotient w.r.t. its atom is the given one. The tool that we have employed is the level-set representation of tomonoids, as proposed in [3].

A further aim will be the formulation of the procedure for any, not only Archimedean tomonoids. Furthermore, for the sake of a classification of finite negative, commutative tomonoids it would certainly be desirable to understand the construction process as a whole rather than only step by step.

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Lattice-valued bornological systems or probabilistic modelling of cancer research?

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1 Introduction

Incidence of various cancer diseases increased more than 6 times since 1900. Our civilization will face to several dilemmas induced by cancer. Cancer is not a simple disease (especially for a Western genom), e.g. [11] surprisingly pointed out that stochasticity of the mutation process is incapable of explaining the spread of times at diagnosis of acute myeloid leukemia in this case; it is necessary to additionally assume a wide spread of proliferative parameters among disease cases. This finding was unexpected but generally consistent with the wide heterogeneity of characteristics of human cancers.

During the talk we will present several examples from clinical practice where need for proper algebraical, topological ([24, 3]) probabilistic ([23, 12]) and fuzzy modelling is needed. The problem with probabilistic modelling are not known forms of anomalous diffusion and in several ways a convenient approximation to the problem could be introducing of bornology, and lattice valued bornological systems, see [21].

Another unsolved problem, touching clinical practice in a drastic way, is unknown relationship between 2dimensional fractal dimension of 2d slices and 3d fractal dimension of original tissue, from which these slices are taken. Such problem is formulated e.g. in [13]. Here bornological approach could be of, at least proxy, usage. These issues can bring a potential to better understand relationship between blasthoma and tumor, e.g. in Wilms cancer of small children. Moreover, the developed theory of bornological systems could find its applications in cancer research. To be more precise, tumors in humans can be conveniently modeled by fractals, the "dimension" of which is non-integer [3]. One of the ways how to get these non-integers in metric spaces is to employ *Hausdorff dimension* [7], the value of which could tell the level of carcinogenicity of a given tumor. In practical applications, however, one often encounters a bornological space instead of a metric one, which motivated J. Almeida and L. Barreira [2] to introduce the concept of Hausdorff dimension for convex bornological spaces. Having a place for both geometric and algebraic information, bornological systems though seem to be more suitable in many cases.

In 1989, S. Vickers [25] has introduced the notion of *topological system* as a common setting for both topological spaces, and their underlying algebraic structures – locales. Later on, J. T. Denniston, A. Melton, and S. E. Rodabaugh [4] provided the concept of *lattice-valued topological system* as an extension of lattice-valued topological spaces. Lattice-valued analogues of the main system-related procedures, i.e., *spatialization* (a space from a system) and *localification* (a locale from a system) were soon considered in [16, 17, 19], thereby providing a complete fuzzification of the original setting of S. Vickers. At present, the theory of lattice-valued topological systems has already found numerous applications in many fields of mathematics (see, e.g., [5, 6, 8, 18, 20]).

In 2011, M. Abel and A. Šostak [1] came out with a fuzzification of the well-known concept of *bornological space* of functional analysis [9], and considered the category of such structures. The main meta-mathematical difference between topological and bornological spaces is that the former provide a convenient tool to study "continuity", and the latter do the same job for "boundedness". As a result, while both notions are defined through a collection of subsets of a set, the respective axiomatics are different.

Motivated by the theory of topological systems, this talk introduces the notion of *bornological system* and shows its possible fuzzification. In particular, we provide bornological (and, partially, lattice-valued) analogues of system spatialization and localification procedures. The latter procedure though needs the concept of *point-free bornology*. While the theory of point-free topology is well-developed [10], point-free bornology (up to our knowledge) is still non-existent. To fill the gap, we introduce an approach to point-free bornology, e.g., describe the algebraic structure, which underlies bornologies, and show its respective homomorphism. Similar to topological systems, we aim at providing a common setting for both point-set and point-free bornologies.

2 Lattice-valued bornological spaces and systems

This section provides a brief extract from the already obtained theory (cf. [21]).

2.1 Lattice-valued bornological spaces

We recall first the definition of L-bornological space of M. Abel and A. Šostak [1].

Definition 1. *Let L be a complete lattice. An* L-bornological space *is a pair $(X, \mathcal{B})$, where X is a set, and $\mathcal{B}$ (an* L-bornology *on X) is a subfamily of L^X (the elements of which are called* bounded L-sets*), which satisfies the following axioms:*

1. *for every $x \in X$, $\bigvee_{B\in\mathcal{B}} B(x) = \top_L$ (the top element of L);*
2. *given $B \in \mathcal{B}$ and $D \in L^X$ such that $D \leqslant B$, it follows that $D \in \mathcal{B}$;*
3. *if $\mathcal{S} \subseteq \mathcal{B}$ is finite, then $\bigvee \mathcal{S} \in \mathcal{B}$.*

Given L-bornological spaces $(X_1, \mathcal{B}_1)$ and $(X_2, \mathcal{B}_2)$, a map $X_1 \xrightarrow{f} X_2$ is called L-bounded *provided that $f_L^{\rightarrow}(B_1) \in \mathcal{B}_2$ for every $B_1 \in \mathcal{B}_1$.* L-**Born** *stands for the construct of L-bornological spaces and L-bounded maps.*

In the next step, we introduce a variable-basis approach (in the sense of S. E. Rodabaugh [14]) over the category **Sup** of $\bigvee$-semilattices and $\bigvee$-preserving maps. Given a subcategory **L** of **Sup**, there exists a functor $\mathbf{Set} \times \mathbf{L} \xrightarrow{(-)^{\rightarrow}} \mathbf{Sup}$, which is defined by $((X_1, L_1) \xrightarrow{(f,\psi)} (X_2, L_2))^{\rightarrow} = L_1^{X_1} \xrightarrow{(f,\psi)^{\rightarrow}} L_2^{X_2}$, $((f,\psi)^{\rightarrow}(B))(x_2) = \bigvee_{f(x_1)=x_2} \psi \circ B(x_1)$.

Definition 2. *Given a subcategory* $\mathbf{L}$ *of* **Sup**, $\mathbf{L}$-**Born** *is the category, concrete over the product category* $\mathbf{Set} \times \mathbf{L}$, *whose objects are triples* $(X, L, \mathcal{B})$, *where* L *is an* $\mathbf{L}$-*object, and* $(X, \mathcal{B})$ *is an* L-*bornological space; and whose morphisms* $(X_1, L_1, \mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2, L_2, \mathcal{B}_2)$ *(called* $\mathbf{L}$-bounded maps*) consist of a map* $X_1 \xrightarrow{f} X_2$ *and an* $\mathbf{L}$-*morphism* $L_1 \xrightarrow{\psi} L_2$ *such that* $(f,\psi)^{\rightarrow}(B) \in \mathcal{B}_2$ *for every* $B \in \mathcal{B}_1$.

2.2 Lattice-valued bornological systems

We begin again with the fixed-basis approach over a complete lattice L.

Definition 3. *A* b_e-lattice *is a poset* C, *which has finite* $\bigvee$ *and non-empty* $\bigwedge$. *Given* b_e-*lattices* C_1 *and* C_2, *a* b_e-lattice homomorphism *is a map* $C_1 \xrightarrow{\varphi} C_2$, *which preserves finite* $\bigvee$. b_e-**Lat** *stands for the construct of* b_e-*lattices and* b_e-*lattice homomorphisms.*

The category b_e-**Lat** provides a possible approach to point-free bornology. Additional conditions on b_e-lattices (off this abstract) make them almost dual to *frames* [10].

Definition 4. *Given a complete lattice* L, *an* L-bornological system *is a triple* $(X, C, \models)$, *where* X *is a set,* C *is a* b_e-*lattice, and* $X \times C \xrightarrow{\models} L$ *is a map (*L-satisfaction relation *on* (X, C)*), which fulfills the following axioms:*

1. *for every* $x \in X$, $\bigvee_{c \in C} \models(x,c) = \top_L$;
2. *if* $c \in C$ *and* $D \in L^X$ *are such that* $D \leqslant \models(-,c)$, *then there exists* $c' \in C$ *such that* $D = \models(-,c')$;
3. *for every* $x \in X$, *the map* $C \xrightarrow{\models(x,-)} L$ *preserves finite* $\bigvee$.

Given L-*bornological systems* $(X_1, C_1, \models_1)$ *and* $(X_2, C_2, \models_2)$, *an* L-*bornological system morphism (also called* L-bounded map*)* $(X_1, C_1, \models_1) \xrightarrow{(f,\varphi)} (X_2, C_2, \models_2)$ *consists of a map* $X_1 \xrightarrow{f} X_2$ *and a* b_e-*lattice homomorphism* $C_1 \xrightarrow{\varphi} C_2$ *such that for every* $x_2 \in X_2$ *and every* $c_1 \in C_1$, $\models_2(x_2, \varphi(c_1)) = \bigvee_{f(x_1)=x_2} \models_1(x_1, c_1)$. L-**BornSys** *is the category of* L-*bornological systems and* L-*bounded maps, concrete over the category* $\mathbf{Set} \times b_e$-**Lat**.

In the next step, we introduce a variable-basis approach over the category **Sup**.

Definition 5. *Given a subcategory* $\mathbf{L}$ *of* **Sup**, $\mathbf{L}$-**BornSys** *is the category, concrete over the product category* $\mathbf{Set} \times \mathbf{L} \times b_e$-**Lat**, *whose objects are tuples* $(X, L, C, \models)$, *where* L *is an* $\mathbf{L}$-*object, and* $(X, C, \models)$ *is an* L-*bornological system; and whose morphisms* $(X_1, L_1, C_1, \models_1) \xrightarrow{(f,\psi,\varphi)} (X_2, L_2, C_2, \models_2)$ *(called* $\mathbf{L}$-bounded maps*) consist of a map* $X_1 \xrightarrow{f} X_2$, *an* $\mathbf{L}$-*morphism* $L_1 \xrightarrow{\psi} L_2$, *and a* b_e-*lattice homomorphism* $C_1 \xrightarrow{\varphi} C_2$ *such that for every* $x_2 \in X_2$ *and every* $c_1 \in C_1$, $\models_2(x_2, \varphi(c_1)) = \bigvee_{f(x_1)=x_2} \psi \circ \models_1(x_1, c_1)$.

We notice an important difference between continuous (topology) and bounded (bornology) maps, namely, while the former depend on the backward powerset operator, the latter employ the forward one [15]. This difference is reflected in the underlying algebraic structure of point-free topology (the category **Loc** of locales, which is, moreover, a dual category) and bornology (the category b_e-**Lat** of b_e-lattices).

2.3 Spaces versus systems

In this subsection, we provide a spatialization procedure for bornological systems.

Theorem 1.

1. *There exists a full embedding* **L-Born** $\xhookrightarrow{E}$ **L-BornSys**, *which is defined by* $E((X_1,L_1,\mathcal{B}_1) \xrightarrow{(f,\psi)} (X_2,L_2,\mathcal{B}_2)) = (X_1,L_1,\mathcal{B}_1,\models_1) \xrightarrow{(f,\psi,\overline{(f,\psi)}^{\rightarrow})} (X_2,L_2,\mathcal{B}_2,\models_2)$, *where* $\models_i(x,B) = B(x)$, *and* $\overline{(f,\psi)}^{\rightarrow}$ *is the restriction of* $(f,\psi)^{\rightarrow}$ *to* $\mathcal{B}_1$ *and* $\mathcal{B}_2$.
2. *There exists a functor* **L-BornSys** $\xrightarrow{Spat}$ **L-Born**, *which is defined by the formula* $Spat((X_1,L_1,C_1,\models_1) \xrightarrow{(f,\psi,\varphi)} (X_2,L_2,C_2,\models_2)) = (X_1,L_1,\{Ext_1(c)\,|\,c \in C_1\}) \xrightarrow{(f,\psi)} (X_2,L_2,\{Ext_2(c)\,|\,c \in C_2\})$, *where* $(Ext_i(c))(x) = \models_i(x,c)$.
3. *The functor Spat is a left-adjoint-left-inverse to the embedding E.*

Corollary 1. *The category* **L-Born** *is isomorphic to a full reflective subcategory of the category* **L-BornSys**.

The above embedding is not concrete, since the concrete categories **L-Born** and **L-BornSys** have different ground categories (**Set**×**L** and **Set**×**L**×b_e-**Lat**, respectively).

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Partial metric spaces as enriched categories

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1.—Following Fréchet [1], a metric space (X,d) is a set X together with a real-valued function d on $X \times X$ such that the following axioms hold:

[M0] $d(x,y) \geq 0$,
[M1] $d(x,y) + d(y,z) \geq d(x,z)$,
[M2] $d(x,x) = 0$,
[M3] if $d(x,y) = 0 = d(y,x)$ then $x = y$,
[M4] $d(x,y) = d(y,x)$,
[M5] $d(x,y) \neq +\infty$.

The categorical content of this definition, as first observed by Lawvere [3], is that (i) the extended real interval $[0,+\infty]$ underlies a quantale $([0,+\infty]^{\mathsf{op}},+,0)$, so that (ii) a "generalised metric space" (i.e. a structure as above, minus the axioms M3-M4-M5) is exactly a category enriched in that quantale.

For clarity's sake, let us expand a bit on this.

2.—Generally speaking, a *quantale* Q is an ordered set $(Q,\leq)$ admitting all suprema, combined with a monoid structure $(Q,\cdot,1)$, in such a way that multiplication distributes on both sides over suprema. Lawvere's quantale of extended positive real numbers is formed by the *opposite* of the natural order on $[0,+\infty]$, together with addition as "multiplication".

In general, a *category* $\mathbb{C}$ *enriched in a quantale* Q consists of a set $\mathbb{C}_0$ together with a Q-valued binary predicate $\mathbb{C}(-,-)$ on $\mathbb{C}_0 \times \mathbb{C}_0$, in such a way that the axioms

[C1] $\mathbb{C}(x,y) \cdot \mathbb{C}(y,z) \leq \mathbb{C}(x,z)$,
[C2] $1 \leq \mathbb{C}(x,x)$

hold. Such a Q-category $\mathbb{C}$ is said to be *separated* (or *skeletal*) if moreover

[C3] if $1 \leq \mathbb{C}(x,y)$ and $1 \leq \mathbb{C}(y,x)$ then $x = y$

holds, and *symmetric* if we have

[C4] $\mathbb{C}(x,y) = \mathbb{C}(y,x)$.

Finiteness of $\mathbb{C}(x,y)$ can be formulated for general Q-enriched categories too, insofar as the quantale itself sports a notion of finiteness: if $\mathsf{Fin}(Q)$ is a suitable subset of Q of "finite elements", then one can require

[C5] $\mathbb{C}(x,y) \in \mathsf{Fin}(Q)$.

Reckoning with the particularities of Lawvere's quantale $([0,+\infty]^{\mathsf{op}},+,0)$, it is now easy to check the correspondence between the axioms for metric spaces and those for enriched categories.

3.—More recently, see e.g. [4], the notion of a partial metric space (X,p) has been proposed to mean a set X together with a real-valued function p on $X \times X$ satisfying the following axioms:

[P0] $p(x,y) \geq 0$,

[P1] $p(x,y) + p(y,z) - p(y,y) \geq p(x,z)$,

[P2] $p(x,y) \geq p(x,x)$,

[P3] if $p(x,y) = p(x,x) = p(y,y) = p(y,x)$ then $x = y$,

[P4] $p(x,y) = p(y,x)$,

[P5] $p(x,y) \neq +\infty$.

The categorical content of *this* definition was discovered by Höhle and Kubiak [2], who showed that (i) there is a particular quantaloid of positive real numbers, such that (ii) categories enriched in that quantaloid correspond to ("generalised") partial metric spaces. We realised in [5] that (iii) Höhle and Kubiak's quantaloid of real numbers is actually a universal construction on Lawvere's quantale of real numbers.

Let us give some detail.

4.—A *quantaloid* $\mathcal{Q}$ is a category in which all hom-sets are complete lattices and such that composition distributes on both sides over arbitrary suprema. (In other words, a quantaloid $\mathcal{Q}$ is precisely a category enriched in the category of complete lattices and supremum-preserving functions.) As per usual, we shall write $\mathcal{Q}_0$ for the collection of objects of $\mathcal{Q}$, and $\mathcal{Q}_1$ for the collection of morphisms. Note how a quantaloid with one object is the same thing as a quantale!

The definition of a *category* $\mathbb{C}$ *enriched in a quantaloid* $\mathcal{Q}$ now goes as follows: it consists of a set $\mathbb{C}_0$ together with a $\mathcal{Q}_0$-valued unary predicate t on $\mathbb{C}_0$ and a $\mathcal{Q}_1$-valued binary precidate $\mathbb{C}(-,-)$ on $\mathbb{C}_0 \times \mathbb{C}_0$, such that the following conditions hold:

[$\tilde{\text{C}}$0] $\mathbb{C}(x,y) \colon ty \to tx$,

[$\tilde{\text{C}}$1] $\mathbb{C}(x,y) \circ \mathbb{C}(y,z) \leq \mathbb{C}(x,z)$,

[$\tilde{\text{C}}$2] $1_{tx} \leq \mathbb{C}(x,x)$.

Such a $\mathcal{Q}$-category $\mathbb{C}$ is said to be *separated* (or *skeletal*) if moreover

[$\tilde{\text{C}}$3] if $tx = ty$ and $1_{tx} \leq \mathbb{C}(x,y)$ and $1_{tx} \leq \mathbb{C}(y,x)$ then $x = y$,

and *symmetric* if we have

[$\tilde{\text{C}}$4] $\mathbb{C}(x,y) = \mathbb{C}(y,x)$.

As before, if the quantaloid $\mathcal{Q}$ itself comes with a multiplicative subset $\mathsf{Fin}(\mathcal{Q}_1)$ of distinguished "finite morphisms", then it makes sense to say that $\mathbb{C}$ is *locally finite* if

[$\tilde{\text{C}}$5] $\mathbb{C}(x,y) \in \mathsf{Fin}(\mathcal{Q})$.

5.—It often happens in practice that quantaloids arise from quantales by one or another universal construction. We shall describe one such case, which will turn out to be crucial to describe the categorical content of partial metric spaces.

Fixing two morphisms $f \colon A \to B$ and $g \colon C \to D$ in a quantaloid $\mathcal{Q}$, we say that a third morphism $d \colon A \to D$ in $\mathcal{Q}$ is a *diagonal from* f *to* g if any (and thus both) of the following equivalent conditions holds:

[D1] there exist $x \colon A \to C$ and $y \colon B \to D$ in $\mathcal{Q}$ such that $y \circ f = d = g \circ x$,

[D2] $g \circ (g \searrow d) = d = (d \swarrow f) \circ f$.

In the second condition, we made use of the right adjoints to pre- and post-composition with d (technically speaking, "lifting" and "extension" through d) in the quantaloid $\mathcal{Q}$. It now so happens that a new quantaloid $\mathcal{D}(\mathcal{Q})$ of *diagonals in* $\mathcal{Q}$ can be built:

- the composition of two diagonals $d\colon f \to g$ and $e\colon g \to h$ is defined to be

$$e \circ_g d := (e \swarrow g) \circ g \circ (g \searrow d);$$

- the identity on f is $f\colon f \to f$ itself;
- and the supremum of a set of diagonals $(d_i\colon f \to g)_{i \in I}$ is computed "as in $\mathcal{Q}$".

6.—Combining the above, we may now consider categories enriched in $\mathcal{D}([0,+\infty])$, the quantaloid of diagonals in Lawvere's quantale $([0,+\infty]^{\mathsf{op}},+,0)$. Carefully checking all conditions, and weeding out redundancies (due to the particularity of Lawvere's quantale!), it turns out that partial metric spaces are exactly the separated, symmetric, locally finite $\mathcal{D}([0,+\infty])$-enriched categories.

7.—The aim of my talk is not only to explain in detail all the above, but also (i) to indicate some advantages of this categorical approach to partial metrics, more precisely when considering "completions"; (ii) to give other examples of the phenomena described here, notably in sheaf theory; and (iii) to comment on the construction of the quantaloid of diagonals, showing in particular the relation with divisible quantales, BL-algebras and t-norms.

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Goldblatt-Thomason Theorem for Łukasiewicz finitely-valued modal language

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1 Introduction and overview

Propositional modal logic is often advertised as being a way to talk about relational structures and conversely. One can indeed consider two types of problems depending on what one wants to focus is attention on.

On the one hand, if one is interested in the deductive aspects of the modal language, then one can study relational semantics in order to build completeness results. On the other hand, if one is interested in the descriptive power of the modal language, then one can try to characterize classes of relational structures that are modally definable.

We are interested in languages $\mathcal{L}$ that are modal extensions of the language of ŁUKASIEWICZ logic (it means that connectives $\neg$ and $\rightarrow$ are intended to be interpreted in a ŁUKASIEWICZ way). Several authors have considered the deductive aspects of such languages ([1–4]). Among the crisp structures, it turned out that there are two classes of relational structures that are relevant to interpret these languages. The first one is the class of $\mathcal{L}$-frames and the second one is the class of L_n-valued $\mathcal{L}$-frames. The latter are KRIPKE frames in which the set of allowed truth values is specified in each world of the frame. These two classes of structures give rise to two different notions of KRIPKE completeness ([4]).

In this talk, we study the descriptive power of such languages $\mathcal{L}$ with regards to these two types of relational structures. More precisely, we give many-valued generalizations of the celebrated GOLDBLATT – THOMASON characterization of modally definable classes of KRIPKE frames that are closed under ultrapowers ([5]).

Hence, our two main results are the following. They involve new notions that are introduced in the remainder of the paper.

Theorem 1. *Assume that $\mathcal{C}$ is a class of $Ł_n$-valued $\mathcal{L}$-frames that contains ultrapowers of its elements. Then $\mathcal{C}$ is definable if and only if the following two conditions are satisfied.*

1. *The class $\mathcal{C}$ contains $Ł_n$-valued generated subframes, disjoint unions and $Ł_n$-valued bounded morphic images of its members.*
2. *For any $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$, if $\mathfrak{Ce}(\mathfrak{F}) \in \mathcal{C}$ then $\mathfrak{F} \in \mathcal{C}$.*

Theorem 2. *Assume that $\mathcal{C}$ is a class of $\mathcal{L}$-frames that contains ultrapowers of its elements. Then $\mathcal{C}$ is $Ł_n$-definable if and only if the following two conditions are satisfied.*

1. *The class $\mathcal{C}$ contains generated subframes, disjoint union and bounded morphic images of its members.*
2. *For any $\mathcal{L}$-frame $\mathfrak{F}$, if $\mathfrak{Ce}(\mathfrak{F}) \in \mathcal{C}$ then $\mathfrak{F} \in \mathcal{C}$.*

2 Language and notations

Let $\mathcal{L} = \{\neg, \to, 0\} \cup \{\nabla_i \mid i \in I\}$ be a language, where $\neg$ is unary, $\to$ is binary, 0 is constant and ∇_i is n_i-ary for any $i \in I$. The set $\mathsf{Form}_{\mathcal{L}}$ of formulas is defined by induction from a countably infinite set of propositional variables Prop using the grammar

$$\phi ::= p \in \mathsf{Prop} \mid 0 \mid \neg\phi \mid \phi \to \phi \mid \nabla_i(\phi, \ldots, \phi).$$

Elements of $\{\nabla_i \mid i \in I\}$ are called a *modalities* (our modalities are universal ones). We sometimes write $\phi(p_1, \ldots, p_k)$ to stress that ϕ is a formula whose propositional variables are among $p_1, \ldots, p_k$.

We use bold letters to denote tuples (arity is given by the context). Hence, we denote by $\boldsymbol{\phi}, \boldsymbol{\psi}, \ldots$ tuples of formulas and by ϕ_i the ith component of $\boldsymbol{\phi}$. If $R \subseteq W^n$, we write $\mathbf{u} \in R$ for $(u_1, \ldots, u_n) \in R$ and $\mathbf{w} \in Ru$ for $(u, w_1, \ldots, w_{n-1}) \in R$.

We use standard abbreviations: $\phi \oplus \psi$ stands for $\neg\phi \to \psi$, $\phi \odot \psi$ stands for $\neg(\neg\phi \oplus \neg\psi)$, $\phi \vee \psi$ stands for $(\psi \odot \neg\phi) \oplus \phi$, $\phi \wedge \psi$ stands for $(\psi \oplus \neg\phi) \odot \phi$, if k is an nonnegative integer then ϕ^k stands for $\phi \odot \cdots \odot \phi$ (with k factors ϕ).

We use n to denote a positive integer and $Ł_n$ to denote the sub-MV-algebra $\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\}$ of the standard MV-algebra $[0,1]$.

3 $\mathcal{L}$-frames and $Ł_n$-valued $\mathcal{L}$-frames

Definition 1 **($\mathcal{L}$-frame, $Ł_n$-model).** *An $\mathcal{L}$-frame, is a tuple $(W, (R_i)_{i \in I})$ where W is a nonempty set and R_i is an $n_i + 1$-ary relation for any $i \in I$. Elements of the set W are called worlds. We denote by $\mathcal{FR}_{\mathcal{L}}$ the class of $\mathcal{L}$-frames.*

An $Ł_n$-model is a couple $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ where $\mathfrak{F} = (W, (R_i)_{i \in I})$ is an $\mathcal{L}$-frame and $\mathrm{Val} : W \times \mathsf{Prop} \to Ł_n$. We say that $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ is based on $\mathfrak{F}$.

In an $Ł_n$-model $\mathcal{M}$, the valuation map Val is extended inductively to $\mathsf{Form}_{\mathcal{L}}$ using ŁUKASIEWICZ interpretation of the connectors $\neg$ and $\to$ in $[0,1]$ and the rules $\mathrm{Val}(u, \nabla_i(\boldsymbol{\phi})) = \bigwedge\{\bigvee_{1 \leq k \leq n_i} \mathrm{Val}(w_k, \phi_k) \mid \mathbf{w} \in Ru\}$ for any $i \in I$.

Definition 2 **(True, $Ł_n$-valid).** *If $\mathcal{M} = (\mathfrak{F}, \mathrm{Val})$ is an L_n-model and if $\phi \in \mathrm{Form}_{\mathcal{L}}$, we note $\mathcal{M} \models_n \phi$ if $\mathrm{Val}(u, \phi) = 1$ for any world u of $\mathfrak{F}$. We say that ϕ is true in $\mathcal{M}$.*

If Φ is a set of $\mathcal{L}$-formulas that are true in any $Ł_n$-model based on a frame $\mathfrak{F}$, we write $\mathfrak{F} \models_n \Phi$ and say that Φ is L_n-valid in $\mathfrak{F}$. We write $\mathfrak{F} \models_n \phi$ instead of $\mathfrak{F} \models_n \{\phi\}$.

Definition 3 **($Ł_n$-definable).** *A class $\mathcal{C}$ of $\mathcal{L}$-frames is $Ł_n$-definable if there is a $\Phi \subseteq \mathsf{Form}_{\mathcal{L}}$ such that $\mathcal{C} = \{\mathcal{F} \in \mathcal{FR}_{\mathcal{L}} \mid \mathfrak{F} \models_n \Phi\}$. In that case, we write $\mathcal{C} = \mathrm{Mod}_n(\Phi)$.*

We denote by $\mathsf{PForm}^n_{\mathcal{L}}$ the fragment of $\mathsf{Form}_{\mathcal{L}}$ defined by the grammar

$$\phi ::= p^n \mid 0 \mid \neg\phi \mid \phi \to \phi \mid \nabla_i(\phi,\ldots,\phi)$$

where $p \in \mathrm{Prop}$ and $i \in I$.

Let tr_n be the map $\mathrm{tr}_n : \mathsf{Form}_{\mathcal{L}} \to \mathsf{PForm}^n_{\mathcal{L}} : \phi(p_1,\ldots,p_k) \mapsto \phi(p^n_1,\ldots,p^n_k)$.

Lemma 1. *Let $\mathcal{C}$ be a class of $\mathcal{L}$-frames and $\Phi \subseteq \mathrm{Form}_{\mathcal{L}}$. The following conditions are equivalent.*

1. $\mathcal{C} = \mathrm{Mod}_1(\Phi)$.
2. *There is an $n > 0$ such that $\mathcal{C} = \mathrm{Mod}_n(\mathrm{tr}_n(\Phi))$.*
3. *For any $n > 0$, $\mathcal{C} = \mathrm{Mod}_n(\mathrm{tr}_n(\Phi))$.*

Moreover $\mathrm{Mod}_n(\Phi) \subseteq \mathrm{Mod}_1(\Phi)$ *for any $n > 0$.*

Next example illustrates that $\mathrm{Mod}_n(\Phi)$ may differ from $\mathrm{Mod}_1(\Phi)$.

Example 1. Let $\mathcal{L}_\Box$ be the modal language with a single unary modality $\Box$ and $n > 1$. Then $\mathrm{Mod}_1(\Box(p \vee \neg p)) = \mathcal{FR}_{\mathcal{L}_\Box}$ while $\mathrm{Mod}_n(\Box(p \vee \neg p)) = \{(W,R) \in \mathcal{FR}_{\mathcal{L}_\Box} \mid R = \varnothing\}$.

For any positive integer n, we denote by $\mathrm{div}(n)$ the set of its positive divisors.

Definition 4 ($Ł_n$-valued $\mathcal{L}$-frame). *An $Ł_n$-valued $\mathcal{L}$-frame is a tuple $(W,\{r_m \mid m \in \mathrm{div}(n)\},(R_i)_{i\in I})$ where*

1. *$(X,(R_i)_{i\in I})$ is an $\mathcal{L}$-frame,*
2. *$r_m \subseteq W$ for any $m \in \mathrm{div}(n)$,*
3. *$r_n = W$ and $r_m \cap r_q = r_{\gcd(m,q)}$ for any $m,q \in \mathrm{div}(n)$,*
4. *$R_i u \subseteq r^{n_i}_m$ for any $i \in I$, any $m \in \mathrm{div}(n)$ and any $u \in r_m$.*

We denote by $\mathcal{F}_\sharp$ the underlying $\mathcal{L}$-frame of the $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$ and by $\mathcal{FR}^n_{\mathcal{L}}$ the class of the $Ł_n$-valued $\mathcal{L}$-frames.

The trivial $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}^n_b$ associated to an $\mathcal{L}$-frame $\mathfrak{F}$ is obtained by enriching $\mathfrak{F}$ with $\{r_m \mid m \in \mathrm{div}(n)\}$ where $r_m = \varnothing$ if $m \neq n$ and $r_n = W$.

As explained in the next definition, the structure given by the sets r_m (where $m \in \mathrm{div}(n)$) is used to weaken the validity relation.

Definition 5 (Validity in $Ł_n$-valued $\mathcal{L}$-frames). *An $Ł_n$-model $\mathcal{M} = (\mathfrak{F}', \mathrm{Val})$ is based on the $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F} = (W,\{r_m \mid m \in \mathrm{div}(n)\},(R_i)_{i\in I})$ if $\mathfrak{F}' = \mathfrak{F}_\sharp$ and $\mathrm{Val}(u,\mathsf{Prop}) \subseteq Ł_m$ for any $m \in \mathrm{div}(n)$ and any $u \in r_m$.*

If Φ is a set of $\mathcal{L}$-formulas that are true in any $Ł_n$-model based on a $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$, we write $\mathfrak{F} \models \Phi$ and say that Φ is valid in $\mathfrak{F}$. We write $\mathfrak{F} \models \phi$ instead of $\mathfrak{F} \models \{\phi\}$.

Definition 6 (Definability). *A class $\mathcal{C}$ of $Ł_n$-valued $\mathcal{L}$-frames is definable if there is a $\Phi \subseteq \mathrm{Form}_{\mathcal{L}}$ such that $\mathcal{C} = \{\mathcal{F} \in \mathcal{FR}^n_{\mathcal{L}} \mid \mathfrak{F} \models \Phi\}$. In that case, we write $\mathcal{C} = \mathrm{Mod}(\Phi)$.*

Example 2. One can check that $\mathrm{Mod}(\Box(p \vee \neg p)) = \{\mathfrak{F} \in \mathcal{FR}^n_{\mathcal{L}_\Box} \mid \forall u\, Ru \subseteq r_1\}$. Moreover $\{\mathfrak{F} \in \mathcal{FR}^n_{\mathcal{L}_\Box} \mid \forall u\, u \notin r_m\}$ is not definable if m is a strict divisor of n.

4 $Ł_n$-valued $\mathcal{L}$-frame constructions

$\mathcal{L}$-frame constructions used in the statement of Theorem 2 are classical in modal logic (see [6] for example). We define the $Ł_n$-valued $\mathcal{L}$-frame constructions needed to understand statement of Theorem 1.

Definition 7 ($Ł_n$-valued bounded morphism). *A map $f : \mathfrak{F} \to \mathfrak{F}'$ between two $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F} = (W, \{r_m \mid m \in \mathrm{div}(n)\}, (R_i)_{i\in I})$ and $\mathfrak{F}' = (W', \{r'_m \mid m \in \mathrm{div}(n)\}, (R'_i)_{i\in I})$ is an $Ł_n$-valued bounded morphism if f is a bounded morphism between $\mathfrak{F}_\sharp$ and $\mathfrak{F}'_\sharp$ and if $f(r_m) \subseteq r'_m$ for any $m \in \mathrm{div}(n)$.*

Definition 8 ($Ł_n$-valued generated subframe). *An $\mathcal{L}$-substructure $\mathfrak{F}'$ of an $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$ is called an $Ł_n$-valued generated subframe of $\mathfrak{F}$ if the inclusion map $\iota : \mathfrak{F}' \to \mathfrak{F}$ is an $Ł_n$-valued bounded morphism.*

If u is a world of an $Ł_n$-valued $\mathcal{L}$-frame $\mathfrak{F}$, we denote by s_u the integer $\gcd\{m \in \mathrm{div}(n) \mid u \in r_m\}$.

Definition 9 (Canonical extension). *Let $\mathfrak{F} = (W, \{r_m \mid m \in \mathrm{div}(n)\}, (R_i)_{i\in I})$ be an $Ł_n$-valued $\mathcal{L}$-frame. We denote by $\mathfrak{F}_\times$ the $\mathcal{L}$-algebra whose universe is $\prod_{u\in W} Ł_{s_u}$ with operations 0, $\neg$ and $\to$ defined pointwise and ∇_i defined by*

$$\nabla_i(\alpha)(u) = \bigwedge_{\mathbf{w}\in R_i u} \bigvee_{1\leq k\leq n_i} \alpha_k(w_k),$$

for any $i \in I$.

The canonical extension of $\mathfrak{F}$, in notation $\mathfrak{Ce}(\mathfrak{F})$ is the structure $(W^e, \{r^e_m \mid m \in \mathrm{div}(n)\}, (R^e_i)_{i\in I})$ defined by:

- *$W^e = \mathcal{MV}(\mathfrak{F}_\times, Ł_n)$ is the set of MV-homomorphisms from $\mathfrak{F}_\times$ to $Ł_n$,*
- *$u \in r^e_m$ if $u(\mathfrak{F}_\times) \subseteq Ł_m$,*
- *$(u, \mathbf{w}) \in R^e_i$ if $\bigvee_{1\leq k\leq n_i} w_k(\alpha_k) = 1$ for any $\alpha \in \mathfrak{F}_\times^{n_i}$ such that $u(\nabla_i\alpha) = 1$.*

If $\mathfrak{F}$ is an $\mathcal{L}$-frame, the canonical extension of $\mathfrak{F}$, in notation $\mathfrak{Ce}(\mathfrak{F})$, is the $\mathcal{L}$-frame $(\mathfrak{Ce}(\mathfrak{F}^1_b))_\sharp$.

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A product modal logic

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Fuzzy modal logics are a family of logics that are still under research for their understanding. Several papers have been published on this issue, treating different problems about the fuzzy modal logics (see for instance [CR10], [CR11], [BEGR11], [HT06], [HT13], or [CMRR13]). However, the study of the product modal logics, which we understand as logics that arise from Kripke structures whose relation and universes are evaluated over the product standard algebra, has remained undone. We present here some results to partially fill that gap for Kripke semantics with crisp accessibility relations, together with a characterization of a strongly complete infinitary product logic with truth-constants. We consider that the characterization and understanding of the product modal logics could open the possibility of studying the more general case of BL modal logics.

1 Propositional strong completeness

Propositional Product logic is finite strong complete but not strong complete with respect to the standard product chain over the real unit interval as proved in [Háj98]. In [Mon06], Montagna defined a logical system, an axiomatic extension of the BL logic with storage operator $*$ and an infinitary rule

$$(R_M)\ \frac{\chi \vee (\varphi \to \psi^k),\ \text{for all k}}{\chi \vee (\varphi \to \psi^*)},$$

that is proved to be strong complete (for infinite theories) with respect to the standard BL chains over the real unit interval. In particular, the expansion of Product Logic with the infinitary rule and Monteiro-Baaz Delta operator is complete with respect to the standard Product algebra over the real unit interval with Delta.

On the other hand, in [SCE$^+$06], the addition of rational truth constants to product logic was studied, and it was proven that the extension of product logic with the Δ axioms from before and natural axiomatization for the constants is finitely strong standard complete with respect to the canonical standard product algebra (where the rational constants are interpreted by its name, in $[0,1)]_{\mathbb{Q}}$).

We let Π^* be the logic defined by the following axioms and rules:

- Axioms of Π (product propositional logic) (see for instance [Háj98])
- Axioms referring to rational constants over product logic [SCE$^+$06]

– Axioms of the Δ operator ([Háj98]) plus
$\Delta\overline{c} \leftrightarrow \overline{\delta(c)}$, for each $c \in [0,1]_{\mathbb{Q}}$
with $\delta(1) = 1$ and $\delta(x) = 0$ for $x < 1$.
– Rules of modus ponens and necessitation for Δ: from φ derive $\Delta\varphi$
– The infinitary rules

$$(\mathbf{R_1})\ \frac{\overline{c} \to \varphi, \text{ for all } c \in (0,1)_{\mathbb{Q}}}{\varphi} \qquad (\mathbf{R_2})\ \frac{\varphi \to \overline{c}, \text{ for all } c \in (0,1)_{\mathbb{Q}}}{\neg\varphi}$$

Considering that these two last rules imply the archimedeanicity of the algebras associated to a logic closed by them, if we follow the usual precourse of extending a theory to another one complete and we let it be closed under R_1 and R_2, we obtain that its Lindembaum sentence algebra is an archimedean product chain (with canonical constants). Following some ideas from [Mon06] and the results about product algebras available at [CT00], we can equally prove the strong completeness of Π^* with respect to the canonical standard product algebra,

$$\Gamma \vdash_{\Pi^*} \varphi \text{ iff } \Gamma \models_{[\mathbf{0},\mathbf{1}]_\Pi} \varphi.$$

It is interesting that this logic has a natural behaviour, in the sense that the Deduction Theorem (with Δ) is still valid, i.e. $\Gamma \cup \{\alpha\} \vdash_{\Pi^*} \varphi$ iff $\Gamma \vdash_{\Pi^*} \Delta\alpha \to \varphi$.

2 A modal product logic

Our aim is to define a modal logic over the standard product algebra with canonical rational truth-constants and the Δ operator, by introducing the two usual modalities $\Box$ and $\Diamond$, and with Kripke semantics defined by structures with crisp accessibility relations.

Definition 1. *A Crisp Product Kripke model (PK-model) is a structure $\mathcal{M} = \langle W, R, e\rangle$ where:*

- *W a non-empty set of objects (worlds);*
- *$R \subseteq W \times W$ (a crisp accessibility relation);*
- *$e\colon W \times \mathcal{V} \to [0,1]$ a truth-evalution of propositional variables $\mathcal{V}$ in each world $[0,1]_\Pi$.*

The evaluation e is extended to all modal formula in $Fm_\Box$ (with constants) by defining inductively at each world w, the evaluation of propositional connectives by their corresponding operation in the algebra (over the evaluated terms), and the evaluation of modal formulas as follows:

$$e(w, \Box\varphi) := \inf\{e(v,\varphi) : \ Rwv = 1\}; \quad e(w, \Diamond\varphi) := \sup\{e(v,\varphi) : \ Rwv = 1\}$$

We can consider different notions of truth, depending on the locality. We say φ is true in M at w, and write $M \models_w \varphi$ iff $e(w,\varphi) = 1$. φ is valid in M ($M \models \varphi$) iff $M \models_w \varphi$ for any $w \in W$, and finally, φ is PK-valid iff $M \models \varphi$ for any PK-model M.

Then, at a semantic level we will study the *local (product crisp) modal logic PK_l*, defined by letting for all sets $\Gamma \cup \{\varphi\}$ of modal formulas without canonical constants $\Gamma \models_{PK_l} \varphi$ iff for every CPK-model M and for any world $w \in W$, if $M \models_w \Gamma$ then $M \models_w \varphi$. To provide an axiomatization for PK_l, we will consider the following logic K_Π:

– Axioms and rules from Π^*.
– $(K) : \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
$(\mathbf{A}_\Box 1) : (\overline{c} \to \Box\varphi) \leftrightarrow \Box(\overline{c} \to \varphi)$;
$(\mathbf{A}_\Box 2) : \Delta\Box\varphi \leftrightarrow \Box\Delta\varphi$;
$(\mathbf{A}_\Diamond 1) : \Box(\varphi \to \overline{c}) \leftrightarrow (\Diamond\varphi \to \overline{c})$;
$(\mathbf{A}_\Diamond 2) : \Diamond\Delta\varphi \to \Delta\Diamond\varphi$;
$(\mathbf{NR}_\Box) : \frac{\varphi}{\Box\varphi}$, applied only over theorems

The notion of proof in K_Π, denoted $\vdash_{K_\Pi}$ is defined as follows: for any set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \vdash_{K_\Pi} \varphi$ iff there is a (possibly infinite) proof from Γ to φ using axioms and rules from K_Π.

In the logic K_Π, the Deduction Theorem keeps holding because the new modal rule only affects theorems, but having an infinitary logic, to proceed towards a completeness theorem it is necessary to prove that if $\Gamma \vdash_{K_\Pi} \varphi$ holds, then $\Box\Gamma \vdash_{K_\Pi} \Box\varphi$ holds as well, where $\Box\Gamma$ is a shorthand for $\{\Box\psi \mid \psi \in \Gamma\}$.

Since we only added as modal rule one just affects theorems, we can move from modal derivations to propositional ones, just adding a new set of premises (modal theorems): $\Gamma \vdash_{K_\Pi} \varphi$ iff $\Gamma \cup Th_{K_\Pi} \vdash_{\Pi^*} \varphi$ where $Th_{K_\Pi} := \{\theta : \emptyset \vdash_{K_\Pi} \theta\}$. This result is crucial for being able to use a canonical model to prove completeness.

The canonical model can be defined as usual, fixing as universes all the Π^*-evaluations into the canonical standard product algebra (of variables and modal formulae) that satisfy the modal theorems, and defining the relation by $R_c vw = 1$ if and only if

– $v(\Box\theta) = 1 \Rightarrow w(\theta) = 1$ for all $\theta \in Fm$; and
– $v(\Diamond\theta) < 1 \Rightarrow w(\theta) < 1$ for all $\theta \in Fm$;

It can be proven that the extension of the evaluation to modal formulae keeps satisfying the condition $e(v,\varphi) = v(\varphi)$ for every φ, i.e., that in particular both $v(\Box\varphi) = \inf\{w(\varphi) : R_c vw = 1\}$ and $v(\Diamond\varphi) = \sup\{w(\varphi) : R_c vw = 1\}$, and so, the model we define works properly for the completeness proof.

With this, we obtain that K_Π is a modal logic with truth-constants, complete with respect to the class of crisp Kripke models whose worlds evaluate formulas over the canonical standard product algebra.

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Categories of relations for variable-basis fuzziness

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Abstract. Arrow categories as a suitable categorical and algebraic description of $\mathcal{L}$-fuzzy relations have been used to specify and describe fuzzy controllers in an abstract manner. The theory of arrow categories has also been extended to include higher-order fuzziness. In this paper we want to use this theory in order to develop an appropriate description of type-2 fuzzy controllers. An overview of the relational representation of a type-1 fuzzy controller is given before discussing the extension to a type-2 controller. We discuss how to model type reduction, an essential component of any type-2 controller. In addition, we provide a number of examples of general type reducers.

1 Introduction

Allegories and Dedekind categories provide a suitable abstract framework to reason about binary relations. In addition to the standard model of binary relations, i.e., the category Rel of sets and binary relations, these categories are also suitable for $\mathcal{L}$-fuzzy relations. In such a relation every pair of elements is related up to a certain degree indicated by a membership value from the complete Heyting algebra $\mathcal{L}$. Formally, an $\mathcal{L}$-fuzzy relation R (or $\mathcal{L}$-relation for short) between a set A and a set B is a function $R : A \times B \to \mathcal{L}$. However, the theory of those categories is too weak to express certain notions important in the case of $\mathcal{L}$-fuzzy relations. For example, the notion of crispness cannot be expressed in the language of Dedekind categories. A crisp relation is a relation that assigns either 0 (least element of $\mathcal{L}$) or 1 (greatest element of $\mathcal{L}$) to each pair as a membership value. In order to overcome this deficiency the theory of arrow and Goguen categories has been established as an algebraic and categorical framework to reason about these $\mathcal{L}$-fuzzy relations [9].

The theory of arrow and Goguen categories has been studied intensively [3, 9–12, 14]. This includes investigations into higher-order fuzziness [15, 16], i.e., fuzzy relation that are based on fuzzy membership values. A fuzzy membership value is a function $f : \mathcal{L} \to \mathcal{L}$ indicating for every $x \in \mathcal{L}$ up to which degree $f(x)$ the value x is the membership value in question. In addition to the theoretical studies, the theory has been used to model and specify type-1 and type-2 $\mathcal{L}$-fuzzy controllers [13, 17].

Scalar relations and/or ideal relations can be used to identify the underlying lattice $\mathcal{L}$ of membership values even in the case of abstract arrow or Goguen categories. However, these categories are uniform - a properties that implies that all relations of the category are based on the same lattice $\mathcal{L}$. This means that the theory models the fixed-base case.

Indeed, higher-order fuzziness is modeled via multiple arrow categories. It is based on an abstract definition of a type-2 arrow category over a ground arrow category.

In this paper we are interested in the variable-base case, i.e., the case where relations between different objects may use different membership values. Such a theory is interesting for multiple reasons. First of all, it will provide more inside into the relationship between the fixed- and variable-base case. Second, it can also serve as foundation for an internal version of higher-order fuzziness rather than an approach involving multiple categories. Last but not least, this theory may motivate fuzzy controllers that use different membership values within different components.

In this paper we will introduce the notion of a membership basis $\mathfrak{L}$, which mainly consists of a collection of complete Heyting algebras. We will show that every membership basis gives rise to a Dedekind category of fuzzy relations using membership values from the lattices of $\mathfrak{L}$. On the other hand, the collection of the lattices of scalar relations of a Dedekind category forms a membership basis. In addition, we will investigate membership bases that originate from a single lattice.

2 Mathematical Preliminaries

In this section we want to recall some basic notions from lattice, category and allegory theory. For further details we refer to [1, 2].

We will write $R : A \to B$ to indicate that a morphism R of a category $\mathcal{R}$ has source A and target B. We will use ; to denote composition in a category, which has to be read from left to right, i.e., $R;S$ means R first, and then S. The collection of all morphisms $R : A \to B$ is denoted by $\mathcal{R}[A,B]$. The identity morphism on A is written as $\mathbb{I}_A$.

A distributive lattice $\mathcal{L}$ is called a complete Heyting algebra (or frame) iff $\mathcal{L}$ is complete and $x \sqcap \bigsqcup M = \bigsqcup\limits_{y \in M} (x \sqcap y)$ holds for all $x \in \mathcal{L}$ and $M \subseteq \mathcal{L}$.

We will use the framework of Dedekind categories [7, 8] throughout this paper as a basic theory of relations. Categories of this type are called locally complete division allegories in [2].

Definition 1. *A Dedekind category $\mathcal{R}$ is a category satisfying the following:*

1. *For all objects A and B the collection $\mathcal{R}[A,B]$ is a complete Heyting algebra. Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$, respectively.*
2. *There is a monotone operation $\smile$ (called converse) mapping a relation $Q : A \to B$ to $Q^\smile : B \to A$ such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds: $(Q;R)^\smile = R^\smile;Q^\smile$ and $(Q^\smile)^\smile = Q$.*
3. *For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law $(Q;R) \sqcap S \sqsubseteq Q;(R \sqcap (Q^\smile;S))$ holds.*
4. *For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $X : A \to B$ the following holds: $X;R \sqsubseteq S \iff X \sqsubseteq S/R$.*

As mentioned in the introduction the collection of binary relations between sets as well as the collection of $\mathcal{L}$-relations between sets form a Dedekind category normally called $\mathcal{L}$-Rel.

The relational version of a terminal object is a unit. A unit 1 is an object of a Dedekind category so that $\mathbb{I}_1 = \mathbb{T}_{11}$ and $\mathbb{T}_{A1}$ is total for all objects A. Notice that a unit is a terminal object in the subcategory of mappings.

In a Dedekind category one can identify the underlying lattice $\mathcal{L}$ of membership values by the scalar relations on an object.

Definition 2. *A relation $\alpha : A \to A$ is called a scalar on A iff $\alpha \sqsubseteq \mathbb{I}_A$ and $\mathbb{T}_{AA};\alpha = \alpha;\mathbb{T}_{AA}$.*

The notion of scalars was introduced by Furusawa and Kawahara [5] and is equivalent to the notion of ideals, i.e., relations $R : A \to B$ that satisfy $\mathbb{T}_{AA};R;\mathbb{T}_{BB} = R$, which were introduced by Jónsson and Tarski [4]. We denote the complete Heyting algebra of scalar relations on A by $\mathrm{Sc}(A)$.

3 The Category $\mathfrak{L}$-Rel

Let $\mathfrak{L}$ be a collection of complete Heyting algebras with mappings $f_{\mathcal{L}\mathcal{M}} : \mathcal{L} \to \mathcal{M}$ for all $\mathcal{L}$ and $\mathcal{M}$ in $\mathfrak{L}$. Then we call $\mathfrak{L}$ a membership basis iff we have for all $\mathcal{L}, \mathcal{M}$ and $\mathcal{P}$ from $\mathfrak{L}$:

1. $f_{\mathcal{L}\mathcal{L}}$ is the identity, i.e., $f_{\mathcal{L}\mathcal{L}}(x) = x$ for all $x \in \mathcal{L}$,
2. $f_{\mathcal{L}\mathcal{M}}$ is a dense mapping, i.e., if $z \sqsubseteq f_{\mathcal{L}\mathcal{M}}(x)$ for some $x \in \mathcal{L}$, then there is a $y \in \mathcal{L}$ with $f_{\mathcal{L}\mathcal{M}}(y) = z$,
3. $f_{\mathcal{L}\mathcal{M}}$ preserves all suprema, i.e., $f_{\mathcal{L}\mathcal{M}}(\bigsqcup M) = \bigsqcup_{x \in M} f_{\mathcal{L}\mathcal{M}}(x)$ for all $M \subseteq \mathcal{L}$,
4. $f_{\mathcal{L}\mathcal{M}}$ preserves all non-empty infima, i.e., $f_{\mathcal{L}\mathcal{M}}(\bigsqcap M) = \bigsqcap_{x \in M} f_{\mathcal{L}\mathcal{M}}(x)$ for all $\emptyset \neq M \subseteq \mathcal{L}$,
5. $f_{\mathcal{L}\mathcal{M}};f_{\mathcal{M}\mathcal{L}};f_{\mathcal{L}\mathcal{M}} = f_{\mathcal{L}\mathcal{M}}$, i.e., $f_{\mathcal{L}\mathcal{M}}(f_{\mathcal{M}\mathcal{L}}(f_{\mathcal{L}\mathcal{M}}(x))) = f_{\mathcal{L}\mathcal{M}}(x)$ for all $x \in \mathcal{L}$,
6. $f_{\mathcal{M}\mathcal{P}}(f_{\mathcal{L}\mathcal{M}}(x)) \sqcap f_{\mathcal{M}\mathcal{P}}(y) = f_{\mathcal{L}\mathcal{P}}(x) \sqcap f_{\mathcal{M}\mathcal{P}}(y)$ for all $x \in \mathcal{L}$ and $y \in \mathcal{M}$.

Notice that 6. from above implies $f_{\mathcal{L}\mathcal{M}};f_{\mathcal{M}\mathcal{P}} \sqsubseteq f_{\mathcal{L}\mathcal{P}}$, i.e., $f_{\mathcal{M}\mathcal{P}}(f_{\mathcal{L}\mathcal{M}}(x)) \sqsubseteq f_{\mathcal{L}\mathcal{P}}(x)$ for all $x \in \mathcal{L}$. Furthermore, we have the following lemma.

Lemma 1. *Suppose $\mathfrak{L}$ is a membership basis. Then the complete Heyting algebras $f_{\mathcal{L}\mathcal{M}}(\mathcal{L}) \subseteq \mathcal{M}$ and $f_{\mathcal{M}\mathcal{L}}(\mathcal{M}) \subseteq \mathcal{L}$ are isomorphic (via $f_{\mathcal{M}\mathcal{L}}$ and $f_{\mathcal{L}\mathcal{M}}$).*

Suppose $\mathfrak{L}$ is a membership basis. Then we define the category $\mathfrak{L}$-Rel by

1. The objects of $\mathfrak{L}$-Rel are pairs $(A, \mathcal{L})$ where A is a set and $\mathcal{L}$ is from $\mathfrak{L}$.
2. A relation R from $(A, \mathcal{L})$ to $(B, \mathcal{M})$ is a function $R : A \times B \to f_{\mathcal{L}\mathcal{M}}(\mathcal{L})$, i.e., a $f_{\mathcal{L}\mathcal{M}}(\mathcal{L})$-fuzzy relation.
3. The identity relation $\mathbb{I} : A \times A \to \mathcal{L}$ on $(A, \mathcal{L})$ is defined as usual.
4. Composition of two relations $Q : (A, \mathcal{L}) \to (B, \mathcal{M})$ and $R : (B, \mathcal{M}) \to (C, \mathcal{P})$ is defined by
$$(Q;R)(x,z) = \bigsqcup_{y \in B} f_{\mathcal{M}\mathcal{P}}(Q(x,y)) \sqcap R(y,z).$$

We obtain the following theorem.

Theorem 1. *$\mathfrak{L}$-Rel is a Dedekind category.*

A special case of the theorem above is obtained when $\mathfrak{L}$ is generated by a single lattice. Any element $a \in \mathcal{L}$ induces a complete Heyting algebra on its down-set $a{\downharpoonleft} = \{x \in \mathcal{L} \mid x \leq a\}$. Furthermore, we define mappings $f_{a{\downharpoonleft}b{\downharpoonleft}} : a{\downharpoonleft} \to b{\downharpoonleft}$ by $f_{a{\downharpoonleft}b{\downharpoonleft}}(x) = x \sqcap b$.

Lemma 2. *Let $\mathcal{L}$ be a complete Heyting algebra. Then the down-sets of $\mathcal{L}$ together with the mappings $f_{a{\downharpoonleft}b{\downharpoonleft}}$ form a membership basis.*

We call the structure above the membership basis induced by $\mathcal{L}$. The next theorem shows that a lot of membership bases are actually induced by a single lattice.

Theorem 2. *Let $\mathfrak{L}$ be a membership basis. If $\mathfrak{L}$ is small, i.e., the collection of lattices is a set, then there is a complete Heyting algebra $\mathcal{L}$ so that $\mathfrak{L}$ is a subset of the membership basis induced by $\mathcal{L}$.*

4 The Membership Basis of Scalar Relations

In this section we want to study the lattices of scalar elements in an arbitrary Dedekind category.

Theorem 3. *Let $\mathcal{R}$ be a Dedekind category. Then the collection of all lattices $\mathrm{Sc}(A)$ of scalars on A together with the mappings $f_{\mathrm{Sc}(A)\mathrm{Sc}(B)} : \mathrm{Sc}(A) \to \mathrm{Sc}(B)$ defined by $f_{\mathrm{Sc}(A)\mathrm{Sc}(B)}(\alpha) = \mathbb{T}_{BA};\alpha;\mathbb{T}_{AB} \sqcap \mathbb{I}_B$ forms a membership basis.*

We call the structure above the membership basis of $\mathcal{R}$.

The final theorem of this paper shows that whether a membership basis is generated by a single lattice is tightly connected to the existence of a unit.

Theorem 4. *If $\mathcal{R}$ has a unit, then the membership basis of $\mathcal{R}$ is induced by the complete Heyting algebra $\mathrm{Sc}(1)$.*

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Inconsistency management based on relevance degrees

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Inconsistency handling in constrained databases is a primary issue in the context of consistent query answering, data integration, and data exchange. The standard approaches to this issue are usually based on the principle of minimal change, aspiring to achieve consistency via a minimal amount of data modifications (see, e.g., [2, 6, 7]). A key question in this respect is how to *choose* among the different possibilities of restoring the consistency of ('repairing') a database.

Earlier approaches to inconsistency management were based on the assumption that there should be some fixed, pre-determined way of repairing a database. Recently, there has been a paradigm shift towards user-controlled inconsistency management policies. Works taking this approach provide a possibility for the user to express some *preference* over all possible database repairs, preferring certain repairs to others[4] (see [12] for a survey and further references). While such approaches provide the user with flexibility and control over inconsistency management, in reality they entail a considerable technical burden on the user's shoulders of calibrating, updating and maintaining preferences or policies. Moreover, in many cases these preferences may be *dynamic*, changing quickly on the go (e.g., depending on the user's geographical location). In the era of ubiquitous computing, users want *easy* – and sometimes even *fully automatic* – inconsistency management solutions with little cognitive load, while still expecting them to be *personalized* to their particular needs. This leads to the idea of introducing *context-awareness* into inconsistency management. In recent years there is a dramatic increase in the interest in context-aware systems. Context is usually defined as "any information that can be used to characterize the situation of an entity, where an entity can be a person, place, or physical or computational object" (see [1]); context-awareness means thus making use of this context. We believe that the idea of context-awareness can be very naturally extended to the context of inconsistency management.

[4] Some notable examples are [8–10, 14].

In this talk we will report on a work in progress, in which we aim at developing a general theoretical framework for capturing context-aware inconsistency management. To this end we incorporate notions and techniques from context-aware systems and database repair by combining the following two ingredients:

- *Distance-based semantics* ([3–5]) for restoring the consistency inconsistent databases according to the principle of minimal change, and
- *Context-awareness considerations* ([11, 13]), captured by real-valued relevance degrees for incorporating user context and preferences.

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