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Functional Equations and Inequalities

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Abstracts

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Editors

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FUNCTIONAL EQUATIONS AND INEQUALITIES

ABSTRACTS

Bernard De Baets, Radko Mesiar,
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Since their inception in 1979, the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide directions for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2016 will be the 36th seminar carrying on this tradition and is devoted to the theme “Functional Equations and Inequalities”. The goal of the seminar is to present and to discuss recent advances on (algebraic) functional equations and inequalities and their applications in pure and applied mathematics, with special emphasis on many-valued logics, multicriteria decision aid and preference modelling.

A considerable amount of interesting contributions were submitted for possible presentation at LINZ 2016 and subsequently reviewed by PC members. This volume contains the abstracts of the accepted contributions. These regular contributions are complemented by five invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

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Contents

Construction of flipping-invariant functions in higher dimensions

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Recently, there have been several studies of transformations called 'flippings', which map n-copulas to n-copulas. The resulting transforms can be thought of as the multivariate cumulative distribution functions of random vectors that are obtained by replacing (called flipping) each of the original random variables from a given subset of the random vector by a countermonotonic counterpart. It is important to note that if all the variables are flipped, the resulting transform is the well-known survival n-copula (for more details, see [9]). In the bivariate case, these transformations have been studied from the algebraic point of view in [7], and have been further generalized to binary aggregation functions in [2, 3]. In the multivariate case, these operations have been studied in [5] for n-copulas, while in [4] the authors have studied the case of multivariate aggregation functions.

Inspired by the above results and the notion of invariant copula (i.e., a copula that coincides with one of its transforms [8]), we present two methods to construct flipping-invariant copulas in higher dimensions, given a lower-dimensional marginal copula. Both methods are partially based on an associative extension of an aggregation function, although not in the way that it is usually done, as it can be easily seen that there is no associative solution to the Frank functional equation in the n-dimensional case for $n \geqslant 3$ (see [1, 5]).

In the first method, we construct a 3-dimensional function that is flipping invariant, starting from a bivariate flipping-invariant symmetric copula. We show that if the function that is obtained by this transformation is increasing, then it is a 3-quasi-copula. We also present some numerical examples of this method for well-known families of flipping-invariant 2-copulas, such as the Frank copula family and the Farlie-Gumbel-Morgenstern copula family. In the second method, we construct a 3-dimensional aggregation function that it is flipping invariant in the last variable starting from an arbitrary 2-copula. We study some properties of the aggregation function that is obtained by this transformation, as well as conditions that guarantee that it is a 3-(quasi)-copula. Finally, we discuss several possible generalizations of both methods in higher dimensions.

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Functional equations related to the distributivity of fuzzy implications

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Abstract. In this contribution we discuss solutions of different functional equations, which appear during the investigations connected with the distributivity of fuzzy implications.

In classical logic conjunction distributes over disjunction and disjunction distributes over conjunction. Moreover, implication is left-distributive over conjunction and disjunction:

$$p \to (q \wedge r) \equiv (p \to q) \wedge (p \to r),$$
$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r).$$

At the same time it is neither right-distributive over conjunction nor over disjunction. However, the following two equalities, that are kind of right-distributivity of implications, hold:

$$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r), \tag{1}$$
$$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r).$$

We can rewrite the above four classical tautologies in fuzzy logic to obtain the following functional equations, called the distributivity equations for multivalued implications:

$$I(x, C_1(y, z)) = C_2(I(x, y), I(x, z)), \tag{D1}$$
$$I(x, D_1(y, z)) = D_2(I(x, y), I(x, z)), \tag{D2}$$
$$I(C(x, y), z) = D(I(x, z), I(y, z)), \tag{D3}$$
$$I(D(x, y), z) = C(I(x, z), I(y, z)), \tag{D4}$$

that are satisfied for all $x, y, z \in [0, 1]$, where I is some generalization of the classical implication, C, C_1, C_2 are some generalizations of the classical conjunction and D, D_1, D_2 are some generalizations of the classical disjunction. We can define and study those equations in any lattice $\mathcal{L} = (L, \leqslant_L)$ instead of the unit interval $[0, 1]$ with regular order „$\leqslant$” on the real line, as well.

From the functional equation's point of view J. Aczél was probably the one that studied right-distributivity first (see [1, Section 7.1.3, Th. 6]). He characterized solutions of the functional equation (D3) in the case when $C = D$, among functions I that are bounded from below and functions C that are continuous, strictly increasing, associative and have neutral element. The importance of these equations in fuzzy logic has been introduced by Combs and Andrews [17], wherein they exploit the classical tautology (1) in their inference mechanism towards reduction in the complexity of fuzzy "IF-THEN" rules. Subsequently, there were many discussions in [15, 16, 18, 20], most of them pointing out the need for a theoretical investigation required for employing such equations, as concluded by Dick and Kandel [18], "Future work on this issue will require an examination of the properties of various combinations of fuzzy unions, intersections and implications" or by Mendel and Liang [20], "We think that what this all means is that we have to look past the mathematics of IRC$\Leftrightarrow$URC and inquire whether what we are doing when we replace IRC by URC makes sense."

We can divide the investigations on these distributivity equations into two streams: first, in which the implication I is given and second, when all other operations except the implication I are given. In the first group the most important article has been written by Trillas and Alsina [26], where Eq. (D3) has been investigated for three main families of fuzzy implications (S-implications, R-implications and QL-implications) — here C is an unknown t-norm and D is an unknown t-conorm. In particular, they showed that in the case of S-implications and R-implications, Eq. (D3) holds if and only if $T = \min$ and $S = \max$. In a similar way Balasubramaniam and Rao [14] considered the other equations for different types of fuzzy implications.

From the other side, Eq. (D1) with $C_1 = C_2$ the product t-norm is one of the characteristic properties in the class of fuzzy implications introduced by Türksen et al. [19]. This equation has been considered also by Baczyński in [2, 3] along with other equations (like contrapositive symmetry), and he has characterized fuzzy implications I in the case when $C_1 = C_2$ is a strict t-norm. It should be noted here that this equation, when $C_1 = C_2$ is a nilpotent t-norm, has been also investigated by Qin and Yang [23] (also with contrapositive symmetry). Baczyński and Jayaram in [8, 4] have examined solutions of Eq. (D2), when D_1, D_2 are continuous, Archimedean t-conorms and I is an unknown function. In [6] one can find solutions of Eqs. (D3) and (D4), when C (respectively D) is a continuous, Archimedean t-norm (respectively t-conorm). Recently published papers [22, 7, 21] weaken the assumptions and consider only continuous t-norms and/or t-conorms.

The above equations are also considered for other types of fuzzy connectives. Eq. (D3) has been studied by Ruiz-Aguilera and Torrens in [24] for the major part of known classes of uninorms with continuous underlying t-norm and t-conorm and for strong implications derived from uninorms, while in [25], they also studied Eq. (D3), but with the assumption that I is a residual implication derived from a given uninorm. Of course in this case C is a conjunctive uninorm, while D is a disjunctive uninorm. These investigations were continued by one of the author in [5], where he considered all above equations for representable uninorms.

What is really interesting from mathematical point of view is that many of the important results obtained in the above mentioned articles are connected with different functional equations. Our main goal of this contribution is to discuss solutions of the following functional equations, which appear during the investigations connected with the distributivity of fuzzy implications:

$$
\begin{aligned}
F(\min(x+y,r_1)) &= \min(F(x)+F(y),r_2), && x,y\in[0,r_1]\\
f(m_1(x+y)) &= m_2(f(x)+f(y)), && x,y\in[0,r_1],\\
g(u_1+v_1,u_2+v_2) &= g(u_1,u_2)+g(v_1,v_2), && (u_1,u_2),(v_1,v_2)\in L^{\overline{\infty}},\\
h(xc(y)) &= h(x)+h(xy), && x,y\in(0,\infty),\\
k(\min(j(y),1)) &= \min(k(x)+k(xy),1), && x\in[0,1], y\in(0,1],
\end{aligned}
$$

where:

- $F\colon [0,r_1]\to[0,r_2]$, for some $r_1,r_2\in(0,\infty]$;
- $f\colon [0,r_1]\to[0,r_2]$, for some constants r_1,r_2 that may be finite or infinite, and for functions m_2 that may be injective or not;
- $g\colon L^{\overline{\infty}}\to[-\infty,\infty]$, for $L^{\overline{\infty}}=\{(u_1,u_2)\in[-\infty,\infty]^2 \mid u_1\le u_2\}$;
- $h,c\colon (0,\infty)\to(0,\infty)$ and function h is continuous or is a bijection;
- $k\colon [0,1]\to[0,1]$, $g\colon (0,1]\to[1,\infty)$ and function k is continuous.

Many of newly discussed results have been obtained by the authors in collaboration with R. Ger, M. E. Kuczma or T. Szostok. Part of them have been already published either in scientific journals (see [13]) or in refereed proceedings (see [12, 9–11]).

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Functional equations in decision analysis under Cumulative Prospect Theory

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Abstract. We present some natural problems, stemming from decision analysis under Cumulative Prospect Theory, leading to functional equations and inequalities.

1 Introduction

The classical model of decision making under risk, namely the Expected Utility model, is founded on a system of axioms. This fact lead many to believe that this is the only appropriate tool for the decision making under risk. However, this classical model has been violated by observed behaviors (e.g. the Allais paradox). Several authors presented the alternative versions of the Expected Utility models which explain the paradoxes. One of them, namely *the Cumulative Prospect Theory*, has been created by Tversky and Kahneman [11]. It is based on experiments carried out by Tversky and Kahneman, showing that, making decisions under risk, people set a reference point and consider the lower outcomes as losses and larger ones as gains. Furthermore, people distort probabilities and, in general, the probabilities of gains and losses are distorted in a different way. It turns out that many problems, having their origins in the Cumulative Prospect Theory, lead to functional equations and inequalities. In this talk we present some examples of such problems and their solutions. Several applications of functional equations and inequalities in various problems stemming from the Expected Utility Theory can be found in a survey paper [1].

2 Choquet integral

Assume that $L^\infty(\Omega, \Sigma, P)$ is a family of bounded random variables on a probability space (Ω, Σ, P). Let $g : [0,1] \to [0,1]$ be *a distortion function*, that is a non-decreasing function with $g(0) = 0$ and $g(1) = 1$. For $X \in L^\infty(\Omega, \Sigma, P)$, *the Choquet integral* related to g is defined as follows

$$E_g(X) = \int_{-\infty}^{0} (g(P(X > t)) - 1)\, dt + \int_{0}^{\infty} g(P(X > t))\, dt.$$

The Choquet integral has several interesting properties. In particular, it is additive for comonotonic risks, positively homogeneous and monotone. For more details concerning the properties of the Choquet integral we refer to [4]. Under the Cumulative Prospect Theory a preference relation $\preceq$ of a decision maker is represented in the following way

$$X \preceq Y \quad \Leftrightarrow \quad E_{gh}u(X) \leq E_{gh}u(Y),$$

where $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and continuous function with $u(0) = 0$, called *a value function*, and E_{gh} is *the generalized Choquet integral* related to the distortion functions g (for gains) and h (for losses), defined as follows

$$E_{gh}(X) = E_g(\max\{X, 0\}) - E_h(\max\{-X, 0\}).$$

Note that If $x < 0 < y$, $p \in [0,1]$ and a random variable X takes the values x and y with probabilities $P(X = x) = 1 - p$ and $P(X = y) = p$, then

$$E_{gh}(X) = g(1-p)u(y) + h(p)u(x). \tag{1}$$

3 Properties of the Premium Principles under Cumulative Prospect Theory

Consider an insurance company having the initial wealth w, a value function u and the probability distortion functions g (for gains) and h (for losses). The company covers a risk treated as a non-negative random variable. Roughly speaking, a premium principle is a rule for assigning a premium to an insured risk. One of the frequently applied methods of pricing insurance contracts is the *Principle of Equivalent Utility*. Under the Expected Utility Theory a premium $H(X)$ for risk X is a solution of the equation

$$u(w) = E[u(w + H(X) - X)]. \tag{2}$$

A solution $H(X)$ of (1) in the case $w = 0$ is called the *zero utility principle*. Equation (2) has the following interpretation: a value of $H(X)$ is such that the insurer is indifferent between not accepting and accepting the insurance risk. The Principle of Equivalent Utility under the Rank Dependent Utility Theory has been considered in [5]. In a recent paper [7] a modification of the Principle of Equivalent Utility adjusted to the Cumulative Prospect Theory has been introduced. This approach leads to the equation

$$u(w) = E_{gh}[u(w + H(X) - X)]. \tag{3}$$

In [7] several properties of the premium have been considered. One of them is a positive homogeneity. Let us recall that, for any $a > 0$, a premium principle H is said to be *a-homogeneous* provided $H(aX) = aH(X)$ for all feasible risks X. A premium principle H is said to be *positively homogeneous* if it is a-homogeneous for every $a > 0$. Let, for every $x > 0$ and $p \in [0,1]$, (x,p) denotes the random variable X such that $P(X = 0) = 1 - p$ and $P(X = x) = p$. Furthermore, let $\mathcal{X}_2 := \{(x,p) : x > 0, p \in [0,1]\}$. It can be proved that $H(x,p) \in [0,x]$ for $x > 0$ and $p \in [0,1]$. Therefore, making use of (1), from (3) we derive that

$$g(1-p)u(H((x,p))) + h(p)u((H((x,p)) - x)) = 0 \text{ for } x > 0, p \in [0,1].$$

Moreover, if $a_1, a_2 \in (0,\infty) \setminus \{1\}$, $a_1 \neq a_2$ and H is a_i-homogeneous for $i \in \{1,2\}$, then for $x > 0$, $p \in [0,1]$ and $i \in \{1,2\}$, we get

$$g(1-p)u(a_i H((x,p))) + h(p)u(a_i(H((x,p)) - x)) = 0.$$

Solving this system of functional equations we obtain the following result (cf. [3]).

Theorem 1. *Assume that $a_1, a_2 \in (0,\infty) \setminus \{1\}$ are such that $\frac{\ln a_1}{\ln a_2}$ is irrational and $g(1-p)h(p) > 0$ for $p \in (0,1)$. If* (3) *holds for $w = 0$ and H is a_i-homogeneous for $i \in \{1,2\}$ and every $X \in \mathcal{X}_2$, then there exist $b, c, d > 0$ such that*

$$u(x) = \begin{cases} -b(-x)^d & \text{for } x \in (-\infty, 0), \\ cx^d & \text{for } x \in [0,\infty). \end{cases} \tag{4}$$

Moreover, if (3) *holds also for some $w > 0$ then*

$$h(p) = 1 - g(1-p) \quad \text{for} \quad p \in [0,1] \tag{5}$$

and there exists an $a > 0$ such that

$$u(x) = ax \quad \text{for} \quad x \in \mathbb{R}. \tag{6}$$

If (3) *holds for $w = 0$ and u is of the form* (4) *with some $b, c, d > 0$, then H is positively homogeneous.*

A similar problem under the Expected Utility Theory has been investigated in [10].

4 A class of risk measures under Cumulative Prospect Theory

Assume that $u : \mathbb{R} \to \mathbb{R}$ is a value function and g, h are probability distortion functions for gains and losses, respectively. From the properties of the generalized Choquet integral and the value function u it follows that, for every $X \in L^\infty(\Omega, \Sigma, P)$, there exists a unique $C(X) \in \mathbb{R}$ such that

$$u(C(X)) = E_{gh}u(X). \tag{7}$$

So, the decision maker is indifferent between playing the lottery X and obtaining the amount $C(X)$ for sure. A functional $C : L^\infty(\Omega, \Sigma, P) \to \mathbb{R}$ defined in this way is called *the certainty equivalent*. A functional $\rho := -C$ is a risk measure, which is closely related to the important notion of the insurance mathematics, namely *the Mean-Value Premium Principle*. More details on this premium principle under Cumulative Prospect Theory can be found in [8]. In view of (7), we get

$$\rho(X) = -u^{-1}(E_{gh}u(X)) \quad \text{for} \quad X \in L^\infty(\Omega, \Sigma, P). \tag{8}$$

It is widely accepted that a risk measure should have the following properties:

(M) monotonicity: $X \leq Y \Rightarrow \rho(Y) \leq \rho(X)$,

(TI) translation invariance: $\rho(X + m) = \rho(X) - m$ for $m \in \mathbb{R}$,

(PH) positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for $\lambda \geq 0$,
(S) subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

Any risk measure satisfying these conditions is called *coherent* (cf. [2]). The properties of ρ in the case where $g(p) = h(p) = p$ for $p \in [0,1]$ have been investigated in [9]. The case where $u(x) = x$ for $x \in \mathbb{R}$ has been studied in [6]. Note that as the Choquet integral and so the generalized Choquet integral, are monotone and the value function u is strictly increasing, for every probability distortion functions g and h, the risk measure ρ defined by (8), has the property (M). It is clear that the problems of translation invariance, positive homogeneity and subadditivity of ρ can be expressed in terms of functional equations and inequalities. Solving them we obtain the following results.

Theorem 2. *A risk measure ρ defined by* (8) *satisfies (TI) if and only if* (5) *holds and u is either of the form* (6) *with some $a > 0$, or*

$$u(x) = b(e^{cx} - 1) \quad \textit{for} \quad x \in \mathbb{R}$$

with some $b, c \in \mathbb{R}$ such that $bc > 0$.

Theorem 3. *A risk measure ρ defined by* (8) *satisfies (PH) if and only if u is of the form* (4) *with some $b, c, d > 0$.*

Theorem 4. *A risk measure ρ defined by* (8) *is coherent if and only if g is convex,* (5) *holds and u is of the form* (6) *with some $a > 0$, that is*

$$\rho(X) = -E_g X \quad \textit{for} \quad X \in L^\infty(\Omega, \Sigma, P)$$

with a convex probability distortion function g.

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Equations and inequalities in quantum logics

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The mathematical formalism of quantum theory has inspired the development of different forms of non-classical logics, called *quantum logics*. In many cases the semantic characterizations of these logics are based on special classes of algebraic structures defined in a Hilbert-space environment. The prototypical example of quantum logic (created by Birkhoff and von Neumann) can be semantically characterized by referring to the class of all Hilbert-space lattices, whose support is the set $\mathcal{P}(\mathcal{H})$ of all projections of a Hilbert space $\mathcal{H}$. The question whether the class of all Hilbert-space lattices can be axiomatized by a set of equations is still open. What is known is that the variety of all orthomodular lattices (which gives rise to a semantic characterization of a logic often termed "orthodox quantum logic") does not represent a faithful abstraction from the class of all Hilbert-space lattices. A characteristic example of an equation that holds in all Hilbert-space lattices, being possibly violated in orthomodular lattices is the *orthoarguesian law*.

Interesting generalizations of Birkhoff and von Neumann's quantum logic are the so called *unsharp (or fuzzy) quantum logics* that can be semantically characterized by referring to different classes of algebraic structures whose support is the set of all *effects* of a Hilbert space. According to the standard interpretation of the quantum formalism, any projection $P \in \mathcal{P}(\mathcal{H})$ represents a *sharp physical event* to which any possible state of a physical system S (associated with the space $\mathcal{H}$) assigns a probability-value. Such events are called "sharp" because they satisfy the non-contradiction principle: $P \wedge P^{\perp} = \mathbb{O}$ (the infimum between P and its orthogonal projection $P^{\perp}$ is the null projection $\mathbb{O}$). Effects, instead, represent *unsharp physical events* that may violate the non-contradiction principle. The set $\mathcal{E}(\mathcal{H})$ of all effects of a Hilbert space $\mathcal{H}$ is defined as the largest set of linear bounded operators E for which a *Born-probability* can be defined. In other words, for any density operator ρ of $\mathcal{H}$ (representing a possible state of a physical system S whose associated Hilbert space is $\mathcal{H}$), we have: $\mathtt{Tr}(\rho E) \in [0, 1]$ (where $\mathtt{Tr}$ is the trace-functional). The number $\mathtt{Tr}(\rho E)$ represents the probability that a quantum system S in state ρ verifies the physical event represented by the effect E. Of course, $\mathcal{E}(\mathcal{H})$ properly includes $\mathcal{P}(\mathcal{H})$. Let $\mathfrak{D}(\mathcal{H})$ represent the set of all density operators of $\mathcal{H}$. A natural partial order relation $\preceq$ can be defined on the set $\mathcal{E}(\mathcal{H})$ in terms of the notion of Born-probability: $E \preceq F := \forall \rho \in \mathfrak{D}(\mathcal{H})[\mathtt{Tr}(\rho E) \leq \mathtt{Tr}(\rho F)]$. The partial order $\preceq$ induces two different algebraic structures on the two sets $\mathcal{P}(\mathcal{H})$ and $\mathcal{E}(\mathcal{H})$. While $(\mathcal{P}(\mathcal{H}), \preceq, \mathbb{O}, \mathbb{I})$ (bounded by the null projection $\mathbb{O}$ and by the identity projection $\mathbb{I}$) is a complete lattice, the structure $(\mathcal{E}(\mathcal{H}), \preceq, \mathbb{O}, \mathbb{I})$ is a bounded poset that is not a lattice. Taking for granted that effects do not have a lattice-structure, different kinds of algebraic structures have been induced on the set $\mathcal{E}(\mathcal{H})$, giving rise to different forms of *unsharp quantum logics* [2].

Strangely enough, for a long time, the logical approaches to unsharp quantum theory have completely neglected an alternative possibility of defining on $\mathcal{E}(\mathcal{H})$ a somewhat natural partial order that (unlike $\preceq$) gives rise to a lattice-structure [6, **?**]. As is well known, according to the quantum formalism the *observables* of a system S are mathematically represented as self-adjoint operators of the Hilbert space $\mathcal{H}$ associated to S. At the same time, self-adoint operators of $\mathcal{H}$ can be equivalently represented as *spectral families* (maps $M : \mathbb{R} \mapsto \mathcal{P}(\mathcal{H})$ that satisfy the following conditions: 1) $\lambda \leq \mu \Rightarrow M(\lambda) \preceq M(\mu)$; 2) $M(\lambda) = \bigwedge_{\mu > \lambda} M(\mu)$; 3) $\bigwedge_{\lambda \in \mathbb{R}} M(\lambda) = \mathbb{O}$ and $\bigvee_{\lambda \in \mathbb{R}} M(\lambda) = \mathbb{I}$). One can prove that any spectral family M (of a space $\mathcal{H}$) uniquely determines a self-adjoint operator A^M; vice versa, any self-adjoint operator A uniquely determines a spectral family M^A. On this basis, the *spectral partial order* $\preceq_S$ (on the set $\mathcal{E}(\mathcal{H})$) can be defined as follows:

$$E \preceq_S F := \forall \lambda \in \mathbb{R}[M^F(\lambda) \preceq M^E(\lambda)].$$

We have: 1) $\forall E, F \in \mathcal{E}(\mathcal{H})[E \preceq_S F \Rightarrow E \preceq F]$, but generally not the other way around; 2) $\forall P, Q \in \mathcal{P}(\mathcal{H})[P \preceq_S Q \Leftrightarrow P \preceq Q]$ (for sharp events, $\preceq_S$ and $\preceq$ coincide); 3) the structure $(\mathcal{E}(\mathcal{H}), \preceq_S, ', \mathbb{O}, \mathbb{I})$ (where $E' = \mathbb{I} - E$) is an *involutive bounded lattice* that is *regular* ($E \wedge_S E' \preceq_S F \vee_S F'$, where $\wedge_S$ and $\vee_S$ are the lattice-theoretic infimum and supremum) and *paraorthomodular* ($E \preceq_S F$ and $E' \wedge_S F = \mathbb{O} \Rightarrow E = F$). In the class of all abstract regular involutive bounded lattices, paraorthomodularity cannot be expressed as an equation. Consequently, the class of all paraorthomodular regular involutive bounded lattices is not a variety (unlike the class of all orthomodular ortholattices, for which orthomodularity and paraorthomodularity are equivalent properties).

A different approach to quantum logic has been developed in the framework of *quantum computational logics*, inspired by the theory of quantum computation [3]. While sharp and unsharp quantum logics refer to possible structures of physical events, the basic objects of quantum computational logics are *pieces of quantum information*: possible states of quantum systems that can store the information in question. The simplest piece of quantum information is a *qubit*: a unit-vector of the space $\mathbb{C}^2$ that can be represented as a superposition $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$. The vectors $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$ (the two elements of the canonical basis of $\mathbb{C}^2$) represent, in this framework, the two classical bits or (equivalently) the two classical truth-values. It is interesting to consider a "many-valued generalization" of qubits, represented by *qudits*: unit-vectors living in a space $\mathbb{C}^d$ (where $d \geq 2$)[5]. The elements of the canonical basis of $\mathbb{C}^d$ can be regarded as different truth-values: $|0\rangle = |\frac{0}{d-1}\rangle = (1, 0, \ldots, 0)$, $|\frac{1}{d-1}\rangle = (0, 1, 0, \ldots, 0)$, $|\frac{2}{d-1}\rangle = (0, 0, 1, 0, \ldots, 0)$,......, $|1\rangle = |\frac{d-1}{d-1}\rangle = (0, \ldots, 0, 1)$. While $|0\rangle$ and $|1\rangle$ represent the truth-values *truth* and *falsity*, all other basis-elements correspond to *intermediate* truth-values. In this framework, any piece of quantum information can be identified with a density operator ρ living in a tensor-product space $\mathcal{H}_d^{(n)} = \underbrace{\mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d}_{n-times}$ (with $n \geq 1$). The canonical basis of $\mathcal{H}_d^{(n)}$ (whose elements are called d-*registers*) is the following set:

$$\left\{ |x_1, \ldots, x_n\rangle : |x_1\rangle, \ldots, |x_n\rangle \text{ are elements of the canonical basis of } \mathbb{C}^d \right\}$$

(where $|x_1, \ldots, x_n\rangle$ is an abbreviation for the tensor product $|x_1\rangle \otimes \ldots \otimes |x_n\rangle$). A *quregister* of $\mathcal{H}_d^{(n)}$ is a pure state, represented by a unit-vector $|\psi\rangle$ or (equivalently) by the corresponding density operator $P_{|\psi\rangle}$ (the projection-operator that projects over the closed subspace determined by $|\psi\rangle$).

In any space $\mathcal{H}_d^{(n)}$, each truth-value $|\frac{j}{d-1}\rangle$ determines a corresponding *truth-value projection* $P^{(n)}_{\frac{j}{d-1}}$, whose range is the closed subspace spanned by the set of all d-registers $|x_1, \ldots, x_n\rangle$ whose last element $|x_n\rangle$ is $|\frac{j}{d-1}\rangle$. From an intuitive point of view, $P^{(n)}_{\frac{j}{d-1}}$ represents the property "being true according to the truth-value $|\frac{j}{d-1}\rangle$" (briefly, $|\frac{j}{d-1}\rangle$-*truth*). On this basis, one can apply the Born-rule and define for any state ρ (of $\mathcal{H}_d^{(n)}$) the probability that ρ satisfies the $|\frac{j}{d-1}\rangle$-truth: $\mathrm{p}^{(d)}_{\frac{j}{d-1}}(\rho) := \mathtt{Tr}\Big(\rho\, P^{(n)}_{\frac{j}{d-1}}\Big)$. The *probability tout court* of ρ can be then identified with the weighted mean of the probabilities of all truth-value properties: $\mathrm{p}^{(d)}(\rho) := \frac{1}{d-1}\sum_{j=1}^{d-1} j\, \mathrm{p}^{(d)}_{\frac{j}{d-1}}(\rho)$. One can prove that: $\mathrm{p}^{(d)}(\rho) = \mathtt{Tr}\Big(\rho\,(\mathbb{I}^{(n-1)} \otimes E)\Big)$, where $\mathbb{I}^{(n-1)}$ is the identity operator of the space $\mathcal{H}_d^{(n-1)}$, while E is the effect (of the space $\mathbb{C}^d$) represented by the following matrix:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

In the particular case where ρ corresponds to the qubit $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we obtain: $\mathrm{p}^{(2)}(\rho) = |c_1|^2$.

For any choice of a truth-value number d, one can naturally define a preorder-relation on the set $\mathfrak{D}^d$ of all density operators ρ living in some space $\mathcal{H}_d^{(n)}$: $\rho \preceq_d \sigma := \mathrm{p}^{(d)}(\rho) \leq \mathrm{p}^{(d)}(\rho)$. This preorder plays an important role in the definition of the logical-consequence relation for quantum computational logics.

Quantum information is processed by *quantum logical gates* (briefly, *gates*): unitary quantum operations that transform density operators in a reversible way. Some gates are called *semiclassical*, because they always transform d-registers (representing classical information) into d-registers. Other gates are called *genuine quantum gates*, because they can create quantum superposition from d-register inputs. Examples of the first kind are: different forms of negation (the *diametrical negation*, an *intuitionistic-like negation*, an *anti-intuitionistic-like negation*), different *modal operators*, different forms of conjunction and disjunction (including a *Łukasiewicz-like conjunction* and *disjunction*). Examples of the second kind are: the *Hadamard-gate* and the *square root of* *negation*. Different choices of gate-systems (which refer to a given truth-value number d) give rise to different *quantum computational algebraic structures*.

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On the convolution of lattice functions

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Abstract. Convolution operations are a matter of interest in the context of information fusion. In this work, we develop an abstract theory for two particular convolutions operations, join and meet, defined over the set of lattice functions.

1 Convolution operations

Classical integration-based convolution operations are frequently used in engineering, in particular in fields such as signal and image processing and control theory. In other fields, such as mathemematical morphology [1, 2] and fuzzy set theory [3, 4], different convolution operations are in use. In both cases, a convolution operation $*$ is an operation that transforms two functions $f, g : P \longrightarrow Q$ into a new one $f * g : P \longrightarrow Q$.

Specifically, in this work we consider the set of functions from a bounded lattice L_1 to a completely distributive lattice L_2, namely, we consider the set

$$\mathcal{F} = \{f | f : L_1 \longrightarrow L_2\}.$$

We define the operations $\sqcup$ and $\sqcap$, called join and meet respectively, which are given by, for all $f_1, f_2 \in \mathcal{F}$:

$$f_1 \sqcup f_2(z) = \bigvee_{x \vee y = z} f_1(x) \wedge f_2(y) \tag{1}$$

$$f_1 \sqcap f_2(z) = \bigvee_{x \wedge y = z} f_1(x) \wedge f_2(y). \tag{2}$$

The aim of the work is to develop an abstract theory of the algebraic structure generated by these two operations on $\mathcal{F}$. Note that a similar study when $L_1 = [0, 1]$ and $L_2 = [0, 1]$ can be found in [3]. However, when general lattices (instead of $[0, 1]$) are considered, some properties such us idempotency do not hold. In order to ensure these properties, some restrictions on the set of functions are required. In particular, the following classes of functions are considered:

(a) $\mathcal{N} = \{f \in \mathcal{F} \mid \bigvee_{x \in L_1} f(x) = 1\}$;

(b) $\mathcal{C} = \{f \in \mathcal{F} \mid$ for all $x_1 \leq x_2 \leq x_3,\ f(x_2) \geq f(x_1) \wedge f(x_3)\}$;
(c) $\mathcal{I} = \{f \in \mathcal{F} \mid$ for all $x_1, x_2 \in L_1$,
$f(x_1 \vee x_2) \geq f(x_1) \wedge f(x_2)$ and $f(x_1 \wedge x_2) \geq f(x_1) \wedge f(x_2)\}$;
(d) $\mathcal{S} = \{f \in \mathcal{F} \mid$ there exists $x^* \in L_1$ such that $f(x^*) = 1$ and
for all $x \neq x^*,\ f(x) = 0\}$.

Taking into account these sets of functions, the following can be stated.

Theorem 1. *The join and meet operations given in Eqs.* (1) *and* (2) *generate a lattice on any closed subset of* $\mathcal{C} \cap \mathcal{I} \cap \mathcal{N}$.

Note that it can be proven that $\mathcal{C} \cap \mathcal{I} \cap \mathcal{N}$ is not a closed set and consequently the join and meet do not generate a lattice. However, we have the following result.

Proposition 1. *The following statements hold:*

(i) $\mathcal{S} \subset \mathcal{C} \cap \mathcal{I} \cap \mathcal{N}$ *is a closed subset;*
(ii) If L_1 *is a distributive lattice, then* $\mathcal{C} \cap \mathcal{I} \cap \mathcal{N}$ *is a closed set.*

From Theorem 1 and Proposition 1 the following can be inferred.

Corollary 1. *The join and meet operations given in Eqs.* (1) *and* (2) *generate a lattice on* $\mathcal{S}$.

Corollary 2. *Let be* L_1 *a distributive lattice. Then the join and meet operations given in Eqs.* (1) *and* (2) *generate a lattice.*

Moreover, it can be shown that the lattice in Corollary 2 is distributive.

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Many-valued domain theory equations

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Abstract. Domain theory provides a foundation for denotational semantics. In classical domain theory, the domains are partially ordered sets with additional structural properties. Generalizations of classical domain theory include using preordered sets or multivalued partially ordered sets. In this paper, we generalize to many-valued preordered sets. Domain theory in naturally presented in the language of category theory. In this paper, however, our emphasis is not on the categorical structures and properties; it is on developing a basic many-valued domain theory for programming semantics.

1 Introduction and Domain Theory Constructors

There are several introductions to programming semantics and domain theory. In this paper, we use the book *The Formal Semantics of Programming Languages An Introduction* by Glynn Winskel [6] and the chapter *Domain Theory* by Samson Abramsky and Achim Jung in the *Handbook of Logic in Computer Science* volume 3 [1].

As stated in [6], "Domain theory is the mathematical foundation of denotational semantics," and as stated above, classical domains are built from partially ordered sets. We build our domains from many-valued preodered sets.

Definition 1. *Let X be a set, and let $(L, \leq)$ be a frame with largest element $\top_L$ and smallest element $\perp_L$. An L-valued relation R on X is a function $R\colon X \times X \to L$.*[4]

Definition 2. Let $R\colon X \times X \to L$ be an L-valued relation on X. R is an *L-preorder* if

- R is *reflexive*, i.e., $\forall x \in X$, $R(x, x) = \top_L$, and
- R is *transitive*, i.e., $\forall x, y, z \in X$, $R(x, y) \wedge R(y, z) \leq R(x, z)$.

If R is an L-preorder on X, then (X, R) or just X is an *L-preordered set* or an *L-preset*.

[4] Some definitions and comments in this paper are borrowed verbatim from [3].

For an L-preset (X, R), when $R(x, y) = \alpha \in L$, we consider x to be R-related to y to degree α. However, when $R(x, y) = \alpha$, then we normally say either x is less than or equal to y to degree α, or y is greater than or equal to x to degree α.

For denotational semantics, we often want our domains to have bottom or least elements. To make sure that our domains have least elements, we can lift the domains by adding to each domain a new element which is less than or equal to each element in the original domain. Thus, we have the next definition.

Definition 3. Let (X, R) be an L-preset. We define a new element $\perp_{(X,R)}$ which is not in X. Thus, we have a new set $X_{\perp_{(X,R)}} = X \cup \{\perp_{(X,R)}\}$, and we extend the ordering R so that

$$R(\perp_{(X,R)}, \perp_{(X,R)}) = \top_L$$

and for each $x \in X$,

$$R(x, \perp_{(X,R)}) = \perp_L \text{ and } R(\perp_{(X,R)}, x) = \top_L.$$

We use the symbol R for both the original and the extended many-valued relation, and we call $(X_{\perp_{(X,R)}}, R)$ the *lift* of (X, R).

Proposition 1. If (X, R) be an L-preset, then $(X_{\perp_{(X,R)}}, R)$ is also an L-preset.

Let (X, R) and (Y, S) be L-presets. We want to be able to form the L-preset product and direct sum of (X, R) and (Y, S). Thus, we have the following two definitions.

Definition 4. Let R be an L-valued relation on X, and let S be an L-valued relation on Y. We define an L-valued relation $R \times S$ on the set $X \times Y$ so that for $x_1, x_2 \in X$ and for $y_1, y_2 \in Y$, we have

$$(R \times S)((x_1, y_1), (x_2, y_2)) = R(x_1, x_2) \wedge S(y_1, y_2).$$

We say $(X \times Y, R \times S)$ is the *product* or *direct product* of (X, R) and (Y, S)

Proposition 2. If (X, R) and (Y, S) are L-presets, then $(X \times Y, R \times S)$ is also an L-preset.

Definition 5. Let R be an L-valued relation on X, and let S be an L-valued relation on Y. We define an L-valued relation $R + S$ on the disjoint union $X \uplus Y$ so that for $w, z \in X \uplus Y$,

$$(R + S)(w, z) = \begin{cases} R(w, z) & \text{if } w, z \in X \\ S(w, z) & \text{if } w, z \in Y \\ \perp_L & \text{if } w \text{ and } z \text{ are not in same set} \end{cases}$$

We say $(X \uplus Y, R + S)$ is the *direct sum* or *coproduct* of (X, R) and (Y, S)

Proposition 3. If (X, R) and (Y, S) are L-presets, then $(X \uplus Y, R + S)$ is also an L-preset.

The most interesting basic domain constructor is the function space constructor. It is also the one which can create cardinality inconsistencies; see, for example, [5, **?**].

Before we continue, we need more definitions. Please see [3] for more details.

Definition 6. Let (D, R) be an L-preset, and let $\alpha \in L$. (D, R) or just D is an α-*directed set* if every finite subset of D has an α-upper bound in D. Since the finite subset may be empty, then D must be non-empty.

If (X, R) is an L-preset, then $D \subset X$ is an *α-directed subset* of X if (D, R_D) is an α-directed set where R_D is the restriction of R to $D \times D$.

Definition 7. Let (X, R) be an L-preset, and let $\alpha \in L$. (X, R) or X is an *α-directed complete preset* or an *α-dcpro* if every α-directed subset D of X has an α-least upper bound or α-supremum $\bigsqcup_\alpha D$ in X.

Definition 8. Let (X, R) be an L-preset, and let $\alpha \in L$. X has an *α-bottom element* if there is an element $\perp_{(X,R)_\alpha} \in X$ such that for each $x \in X$, $R(\perp_{(X,R)_\alpha}, x) \geq \alpha$. An L-preset (X, R) is said to have an *L-bottom element* or simply a *bottom element* if it has a $\top_L$-bottom element.

If $(X_{\perp_{(X,R)}}, R)$ is a lifted domain, then $\perp_{(X,R)}$ is the $\top_L$-bottom element.

Definition 9. An L-preset (X, R) is an *α-complete preset* or an *α-cpro* if it is α-directed complete and if it has an α-bottom element.

Definition 10. Let (X, R) and (Y, S) be L-presets, and let $\alpha \in L$. A function $f : X \to Y$ is *α-order-preserving* if whenever $a, b \in X$ with $R(a, b) \geq \alpha$, then $S(f(a), f(b)) \geq \alpha$. The function f is L-order-preserving if for all $a, b \in X$, $R(a, b) \leq S(f(a), f(b))$.

Definition 11. Let (X, R) and (Y, S) be L-presets, and let $\alpha \in L$. A function $f : X \to Y$ is *α-Scott continuous* if it preserves α-suprema of α-directed sets. That is, if D is an α-directed subset of X and if $\bigsqcup_\alpha D$ exists in X, then $f(\bigsqcup_\alpha D)$ is an α-supremum of $f^{\to}(D)$.

Definition 12. Let (X, R) be an L-preset, and let $\alpha \in L$. A subset $U \subset X$ is *α-Scott open* if U is α-up-closed and if it non-trivially intersects every α-directed set whose limit it contains. Thus, U is α-Scott open if $U = \uparrow_\alpha U$ and whenever D is an α-directed subset of X with $\bigsqcup_\alpha D$ existing and in U, i.e., $\bigsqcup_\alpha D \in U$, then $U \cap D \neq \emptyset$.

Theorem 1. Let (X, R) and (Y, S) be L-presets; let $\alpha \in L$; and let $f \colon X \to Y$ be a function. $f \colon (X, R) \to (Y, S)$ is α-Scott continuous if and only if $f \colon (X, \tau_R) \to (Y, \tau_S)$ is continuous when τ_R and τ_S are the α-Scott topologies on (X, R) and (X, S), respectively.

Theorem 2. Let (X, R) be an L-preset; let $\alpha \in L$; and let (Y, S) be an α-cpro. Then $([(X, R) \to (Y, S)]_\alpha, R_{X \to (Y,S)})$ is also an α-cpro.

In the proof of the previous theorem, the axiom of choice is used.

2 Conclusion and Continuation

Beginning with L-presets, a domain theory can be developed. We continue this development, especially focusing on the the function space constructor and function space domain equations.

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Sierpinski object for affine systems

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Abstract. We provide an analogue of the Sierpinski space for many-valued topological systems and show that it has three important properties of the crisp case.

1 Introduction

The notion of *Sierpinski space* $\mathcal{S} = (\{0, 1\}, \{\varnothing, \{0\}, \{0, 1\}\})$ plays an important role in general topology. In particular, one can show the following results [1, 10]:

1. A topological space is T_0 iff it can be embedded into some power of $\mathcal{S}$.
2. The injective objects in the category $\mathbf{Top}_0$ of T_0 topological spaces are precisely the retracts of powers of $\mathcal{S}$.
3. A topological space is sober iff it can be embedded as a front-closed subspace into some power of $\mathcal{S}$.

Moreover, [8, 9] introduced the concept of *Sierpinski object* in categories of structured sets and structure-preserving maps, and provided a characterization of the category of topological spaces among such categories in terms of Sierpinski object.

Some of the above-mentioned results have already been extended to lattice-valued topology [7, 13, 14]. In particular, one already has a characterization of the category of fuzzy topological spaces in terms of the Sierpinski object of E. G. Manes [13].

In [15], S. Vickers introduced the concept of *topological system* as a common framework for both point-set and point-free topologies. Inspired by this notion, the authors of [11] have recently presented the concept of Sierpinski object in the category **TopSys** of topological systems, providing topological system analogues of items (1), (2) above.

Motivated by the notion of lattice-valued topological system of [2, 12] and the results of [11], we show lattice-valued system analogues of the above three items.

2 Affine systems

To better encompass various many-valued frameworks, we employ a particular instance of the setting of affine sets of [3, 4], which is based in varieties of algebras.

Definition 1. *Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a family of cardinal numbers, which is indexed by a (possibly, proper or empty) class Λ. An Ω-*algebra *is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$, which comprises a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$ (n_λ-*ary primitive operations *on A). An Ω-*homomorphism $(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ *is a map $A \xrightarrow{\varphi} B$, which makes the diagram*

$$\begin{tikzcd} A^{n_\lambda} \arrow[r, "\varphi^{n_\lambda}"] \arrow[d, "\omega_\lambda^A"'] & B^{n_\lambda} \arrow[d, "\omega_\lambda^B"] \\ A \arrow[r, "\varphi"'] & B \end{tikzcd}$$

commute for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ is the category of Ω-algebras and Ω-homomorphisms, concrete over the category **Set** *of sets and maps (with the forgetful functor $|-|$).*

Definition 2. *Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A* variety of Ω-algebras *is a full subcategory of $\mathbf{Alg}(\Omega)$, which is closed under the formation of products, $\mathcal{M}$-subobjects (subalgebras), and $\mathcal{E}$-quotients (homomorphic images). The objects (resp. morphisms) of a variety are called* algebras *(resp.* homomorphisms*).*

From now on, we fix a variety of algebras **A** (one can think of the variety **Frm** of frames [6], which provides an example for all the results in this talk).

Definition 3. *Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{op}$, $\mathbf{Af\,Sys}(T)$ is the comma category $(T \downarrow 1_{\mathbf{B}^{op}})$, concrete over the product category $\mathbf{X} \times \mathbf{A}^{op}$, whose objects ($T$-*affine systems *or T-*systems*) are triples (X, κ, A), made by $\mathbf{A}^{op}$-morphisms $TX \xrightarrow{\kappa} A$; and whose morphisms ($T$-*affine morphisms *or T-*morphisms*) $(X_1, \kappa_1, A_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, A_2)$ are $\mathbf{X} \times \mathbf{A}^{op}$-morphisms $(X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)$, which make the next diagram commute*

$$\begin{tikzcd} TX_1 \arrow[r, "Tf"] \arrow[d, "\kappa_1"'] & TX_2 \arrow[d, "\kappa_2"] \\ A_1 \arrow[r, "\varphi"'] & A_2. \end{tikzcd}$$

In this talk, we will restrict ourselves to the functor T of the following form.

Proposition 1. *Every subcategory **S** of $\mathbf{A}^{op}$ gives rise to a functor $\mathbf{Set} \times \mathbf{S} \xrightarrow{\mathcal{P}_\mathbf{S}} \mathbf{A}^{op}$, $\mathcal{P}_\mathbf{S}((X_1, A_1) \xrightarrow{(f,\varphi)} (X_2, A_2)) = A_1^{X_1} \xrightarrow{\mathcal{P}_\mathbf{S}(f,\varphi)} A_2^{X_2}$, $(\mathcal{P}_\mathbf{S}(f,\varphi))^{op}(\alpha) = \varphi^{op} \circ \alpha \circ f$.*

For the sake of convenience, we will consider the simplest possible case of *fixed-basis* affine systems, which is given by the subcategory **S** of the form $A \xrightarrow{1_A} A$. Thus, from now on, we fix an **A**-algebra L (as a reminder for "lattice-valued"). The case $\mathbf{A} = \mathbf{Frm}$ and $L = \{0, 1\}$ provides the category **TopSys** of S. Vickers.

3 Sierpinski object for affine systems

We start by introducing an affine system analogue of the Sierpinski space. Let there exist a free **A**-algebra S over a singleton $1 = \{*\}$ with the universal map $1 \xrightarrow{\eta} |S|$.

Definition 4. Sierpinski affine system *is the triple* $\mathsf{S} = (|L|, \kappa_S, S)$*, in which the map* $S \xrightarrow{\kappa_S^{op}} L^{|L|}$ *is given by the diagram*

$$\begin{tikzcd}
1 \arrow[rr, "\eta"] \arrow[d, "h_a^L"'] & & {|S|} \arrow[dll, "|\overline{h_a^L}|"'] \arrow[d, dotted, "|\kappa_S^{op}|"] \\
{|L|} & & {|L^{|L|}|,} \arrow[ll, "|\pi_a|"]
\end{tikzcd}$$

where $h_a^L(*) = a$ *for every* $a \in L$*, and* $\overline{h_a^L}$ *is the unique* **A***-homomorphism, obtained from* h_a^L *with the help of the universal map* η*.*

In case of the category **TopSys**, one gets the Sierpinski object of [11]. One of the crucial properties of the Sierpinski system is the following.

Proposition 2. *Given an affine system* (X, κ, A)*, for every* $a \in A$*, there exists a system morphism* $(X, \kappa, A) \xrightarrow{(f_a, \varphi_a)} \mathsf{S}$*, where* $f_a = \kappa^{op}(a)$ *and* $|\varphi_a^{op}| \circ \eta = h_a^A$*. Every affine morphism* $(A, \kappa, A) \xrightarrow{(f, \varphi)} \mathsf{S}$ *has the form* (f_a, φ_a) *for some* $a \in A$*.*

The next are affine analogues of the three items from the first section. We begin with an affine modification of the concept of T_0 topological system of [15].

Definition 5. *An affine system* (X, κ, A) *is* T_0 *provided that for every* $x, y \in X$*, if* $(\kappa^{op}(a))(x) = (\kappa^{op}(a))(y)$ *for every* $a \in A$*, then* $x = y$*.*

Proposition 3. *The Sierpinski affine system* S *is* T_0*.*

Theorem 1. *An affine system* (X, κ, A) *is* T_0 *iff it is embeddable into a power of* S*.*

Definition 6. $\mathbf{Af\,Sys}_0(T)$ *is the full subcategory of* $\mathbf{Af\,Sys}(T)$ *of all* T_0 *affine systems.*

Proposition 4. *If* **A***-epimorphisms are onto, then* S *is an injective object in* $\mathbf{Af\,Sys}_0(T)$*.*

Theorem 2. *Suppose epimorphisms in* **A** *are onto. Then the injective objects in the category* $\mathbf{Af\,Sys}_0(T)$ *are precisely the retracts of powers of* S*.*

The following is an affine modification of the concept of localic system of [15].

Definition 7. *An affine system* (X, κ, A) *is* sober *provided that the map* $X \xrightarrow{\ell} Pt_L(A)$*,* $\ell(x) = (\kappa^{op}(-))(x)$ *is bijective, where* $Pt_L(A) = \mathbf{A}(A, L)$*.*

In our case, we can equivalently describe the condition on an affine morphism $(X_1, \kappa_1, A_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, A_2)$ from Definition 3 by commutativity of the diagram

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d, "\ell_1"'] & X_2 \arrow[d, "\ell_2"] \\ Pt_L(A_1) \arrow[r, "(\varphi^{op})_L^{\leftarrow}"'] & Pt_L(A_2). \end{tikzcd} \tag{D}$$

Definition 8. *An affine monomorphism $(X_1, \kappa_1, A_1) \xrightarrow{(f,\varphi)} (X_2, \kappa_2, A_2)$ is* sober *if (D) is a weak pullback [5] (the canonical map $X \to Pt_L(A_1) \times_{Pt_L(A_2)} X_2$ is onto).*

Theorem 3. *An affine system (X, κ, A) is sober iff it is soberly embeddable into a power of* S.

We would like to conclude our discussion with an obvious open problem.

Problem 1. Extend our fixed-basis results to the variable-basis case of Proposition 1.

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Vagueness measure based convergence theorem of logical expression using inequalities

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In continuous valued logic we can measure how fair are we from the two valued logic. This measure is called the fuzziness measure. We shall present an operator-dependent fuzziness measure called the vagueness measure. We will show that this measure satisfies the usual classical assumptions for the fuzziness measue. In our view, the operator depended fuzziness measure one of the most important concept, because on this basis we can prove "convergence theorems" -based on inequalities- in the sense that "if there is less fuzziness in the input variables, then there will be less fuzziness in the result". In the fuzzy literature we do not find such theorems, because membership functions, operators and fuzziness measures are unrelated so it seems hopeless to prove such convergence theorems.

In the Pliant concept i.e.: for the generator function of the conjuctive and disjunctive operator are in reciprocial relation. We show the following inequalities hold:

$$c(\mathcal{V}(x), \mathcal{V}(y)) \;\leq\; \mathcal{V}(c(x,y)) \;\leq\; d(\mathcal{V}(x), \mathcal{V}(y))$$

$$c(\mathcal{V}(x), \mathcal{V}(y)) \;\leq\; \mathcal{V}(d(x,y)) \;\leq\; d(\mathcal{V}(x), \mathcal{V}(y))$$

Where $c(x,y)$ and $d(x,y)$ are conjuctive and disjunctive operators (strict t-norm and t-conorm) and $\mathcal{V}(x)$ is the Vagueness measure.
We show that a more general theorem is valid:
Let $\mathcal{L}(x)$ be any pliant logical expresssion then

$$c(\mathcal{V}(x_1) \ldots \mathcal{V}(x_n)) \;\leq\; \mathcal{V}(\mathcal{L}(\mathbf{x})) \;\leq\; d(\mathcal{V}(x_1) \ldots \mathcal{V}(x_n)$$

i.e. the vagueness measure of a logical expression always lies between the conjunction of the vagueness measure of the input variables and the disjunction of the vagueness measure of the input variables. If a variable appears k times in $\mathcal{L}(\mathbf{x})$, then on the left and right hand sides it should be used k times.

We prove -in the weighted operator case- that preference (or distance) index between left and right hand-side bound is

$$p\left(c\left(\mathbf{w}, \mathbf{x}\right), d\left(\mathbf{w}, \mathbf{x}\right)\right) < \nu,$$

where

$$\nu = g^{-1}\left(\frac{1}{4}\left(\sqrt{\frac{g(x^*)}{g(x_*)}} + \sqrt{\frac{g(x_*)}{g(x^*)}}\right)\right)$$

and x^*, x_* are the maximum and minimum values of the variables:

$$x^* = \max_{1 \le i \le n} x_i \qquad x_* = \min_{1 \le i \le n} x_i$$

and $g(x)$ the generator function of the operator. In this theorem we are used weighted operators.

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Supermigrative copulas

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Abstract. Supermigrative copulas (and related aggregation functions) are presented and their main properties are studied with a discussion about possible higher dimensional generalizations. Moreover, several applications of these concepts will be presented in reliability theory (in particular, stochastic system of lifetimes), life insurance and economics.

1 Motivation

The problem of determining the dependence among random variables has been long studied in the case of Gaussian distributions (and related generalizations) and contingency tables. A classical way to extend these studies to more general random structures consists in reducing to the class of multivariate distribution functions with identical marginals, i.e. to consider the so–called *Fréchet classes*. In fact, in such a case, meaningful dependence properties can be easily expressed in terms of (functional) inequalities involving the joint distribution (or some of its sections) and the related univariate margins. See, for instance, [21, 23, 25].

The study of the Fréchet classes of distribution functions is made easy (at least in the continuous case) by considering the associated class of *copulas*, which are multivariate probability distribution functions whose univariate marginals are uniformly distributed on $[0, 1]$ (see, for instance, [17]). In particular, since the copula of a random vector is invariant under increasing transformation of the components of the vector, all the concepts of dependence (and measures of association) that are rank–invariant can be conveniently characterized in terms of copulas. Consider, for instance, classical Spearman's and Kendall's correlation measures as well as notions like quadrant dependence and regression dependence that have become nowadays standard tools in stochastic models. For an overview, see [19, 24].

A more recent problem in the study of random vectors is related to the determination of those properties of lifetimes (i.e. non–negative random variables) that may have an interpretation in terms of *aging* of the system. Usually, aging properties are expressed in terms of comparisons between suitable conditional distribution functions derived from the system with respect to different times and/or a different history. See, for instance, [1–3, 26]. Among these studies, Bassan and Spizzichino have introduced a variety of functional inequalities involving copulas and associated functions, like the multivariate aging functions, in order to express suitable aging relations. See, for instance, [4, 5] and also [10]. Here, in particular, we are interested in a special inequality in the class of bivariate copulas, called *supermigrativity*, which will be descibed in detail in the following.

2 Main results

A semi-copula S is a function from $[0,1]^2$ to $[0,1]$ that is increasing in each variable and satisfies $S(x,1) = S(1,x) = x$ for every $x \in [0,1]$, but it may be neither associative nor commutative [14, 16]. As shown in [13], the class of semi-copulas constitutes the lattice completion of the class of triangular norms.

A semi-copula S is called *supermigrative* if it is commutative and satisfies

$$S(\alpha x, y) \geq S(x, \alpha y) \tag{1}$$

for all $\alpha \in [0,1]$ and for all $x, y \in [0,1]$ such that $y \leq x$. This inequality was first considered in [5] in the definition of a novel notion of bivariate aging. The term "supermigrative" appeared for the first time in [11], where it is linked to the notion of *migrativity*, introduced for a special class of associative functions called triangular norms (see, e.g., [8, 9, 15, 18, 22]).

Following [8], we can intuitively interpret supermigrativity as a property of the aggregation process of two inputs into a single output. Basically, supermigrativity refers to the fact that, when the intensity of one input is reduced to $100 \cdot \alpha$ per cent, global evaluation will be greater when the higher input is being reduced. For an economic interpretation in a two-person bargaining problem, see also [6].

In another context, [7] interpreted Eq. (1) as a way to compare the values assumed by the semi–copula along curves where the product xy is constant, and that the value increases as one approaches to the diagonal $y = x$. In particular, they argued that a copula is supermigrative when, roughly speaking, it has the probability mass concentrated around the diagonal of $[0,1]^2$.

Despite its apparent simplicity, the notion of supermigrativity for semi–copulas presents interesting connections with other concepts considered in the literature. For instance, the following results hold:

– Every supermigrative semi–copula is Schur–concave (see [11]).
– Every supermigrative continuous and Archimedean triangular norm is a copula (see [11]).
– Gaussian copulas with positive correlation are supermigrative, while Gaussian copulas with negative correlation are neither super- nor submigrative (see [12]).

Specifically, the latter property was used in [12] to show how the supermigrativity can be used in order to define a notion of positive dependence (in the sense of Kimeldorf and Sampson [20]) for bivariate vectors of continuous random variables. Another interesting application of supermigrativity is related to the comparison of hazard rates for dependent vector of lifetimes (see [7]). All these properties and applications will be reviewed in the present talk.

Finally, notice that, while supermigrativity has been largely investigated in the bivariate case, its extension in higher dimensions merits special attention in view of different possible definitions that can be adopted. For instance, in [11] it was argued that a proper generalization may involve the extension of the characterization given in [11, Proposition 2.7]. Another possible extension has been instead appeared in [10, Definition 2.1] and it is grounded on its probabilistic interpretation in terms of stochastic aging. Both these extensions will be presented and discussed as well.

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Functional equations related to ordered structures: a panoramic view

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Abstract. When trying to convert different kinds of ordered structures, understood as qualitative scales on a nonempty set, into numerical or quantitative scales, we usually look for representations through one or more real valued functions that formally interpret and characterize the given ordered structure and actually constitute an equivalent scale that is based on numbers. However, instead of using directly what is called as a numerical scale based on functions on one single variable, we could try to use other representations using real-valued bivariate maps, that is, functions on several variables defined on the two-fold Cartesian product of the given set by itself. These bivariate maps give rise to functional equations in two variables. Furthermore, in some appealing situations, suitable solutions to those functional equations could be used to interpret or understand better the particular ordered structure to which they are associated. We will analyze functional equations related to the numerical representability of total preorders, interval orders and semiorders, paying attention to the particular features of those equations and analyzing how they can be used to represent several kinds of ordered structures on a nonempty set. Also, we will establish some relationship between several functional equations that could be related someway to the same kind of ordered structure. In particular, an analysis of the classical Abel's equation on one single variable will be introduced to understand better what is a representable semiorder. Several miscellaneous uses of those functional equations that are often encountered when dealing with numerical representability of ordered structures are also commented, so giving a panoramic view of this theory.

Introduction and general scope

Let X be a nonempty set. A binary relation $\mathcal{R}$ on X can be, under certain circumstances and provided that it satisfies some suitable properties, interpreted as a comparison, preference or ordering on X, so that given $x, y \in X$, the fact $x\mathcal{R}y$ represents the idea of the element x being as least as good as the element y. Therefore the binary relation $\mathcal{R}$ represents a qualitative scale of comparison or ranking. However, it is common to try to convert that scale in another equivalent one, but now based on numbers. To put an example, we could think that each element of X could be assigned a real number, say $r(x) \in \mathbb{R}$ such that the claim $x\mathcal{R}y$ is equivalent to the fact $r(x) \leq r(y)$. In this case, we would have represented the binary relation or qualitative scale $\mathcal{R}$ by means of a numerical or quantitative scale. It is not difficult to see that the mere existence of that quantitative scale would carry important restrictions on the binary relation $\mathcal{R}$. Of course, not every binary relation $\mathcal{R}$ will admit that kind of numerical representation.

The theory that tries to convert qualitative scales or ordered structures into quantitative (numerical) ones that are equivalent to the former (qualitative) scale, is known in the specialized literature as the representability theory of ordered structures. Among the typical binary relations to be analyzed here, we have the total preorders, the interval orders and the semiorders.

A total preorder $\precsim$ defined on X is a binary relation that is total (complete) and transitive. The preorder $\precsim$ is said to be representable if there exists a real-valued function $u : X \to \mathbb{R}$ such that $x \precsim y \Leftrightarrow u(x) \leq u(y)$ holds for every $x, y \in X$. This is equivalent to the existence of a bivariate map $F : X \times X \to \mathbb{R}$ that satisfies the so-called Sincov functional equation, namely $F(x, y) + F(y, z) = F(x, z) \ \ (x, y, z \in X)$ and, in addition $x \precsim y \Leftrightarrow F(x, y) \geq 0$ holds true for every $x, y \in X$.

Similarly, an interval order $\prec$ defined on X is an asymmetric binary relation such that $x \prec y$ jointly with $z \prec t$ imply that either $x \prec t$ or $z \prec y$ holds true for every $x, y, z, t \in X$. The interval order $\prec$ is said numerically representable if there exist two real-valued functions $u, v : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow v(x) \leq u(y)$ holds for every $x, y \in X$. This is equivalent to the existence of a bivariate map $F : X \times X \to \mathbb{R}$ that satisfies the separability functional equation, namely $F(x, y) + F(y, z) = F(x, z) + F(y, y) \ \ (x, y, z \in X)$ and, in addition $x \prec y \Leftrightarrow F(x, y) > 0$ holds true for every $x, y \in X$.

Finally, an interval order $\prec$ defined on X is said to be a semiorder whenever for any $x, y, z, a \in X$ it holds true that $x \prec y \prec z$ carries $x \prec a$ or $a \prec z$. A semiorder $\prec$ is representable in the sense of Scott and Suppes ([19]) if there exists a real-valued function $u : X \to \mathbb{R}$ such that $x \prec y \Leftrightarrow u(x) + 1 \leq u(y)$ holds for every $x, y \in X$. This is equivalent to the existence of a bivariate map $F : X \times X \to \mathbb{R}$ that satisfies the functional equation $F(x, y) + F(y, z) = F(x, z) + F(a, a) \ \ (x, y, z, a \in X)$ and, in addition $x \prec y \Leftrightarrow F(x, y) > 0$ holds true for every $x, y \in X$.

Let I denote an open real interval. Let $h : I \to I$ be a continuous and strictly increasing map such that $x < h(x)$ holds true for every $x \in I$. Then there exists a continuous and strictly increasing real-valued function $f : I \to \mathbb{R}$ such that $f((h(x)) = f(x) + 1$ holds true for every $x \in I$, (For the proof, see Theorem 2.1 in [16] or pp. 133 and ff. in [2]. The associated equation in one single variable, namely, $f((h(x)) = f(x) + 1$ is known as the Abel functional equations and was introduced by Niels Henrik Abel in 1824. This equation is also closely related to the numerical representability of semiorders.

The purpose of the lecture will we to give a panoramic view of these equations and their uses in the theory of the numerical representations of ordered structures.

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Fuzzy topology via fuzzy geometric logic with graded consequence

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1 Extended Abstract

The motivation mostly came from the main topic of Vickers book "Topology via Logic" [5], where he introduced a notion of topological system and indicated it's connection with geometric logic. The relationships among topological space, topological system, frame and geometric logic play an important role to study topology through logic (geometric logic). Naturally the question "from which logic fuzzy topology can be studied?" comes in mind. If such a logic is obtained what could be its significance?

To answer these questions, as basic steps, we first introduced some notions of fuzzy topological systems and established the interrelation with appropriate topological spaces and algebraic structures. These relationships are studied in a categorical framework [4].

In [3] another level of generalisation (that is, introduction of many-valuedness) shall take place giving rise to graded fuzzy topological systems (vide Definition 4) and fuzzy topological spaces with graded inclusion (vide Definition 2). It will also be required to generalise the notion of a frame to a graded frame (vide Definition 3).

Geometric logic is endowed with an informal observational semantics [6]: whether what has been observed does satisfy (match) an assertion or not. In fact, from the stand point of observation, negative and implicational propositions and universal quantification face ontological difficulties. On the other hand arbitrary disjunction needs to be included (cf. [6] for an elegant discussion on this issue). Now, observations of facts and assertions about them may corroborate with each other partially. It is a fact of reality and in such cases it is natural to invoke the concept of 'satisfiability to some extent or to some degree'. As a result the question whether some assertion follows from some other assertion might not have a crisp answer 'yes'/'no'. It is likely that in general the 'relation of following' or more technically speaking, the consequence relation turnstile ($\vdash$) may be itself many-valued or graded (vide Definition 1). For an introduction to the general theory of graded consequence relation we refer to [1, 2]. Thus, we have

⋆ The abstract contains some material from the doctoral thesis of the first author.

adopted graded satisfiability as well as graded consequence in [3]. On top of that a further generalised notions such as fuzzy geometric logic with graded consequence, fuzzy topological spaces with graded inclusion, graded frame and graded fuzzy topological systems came into the picture.

In this seminar the main discussion point is to explain fuzzy geometric logic with graded consequence first and then categorical relationships among fuzzy topological spaces with graded inclusion, graded frames and graded fuzzy topological systems. On top of that the transformation of morphisms between the objects will also be explained, which play an interesting role too. Through the categorical study it becomes more clear why the graded inclusion is important in the fuzzy topology to establish the desired interrelations.

Fuzzy geometric logic with graded inclusion

The alphabet, terms, formulae of this logic are the same as of geometric logic. The changes occur in the definition of interpretation and satisfiability relations. That is, the concept of grades takes place in those concepts. The expression $\phi \vdash \psi$, where ϕ, ψ are formulae of fuzzy geometric logic with graded consequence, is called a sequent. It is to be noted that the grade of satisfiability of a geometric formula ϕ by s, a sequence over the domain of interpretation, is denoted by $gr(s\ sat\ \phi)$.

Definition 1. *1.* $gr(s \text{ sat } \phi \vdash \psi)$=$gr(s \text{ sat } \phi) \rightarrow gr(s \text{ sat } \psi)$, *where* $\rightarrow$: $[0,1] \times [0,1] \longrightarrow [0,1]$ *is the Gödel arrow and s is the sequence over the domain of interpretation of the logic. 2.* $gr(\phi \vdash \psi) = inf_s\{gr(s \text{ sat } \phi \vdash \psi)\}$.

Theorem 1. *Graded sequents satisfy the following properties*

1. $gr(\phi \vdash \phi) = 1$,
2. $gr(\phi \vdash \psi) \wedge gr(\psi \vdash \chi) \leq gr(\phi \vdash \chi)$,
3. *(i)* $gr(\phi \vdash \top) = 1$, *(ii)* $gr(\phi \wedge \psi \vdash \phi) = 1$,
   (iii) $gr(\phi \wedge \psi \vdash \psi) = 1$, *(iv)* $gr(\phi \vdash \psi) \wedge gr(\phi \vdash \chi) = gr(\phi \vdash \psi \wedge \chi)$,
4. *(i)* $gr(\phi \vdash \bigvee S) = 1$ *if* $\phi \in S$,
   (ii)$inf_{\phi \in S}\{gr(\phi \vdash \psi)\} \leq gr(\bigvee S \vdash \psi)$,
5. $gr(\phi \wedge \bigvee S \vdash \bigvee\{\phi \wedge \psi \mid \psi \in S\}) = 1$,
6. $gr(\top \vdash (x = x)) = 1$,
7. $gr(((x_1, \ldots, x_n) = (y_1, \ldots, y_n)) \wedge \phi \vdash \phi[(y_1, \ldots, y_n)/(x_1, \ldots, x_n)])$=*1*,
8. *(i)* $gr(\phi \vdash \psi[x/y]) \leq gr(\phi \vdash \exists y \psi)$, *(ii)* $gr(\exists y \phi \vdash \psi) \leq gr(\phi[x/y] \vdash \psi)$,
9. $gr(\phi \wedge (\exists y)\psi \vdash (\exists y)(\phi \wedge \psi)) = 1$.

Definition 2 (Fuzzy topological space with graded inclusion). *Let X be a set, τ be a collection of fuzzy subsets of X s.t.*

1. $\tilde{\emptyset}$, $\tilde{X} \in \tau$, *where* $\tilde{\emptyset}(x) = 0$, *for all* $x \in X$ *and* $\tilde{X}(x) = 1$, *for all* $x \in X$*;*
2. $\tilde{T_i} \in \tau$ *for* $i \in I$ *imply* $\bigcup_{i \in I} \tilde{T_i} \in \tau$, *where* $\bigcup_{i \in I} \tilde{T_i}(x) = sup_{i \in I}\{\tilde{T_i}(x)\}$*;*
3. $\tilde{T_1}$, $\tilde{T_2} \in \tau$ *imply* $\tilde{T_1} \cap \tilde{T_2} \in \tau$, *where* $(\tilde{T_1} \cap \tilde{T_2})(x) = \tilde{T_1}(x) \wedge \tilde{T_2}(x)$,

and $\subseteq$ be a fuzzy inclusion relation for fuzzy sets is defined as $gr(\tilde{T_1} \subseteq \tilde{T_2}) = inf_{x \in X}\{\tilde{T_1}(x) \rightarrow \tilde{T_2}(x)\}$, where $\tilde{T_1}, \tilde{T_2}$ are fuzzy subsets of X and $\rightarrow$ is the Gödel arrow. Then $(X, \tau, \subseteq)$ is called a ***fuzzy topological space with graded inclusion****. $(\tau, \subseteq)$ is called a* ***fuzzy topology with graded inclusion*** *over X.*

It can be shown that fuzzy topological spaces with graded inclusion can be studied via fuzzy geometric logic with graded consequences [3].

Definition 3 (Graded Frame). *A* ***graded frame*** *is a 5-tuple $(A, \top, \wedge, \bigvee, R)$, where A is a non-empty set, $\top \in A$, $\wedge$ is a binary operation, $\bigvee$ is an operation on arbitrary subset of A, R is a $[0, 1]$-valued fuzzy binary relation on A satisfying the following conditions: (1) $R(a, a) = 1$, (2) $R(a, b) = 1 = R(b, a) \Rightarrow a = b$, (3) $R(a, b) \wedge R(b, c) \leq R(a, c)$, (4) $R(a \wedge b, a) = 1 = R(a \wedge b, b)$, (5) $R(a, \top) = 1$, (6) $R(a, b) \wedge R(a, c) = R(a, b \wedge c)$, (7) $R(a, \bigvee S) = 1$ if $a \in S$, (8) $inf\{R(a, b) \mid a \in S\} = R(\bigvee S, b)$, (9) $R(a \wedge \bigvee S, \bigvee\{a \wedge b \mid b \in S\}) = 1$, for any $a, b, c \in A$ and $S \subseteq A$. We will denote a graded frame by (A, R).*

Definition 4 (Graded fuzzy topological system). *A* ***graded fuzzy topological system*** *is a quadruple $(X, \models, A, R)$ consisting of a nonempty set X, a graded frame (A, R) and a fuzzy relation $\models$ from X to A such that (1) $gr(x \models a) \wedge R(a, b) \leq gr(x \models b)$, (2) for any finite subset including null set, S, of A, $gr(x \models \bigwedge S) = inf\{gr(x \models a) \mid a \in S\}$, (3) for any subset S of A, $gr(x \models \bigvee S) = sup\{gr(x \models a) \mid a \in S\}$.*

Let us consider the quadruple $(X, \models, A, R)$ where X be a non empty set of assignments s, A be the set of geometric formulae, $\models$ defined as $gr(s \models \phi) = gr(s \text{ sat } \phi)$ and $R(\phi, \psi) = gr(\phi \vdash \psi) = inf_s\{gr(s \text{ sat } \phi \vdash \psi)\}$.

Definition 5. *$\phi \approx \psi$ iff $gr(s \models \phi) = gr(s \models \psi)$ for any $s \in X$ and $\phi,\ \psi \in A$.*

It can be shown that the above defined "$\approx$" is an equivalence relation. Thus we get $A/_{\approx}$. The following theorems hold.

Theorem 2. *$(X, \models', A/_{\approx}, R)$ is a graded fuzzy topological system, where $\models'$ is defined by $gr(s \models' [\phi]) = gr(s \models \phi)$ and $R([\phi], [\psi]) = inf_s\{gr(s \models' [\phi]) \rightarrow gr(s \models' [\psi])\}$.*

Categories

Definition 6 (Graded Fuzzy Top). *The category* **Graded Fuzzy Top** *is defined thus.*

- *The objects are fuzzy topological spaces with graded inclusion $(X, \tau, \subseteq)$, $(Y, \tau', \subseteq)$ etc.*
- *The morphisms are functions satisfying the following continuity property: If $f : (X, \tau, \subseteq) \longrightarrow (Y, \tau', \subseteq)$ and $\tilde{T'} \in \tau'$ then $f^{-1}(\tilde{T'}) \in \tau$.*
- *The identity on $(X, \tau, \subseteq)$ is the identity function. It can be shown that the identity function is a* **Graded Fuzzy Top** *morphism.*
- *If $f : (X, \tau, \subseteq) \longrightarrow (Y, \tau', \subseteq)$ and $g : (Y, \tau', \subseteq) \longrightarrow (Z, \tau'', \subseteq)$ are morphisms in* **Graded Fuzzy Top***, their composition $g \circ f$ is the composition of functions between two sets.*

Definition 7 (Graded Frm). *The category* $\mathbf{Graded\ Frm}$ *is defined thus.*

- *The objects are graded frames* (A, R)*,* (B, R') *etc.*
- *The morphisms are graded frame homomorphisms defined in the following way: If* $f : (A, R) \longrightarrow (B, R')$ *then (i)* $f(a_1 \wedge a_2) = f(a_1) \wedge f(a_2)$*, (ii)* $f(\bigvee_i a_i) = sup_i\{f(a_i)\}$*, (iii)* $R(a_1, a_2) \leq R'(f(a_1), f(a_2))$*.*
- *The identity on* (A, R) *is the identity morphism. It can be shown by routine check that the identity morphism is a* $\mathbf{Graded\ Frm}$ *morphism.*
- *If* $f : (A, R) \longrightarrow (B, R')$ *and* $g : (B, R') \longrightarrow (C, R'')$ *are morphisms in* $\mathbf{Graded\ Frm}$*, their composition* $g \circ f$ *is the composition of graded homomorphisms between two graded frames.*

Definition 8 (Graded Fuzzy TopSys). *The category of graded fuzzy topological systems,* $\mathbf{Graded\ Fuzzy\ TopSys}$*, is defined thus.*

- *The objects are graded fuzzy topological systems* $(X, \models, A, R)$*,* $(Y, \models, B, R')$ *etc.*
- *The morphisms are pair of maps satisfying the following continuity properties: If* $(f_1, f_2) : (X, \models, A, R) \longrightarrow (Y, \models', B, R')$ *then (i)* $f_1 : X \longrightarrow Y$ *is a set map, (ii)* $f_2 : (B, R') \longrightarrow (A, R)$ *is a graded frame homomorphism, (iii)* $gr(x \models f_2(b)) = gr(f_1(x) \models' b)$*.*
- *The identity on* $(X, \models, A, R)$ *is the pair* (id_X, id_A)*, where* id_X *is the identity map on* X *and* id_A *is the identity graded frame homomorphism.*
- *If* $(f_1, f_2) : (X, \models, A, R) \longrightarrow (Y, \models', B, R')$ *and* $(g_1, g_2) : (Y, \models', B, R') \longrightarrow (Z, \models'', C, R'')$ *are morphisms in* $\mathbf{Graded\ Fuzzy\ TopSys}$*, their composition* $(g_1, g_2) \circ (f_1, f_2) = (g_1 \circ f_1, f_2 \circ g_2)$ *is the pair of composition of functions between two sets and composition of graded homomorphisms between two graded frames.*

Theorem 3. *There exist adjoint functors between the category* $\mathbf{Graded\ Fuzzy\ Top}$ *and the category* $\mathbf{Graded\ Fuzzy\ TopSys}$.

Theorem 4. *There exist adjoint functors between the category* $\mathbf{Graded\ Frm}$ *and the category* $\mathbf{Graded\ Fuzzy\ TopSys}$.

Theorem 5. *There exist adjoint functors between the category* $\mathbf{Graded\ Fuzzy\ Top}$ *and the category* $\mathbf{Graded\ Frm}$.

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Cylindrical algebraic decomposition

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If you need to determine all the numbers z for which the formula

$$\exists\, x\, \forall\, y : (x-1)(y-1)(z-1) \leq 0 \Rightarrow x^2 - y^2 + z^2 \leq 1$$

is true in the theory of real numbers, then Cylindrical Algebraic Decomposition is the technique you need. Cylindrical Algebraic Decomposition is a quite general technique developped by George Collins in the 1970 for answering questions about systems of polynomial equations and inequalities. While it was already shown in the 1930s that quantifier elimination is decidable over the reals, Collins's algorithm was the first that was not only of theoretical interest but also had a chance to solve non-trivial problems in practice.

Even though the complexity of the problem is so high that we must not expect to be able to solve even medium size problems within a reasonable amount of time, there is an increasing number of such medium size problems for which the latest implementations of the algorithm do find answers within a reasonable amount of time. We therefore believe that Cylindrical Algebraic Decomposition is one of the techniques in the area of computer algebra that deserves to be better known outside the computer algebra community. This is the purpose of the talk.

We will explain what the precise problem specification of the algorithm solves, and give some examples for reducing some seemingly different problems to this pattern. We will also make some brief comments on how the algorithm works. Finally, we will mention some mention some recent results which were obtained with the help of this algorithm. For further details on the algorithms, see [1, 3]. The original publication of Collins is [4]. For implementations, see [2, 5, 6].

By the way, the answer to the problem stated at the beginning of this abstract is that the formula is true if and only if $-1 \leq z \leq \sqrt{2}$.

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An extension of the concept of distance as functions of several variables

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Abstract. Extensions of the concept of distance to more than two elements have been recently proposed in the literature to measure to which extent the elements of a set are spread out. Such extensions may be particularly useful to define dispersion measures for instance in statistics or data analysis. In this note we provide and discuss an extension of the concept of distance, called n-distance, as functions of n variables. The key feature of this extension is a natural generalization of the triangle inequality. We also provide some examples of n-distances that involve geometric and graph theoretic constructions.

1 Introduction

The notion of metric space is one of the key ingredients in many areas of pure and applied mathematics, particularly in analysis, topology, and statistics.

Denote the half line $[0,+\infty[$ by $\mathbb{R}_+$. Recall that a *metric space* is a pair (X,d), where X is a nonempty set and d is a distance on X, that is a function $d\colon X^2 \to \mathbb{R}_+$ satisfying the following properties:

(i) $d(x_1,x_2)=0$ if and only if $x_1=x_2$,
(ii) $d(x_1,x_2)=d(x_2,x_1)$ for all $x_1,x_2\in X$,
(iii) $d(x_1,x_2)\leqslant d(x_1,z)+d(z,x_2)$ for all $x_1,x_2,z\in X$.

Property (iii) is often refereed to as *triangle inequality*.

It is natural to generalize the concept of metric space by considering a notion of "distance" among more than two elements of X. The idea behind such a notion is to measure in some sense how spread out the elements of X are. Several attempts in this line have been proposed for instance in [2–4, 6, 8, 9]. For example, Martín and Mayor [6] recently introduced the concept of multidistance as follows. Let S_n denote the set of all permutations on $\{1,\ldots,n\}$. A *multidistance* on a nonempty set X is a function $d\colon \cup_{n\geqslant 1} X^n \to \mathbb{R}_+$ satisfying the following properties for every integer $n\geqslant 1$:

(i) $d(x_1,\ldots,x_n)=0$ if and only if $x_1=\cdots=x_n$,
(ii) $d(x_1,\ldots,x_n)=d(x_{\pi(1)},\ldots,x_{\pi(n)})$ for all $x_1,\ldots,x_n\in X$ and all $\pi\in S_n$,
(iii) $d(x_1,\ldots,x_n)\leqslant \sum_{i=1}^n d(x_i,z)$ for all $x_1,\ldots,x_n,z\in X$.

Properties of multidistances as well as instances such as Fermat multidistance and smallest enclosing ball multidistances have been investigated for example in [5–7, 1]

In this short note we introduce and discuss the following alternative generalization of the concept of metric space by considering the underlying distance as a function of $n \geqslant 2$ variables.

Definition 1. *Let $n \geqslant 2$ be an integer. We say that an n-*metric space *is a pair (X, d), where X is a nonempty set and d is an n-*distance *on X, that is a function $d\colon X^n \to \mathbb{R}_+$ satisfying the following properties:*

(i) $d(x_1, \dots, x_n) = 0$ *if and only if* $x_1 = \dots = x_n$*,*
(ii) $d(x_1, \dots, x_n) = d(x_{\pi(1)}, \dots, x_{\pi(n)})$ *for all* $x_1, \dots, x_n \in X$ *and all* $\pi \in S_n$*,*
(iii) *There exists* $K \in [0, 1]$ *such that* $d(x_1, \dots, x_n) \leqslant K \sum_{i=1}^{n} d(x_1, \dots, x_n)|_{x_i=z}$ *for all* $x_1, \dots, x_n, z \in X$.

We denote by K^* the smallest constant K for which (iii) holds. For $n = 2$, we assume that $K^* = 1$.

Clearly, Definition 1 gives an extension of the concept of metric space. Indeed, a function $d\colon X^2 \to \mathbb{R}_+$ is a distance if and only if it is a 2-distance.

We observe that an important feature of n-distances is that they have a fixed number of arguments, contrary to multidistances (see Martín and Mayor [6]), which have an indefinite number of arguments. In particular, an n-distance can be defined without referring to any given 2-distance.

Example 1 (Drastic n-distance). The function $d\colon X^n \to \mathbb{R}_+$ defined by $d(x_1, \dots, x_n) = 0$, if $x_1 = \dots = x_n$, and $d(x_1, \dots, x_n) = 1$, otherwise, is an n-distance, called the *drastic n-distance*, for which the best constant K^* is given by $\frac{1}{n-1}$ for every $n \geqslant 2$. The function $d'\colon X^n \to \mathbb{R}_+$ defined by $d'(x_1, \dots, x_n) = |\{x_1, \dots, x_n\}| - 1$ is an n-distance for which the best constant is $K^* = 1$.

Proposition 1. *Let d and d' be n-distances on X and let $\lambda > 0$. The following assertions hold.*

(1) $d + d'$ *and* $\lambda\, d$ *are n-distance on X.*
(2) $\frac{d}{1+d}$ *is an n-distance on X, with value in* $[0, 1]$.

2 A generalization of n-distances

Condition (iii) in Definition 1 can be generalized as follows.

Definition 2. *Let $g\colon \mathbb{R}_+^n \to \mathbb{R}_+$ be a symmetric function. We say that a function $d\colon X^n \to \mathbb{R}^+$ is a g-*distance *if it satisfies conditions (i) and (ii) in Definition 1 as well as the condition*

$$d(x_1, \dots, x_n) \;\leqslant\; g\big(d(x_1, \dots, x_n)|_{x_1=z}\,, \dots,\; d(x_1, \dots, x_n)|_{x_n=z}\big)$$

for all $x_1, \dots, x_n, z \in X$.

In view of Proposition 1, it is natural to ask that $d + d'$, $\lambda\, d$, and $\frac{d}{1+d}$ be g-distances whenever so are d and d'. The following proposition provides sufficient conditions on g for these properties to hold. We observe that these conditions are rather strong.

Proposition 2. *Let $g\colon \mathbb{R}_+^n \to \mathbb{R}_+$ be a symmetric function. The following assertions hold.*

(1) *If g is positively homogeneous, i.e., $g(\lambda\,\mathbf{r}) = \lambda\, g(\mathbf{r})$ for all $\mathbf{r} \in \mathbb{R}_+^n$ and all $\lambda > 0$, then for every $\lambda > 0$, $\lambda\, d$ is a g-distance whenever so is d.*
(2) *If g is superadditive, i.e., $g(\mathbf{r} + \mathbf{s}) \geqslant g(\mathbf{r}) + g(\mathbf{s})$ for all $\mathbf{r}, \mathbf{s} \in \mathbb{R}_+^n$, then $d + d'$ is a g-distance whenever so are d and d'.*
(3) *If g is both positively homogeneous and superadditive, then it is concave.*
(4) *If g is bounded from below and additive, that is, $g(\mathbf{r} + \mathbf{s}) = g(\mathbf{r}) + g(\mathbf{s})$ for all $\mathbf{r}, \mathbf{s} \in \mathbb{R}_+^n$, then and only then there exist $\lambda_1, \ldots, \lambda_n \geqslant 0$ such that*

$$g(\mathbf{r}) \;=\; \sum_{i=1}^{n} \lambda_i\, r_i \tag{1}$$

(5) *Suppose that g has the form* (1) *with $\lambda_i \geqslant 1$ for $i = 1, \ldots, n$. Then $\frac{d}{1+d}$ is a g-distance whenever so is d.*

3 Examples

We end this note by considering a few examples of n-distances that arise in different fields of pure and applied mathematics.

Example 2 (Basic examples). Given a metric space (X, d) and an integer $n \geqslant 2$, the maps $d_{\max}\colon X^n \to \mathbb{R}_+$ and $d_{\Sigma}\colon X^n \to \mathbb{R}_+$ defined by

$$d_{\max}(x_1, \ldots, x_n) = \max_{1 \leqslant i < j \leqslant n} d(x_i, x_j)$$
$$d_{\Sigma}(x_1, \ldots, x_n) = \sum_{1 \leqslant i < j \leqslant n} d(x_i, x_j)$$

are n-distances for which the best constants are given by $K^* = \frac{1}{n-1}$.

Example 3 (Geometric constructions). Let $x_1, \ldots, x_n$ be $n \geqslant 2$ arbitrary points in $\mathbb{R}^k$ ($k \geqslant 2$) and denote by $B(x_1, \ldots, x_n)$ the smallest closed ball for the Euclidean distance containing $x_1, \ldots, x_n$. It can be shown that this ball always exists, is unique, and can be determined in linear time.

(1) The radius of $B(x_1, \ldots, x_n)$ is an n-distance whose best constant K^* satisfies $\frac{1}{n-1} \leqslant K^*$ and we conjecture that $K^* = \frac{1}{n-1}$.
(2) The k-dimensional volume of $B(x_1, \ldots, x_n)$ is an n-distance and we conjecture that the best constant K^* is given by $K^* = \frac{1}{n-1-(1/2)^k}$. Actually this value for K^* is correct for $k = 2$.

Example 4 (Fermat point based n-distances). Given a metric space (X, d), and an integer $n \geqslant 2$, the *Fermat set* F_Y of any n-element subset $Y = \{y_1, \ldots, y_n\}$ of X, is defined as

$$F_Y = \Big\{x \in X : \sum_{i=1}^{n} d(x_i, x) \leqslant \sum_{i=1}^{n} d(x_i, z) \text{ for all } z \in X\Big\}.$$

Since the function $h\colon X \to \mathbb{R}_+$ defined by $h(x) = \sum_{i=1}^{n} d(x_i, x)$ is continuous and bounded from below by 0, the Fermat set of an n-element subset of X is never empty. Hence, we can define a function $d_F : X^n \to \mathbb{R}_+$ by setting

$$d_F(x_1, \ldots, x_n) = \min\Big\{\sum_{i=1}^{n} d(x_i, x) : x \in X\Big\}.$$

Thus defined, the map $d_F : X^n \to \mathbb{R}_+$ is an n-distance on X for which the best constant K^* satisfies $K^* \leqslant \frac{1}{\lceil (n-1)/2 \rceil}$.

4 Further research

In this note, we have introduced and discussed an extension of the concept of distance, called n-distance, as functions of n-variables. The key feature of this extension is a natural generalization of the triangle inequality. Finding the best constant for various classes of n-distances and studying their topological properties are topics of current research.

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Some functional equations involving 1-Lipschitz aggregation functions and their duals

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Let us start with the *Frank functional equation*: under which conditions are a binary copula [20] $F\colon [0,1]^2 \to [0,1]$ and its dual $G\colon [0,1]^2 \to [0,1]$ given by $G(x,y) = x + y - F(x,y)$ both associative? The answer given in [7] is that a copula F with this property is either a so-called *Frank copula* [6, 16, 19] or an *ordinal sum* of Frank copulas.

Here we will first study the following:

Problem 1. When are a 1-Lipschitz aggregation function $A\colon [0,1]^2 \to [0,1]$ and its dual $A^*\colon [0,1]^2 \to [0,1]$ given by $A^*(x,y) = x + y - A(x,y)$ both associative?

A function $A\colon [0,1]^2 \to [0,1]$ is called a (*binary*) *aggregation function* if it is monotone non-decreasing and satisfies the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$. We shall denote the set of all binary aggregation functions by $\mathcal{A}$.

A binary aggregation function $A\colon [0,1]^2 \to [0,1]$ is said to be 1-*Lipschitz* [10–12, 17] if, for all $(x,y), (x',y') \in [0,1]^2$ we have

$$|A(x,y) - A(x',y')| \leq |x - x'| + |y - y'|.$$

The set of all binary 1-Lipschitz aggregation functions will be denoted by $\mathcal{A}_1$.

A binary aggregation function $A\colon [0,1]^2 \to [0,1]$ is said to have *neutral element* $e \in [0,1]$ if, for all $x \in [0,1]$, we have $A(x,e) = A(e,x) = x$. The set of all binary 1-Lipschitz aggregation functions with neutral element e by will be denoted by $\mathcal{A}_{1,e}$.

Given a binary function $F\colon [0,1]^2 \to \mathbb{R}$, its *dual* [9] $F^*\colon [0,1]^2 \to \mathbb{R}$ is defined by $F^*(x,y) = x + y - F(x,y)$. Note that, even if $A\colon [0,1]^2 \to [0,1]$ is a binary aggregation function, its dual A^* is not necessarily monotone non-decreasing nor is its range a subset of $[0,1]$. However, for an arbitrary binary function $F\colon [0,1]^2 \to \mathbb{R}$ we have that both F and its dual F^* are monotone non-decreasing if and only if both F and F^* are 1-Lipschitz.

Summarizing (see Theorem 1 in [17]), a binary aggregation function $A \in \mathcal{A}$ is 1-Lipschitz if and only if its dual A^* is a binary 1-Lipschitz aggregation function.

Recall that a (*binary*) *quasi-copula* [2, 8] $Q\colon [0,1]^2 \to [0,1]$ is an element of $\mathcal{A}_{1,1}$, i.e., a binary aggregation function on the unit interval $[0,1]$ which is 1-Lipschitz and has neutral element 1.

A (*binary*) *copula* [1, 19, 20] $C\colon [0,1]^2 \to [0,1]$ is a supermodular quasi-copula, i.e., for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geq C(\mathbf{x}) + C(\mathbf{y}).$$

The set of all binary copulas will be denoted by $\mathcal{C}$.

Coming back to Problem 1, for binary 1-Lipschitz aggregation functions we obtain the following result:

Proposition 1. *A binary* 1*-Lipschitz aggregation function* A *and its dual* A^* *are both associative if and only if* $\{A(0,1), A(1,0)\} \subseteq \{0,1\}$ *and*

(i) *if* $A(0,1) = A(1,0) = 0$ *then* A *is a Frank copula or an ordinal sum of Frank copulas;*
(ii) *if* $A(0,1) = A(1,0) = 1$ *then* A^* *is a Frank copula or an ordinal sum of Frank copulas;*
(iii) *if* $A(0,1) = 0$ *and* $A(1,0) = 1$ *then* A *is the projection onto the first coordinate, i.e., for all* $(x,y) \in [0,1]^2$ *we have* $A(x,y) = x$*;*
(iv) *if* $A(0,1) = 1$ *and* $A(1,0) = 0$ *then* A *is the projection onto the second coordinate, i.e., for all* $(x,y) \in [0,1]^2$ *we have* $A(x,y) = y$*.*

Problem 2. Is there a 1-Lipschitz aggregation function $A\colon [0,1]^2 \to [0,1]$ with neutral element such that also its dual $A^*\colon [0,1]^2 \to [0,1]$ has a neutral element?

In complete analogy to aggregation functions acting on $[0,1]$, it is possible to define aggregation functions acting on an arbitrary interval $[a,b] \subset \mathbb{R}$. For instance, a function $Q\colon [a,b]^2 \to [a,b]$ is a quasi-copula on $[a,b]$ if it satisfies $Q(a,a) = a$, $Q(b,b) = b$ and is monotone non-decreasing in each coordinate (i.e., Q is an aggregation function on $[a,b]$), and if Q is 1-Lipschitz and has neutral element b.

In [3] the ordinal sum $A\colon [0,1]^2 \to [0,1]$, denoted by $A = (\langle 0, b, A_1\rangle, \langle b, 1, A_2\rangle)$, of two aggregation functions $A_1\colon [0,b]^2 \to [0,b]$ and $A_2\colon [b,1]^2 \to [b,1]$ was defined as follows:

$$A(x,y) = \begin{cases} A_1(x,y) & \text{if } (x,y) \in [0,b]^2, \\ A_2(x,y) & \text{if } (x,y) \in]b,1]^2, \\ A_1(x \wedge b, y \wedge b) + A_2(x \vee b, y \vee b) - b & \text{otherwise.} \end{cases}$$

Based on this we obtain the following results (compare also [17]):

Proposition 2. *Let* $A\colon [0,1]^2 \to [0,1]$ *be an aggregation function and* $e \in]0,1[$*. Then we have* $A \in \mathcal{A}_{1,e}$ *if and only if there is a quasi-copula* A_1 *on* $[0,e]$ *and a dual* A_2 *of a quasi-copula on* $[e,1]$ *such that*

$$A = (\langle 0, e, A_1\rangle, \langle e, 1, A_2\rangle) \tag{1}$$

Observe that the function A given by (1) satisfies $A(x,y) = x+y-e$ for each $(x,y) \notin [0,e]^2 \cup [e,1]^2$. Moreover, $A^* = (\langle 0,e,A_1^*\rangle, \langle e,1,A_2^*\rangle)$ is a 1-Lipschitz aggregation function with annihilator e, and we have $A^*(x,y) = e$ whenever $x \wedge y \leq e \leq x \vee y$.

Now we can give the solution of Problem 2:

Corollary 1. *A function $A\colon [0,1]^2 \to [0,1]$ is a 1-Lipschitz aggregation function with neutral element such that also its dual A^* has a neutral element if and only if A is a quasi-copula or the dual of a quasi-copula.*

As a consequence, we obtain the same result for 1-Lipschitz aggregation functions with annihilator:

Corollary 2. *A function $A\colon [0,1]^2 \to [0,1]$ is a 1-Lipschitz aggregation function with annihilator such that also its dual A^* has an annihilator if and only if A is a quasi-copula or the dual of a quasi-copula.*

The third problem we are investigating was also motivated by some result for copulas and their duals. It was shown in [4] (using a stochastic approach) and in [18] (by means of an algebraic proof) that for each copula C also the product $C \cdot C^*$ is a copula.

Problem 3. Fix some set of binary aggregation functions $\mathcal{B} \subseteq \mathcal{A}_1$ and consider, for $F \in \mathcal{A}$ and $A \in \mathcal{B}$, the function $F(A, A^*)\colon [0,1]^2 \to [0,1]$ defined by

$$F(A,A^*)(x,y) = F(A(x,y), A^*(x,y)).$$

For which aggregation functions $F \in \mathcal{A}$ do we have $F(A,A^*) \in \mathcal{B}$ for each $A \in \mathcal{B}$?

Proposition 3. *For binary 1-Lipschitz aggregation functions and for binary 1-Lipschitz aggregation functions with neutral element $e \in [0,1]$ we have:*

(i) $F(A,A^*) \in \mathcal{A}_1$ *for all* $A \in \mathcal{A}_1$ *if and only if* $F \in \mathcal{A}_1$*;*
(ii) $F(A,A^*) \in \mathcal{A}_{1,e}$ *for all* $A \in \mathcal{A}_{1,e}$ *if and only if* $F \in \mathcal{A}_{1,e}$.

Considering the special cases $\mathcal{A}_{1,1}$ and $\mathcal{A}_{1,0}$, Proposition 3(ii) immediately implies the following:

Corollary 3.

(i) $F(A,A^*)$ *is a quasi-copula for each quasi-copula A if and only if F is a quasi-copula;*
(ii) $(F(A,A^*))^*$ *is a quasi-copula for each $A \in \mathcal{A}_1$ such that A^* is a quasi-copula if and only if F^* is a quasi-copula.*

In the case of copulas, two additional properties are needed: ultramodularity [14, 15] and Schur concavity [5, 21] on some subset of $[0,1]^2$.

A function $F\colon [0,1]^2 \to [0,1]$ is said to be *ultramodular* on the upper left triangle

$$\varDelta = \{(x,y) \in [0,1]^2 \mid x \leq y\}$$

if for all $\mathbf{x}, \mathbf{y} \in \Delta$ with $\mathbf{x} \leq \mathbf{y}$ and all $\mathbf{h} \geq \mathbf{0}$ such that $\mathbf{x} + \mathbf{h}, \mathbf{y} + \mathbf{h} \in \Delta$ we have

$$F(\mathbf{x} + \mathbf{h}) - F(\mathbf{x}) \leq F(\mathbf{y} + \mathbf{h}) - F(\mathbf{y}).$$

A copula C is called *Schur concave* on the upper left triangle Δ if, for each $x \in]0,1[$ and for all $(\alpha, \beta) \in \Delta \cap \big[0, \min(x, 1 - x^2)\big]$, we have

$$C(x - \alpha, x + \alpha) \leq C(x - \beta, x + \beta).$$

In [13, Theorem 3.1] the following sufficient condition for $F(C, C^*)$ being a copula for each copula C was given:

Theorem 1. *Let C be a binary copula and let F be a binary copula which is ultramodular and Schur concave on the upper left triangle Δ. Then the function $F(C, C^*)$ is a copula.*

Here one of the results of [4, 18] (namely, that $C \cdot C^*$ is a copula for each copula C) is contained as a special case, since the product copula Π obviously is ultramodular and Schur concave on the upper left triangle Δ.

Note that the ultramodularity and the Schur concavity on the upper left triangle Δ are preserved by our construction (see [13, Proposition 3.2]):

Proposition 4. *Let C, F be binary copulas which are ultramodular and Schur concave on the upper left triangle Δ. Then also the copula $F(C, C^*)$ is ultramodular and Schur concave on Δ.*

It turns out that the ultramodularity of F on the upper left triangle Δ is a necessary condition if we want $F(C, C^*)$ to be a copula for each copula C [13, Theorem 3.3]:

Theorem 2. *Let F be a binary aggregation function such that for each binary copula C the function $F(C, C^*)$ is a copula. Then the function $K\colon [0,1]^2 \to [0,1]$ given by $K(x, y) = F(x \wedge y, x \vee y)$ is a copula and F is ultramodular on the upper left triangle Δ.*

We conclude this paper with several positive and negative examples concerning Theorems 1 and 2 (most of them are taken from [13, Section 4]).

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The mathematics behind the property of associativity

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Abstract. The well-known equation of associativity for binary operations may be naturally generalized to variadic operations. In this talk, we illustrate different approaches that can be considered to study this extension of associativity, as well as some of its generalizations and variants, including barycentric associativity and preassociativity.

1 Introduction

Let X and Y be arbitrary nonempty sets. We regard tuples $\mathbf{x}$ in X^n as n-strings over X. We let $X^* = \bigcup_{n\geqslant 0} X^n$ be the set of all strings over X, with the convention that $X^0 = \{\varepsilon\}$, where ε is called the *empty string*. We denote the elements of X^* by bold roman letters $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. If we want to stress that such an element is a letter of X, we use non-bold italic letters x, y, z, etc. The *length* of a string $\mathbf{x}$ is denoted by $|\mathbf{x}|$. We endow the set X^* with the concatenation operation, for which ε is the neutral element, i.e., $\varepsilon\mathbf{x} = \mathbf{x}\varepsilon = \mathbf{x}$ (in other words, we consider X^* as the free monoid generated by X). Moreover, for every string $\mathbf{x}$ and every integer $n \geqslant 0$, the power $\mathbf{x}^n$ stands for the string obtained by concatenating n copies of $\mathbf{x}$. In particular we have $\mathbf{x}^0 = \varepsilon$.

Let Y be a nonempty set. Recall that, for every integer $n \geq 0$, a function $F\colon X^n \to Y$ is said to be *n-ary*. Similarly, a function $F\colon X^* \to Y$ is said to have an *indefinite arity* or to be *variadic*. A variadic function $F\colon X^* \to Y$ is said to be a *variadic operation on X* (or an *operation* for short) if $\mathrm{ran}(F) \subseteq X \cup \{\varepsilon\}$. It is *standard* if $F(\mathbf{x}) = F(\varepsilon)$ if and only if $\mathbf{x} = \varepsilon$, and *ε-standard* if $\varepsilon \in Y$ and we have $F(\mathbf{x}) = \varepsilon$ if and only if $\mathbf{x} = \varepsilon$.

The main functional properties for variadic functions that we present and investigate in this talk are given in the following definition (see [2, 4, 6]).

Definition 1. *A function $F\colon X^* \to X^*$ is said to be* associative *if, for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$, we have*

$$F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}F(\mathbf{y})\mathbf{z}).$$

It is said to be barycentrically associative *(or* B-associative*) if, for every $\mathbf{x}, \mathbf{y}, \mathbf{z} \in X^*$, we have*

$$F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}F(\mathbf{y})^{|\mathbf{y}|}\mathbf{z}).$$

A variadic function $F\colon X^* \to Y$ is said to be preassociative *if, for every $\mathbf{x}, \mathbf{y}, \mathbf{y}', \mathbf{z} \in X^*$, we have*

$$F(\mathbf{y}) = F(\mathbf{y}') \implies F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z}).$$

It is said to be barycentrically preassociative *(or* B-preassociative*) if for every* $\mathbf{x}, \mathbf{y}$, $\mathbf{y}', \mathbf{z} \in X^*$ *, we have*

$$\left.\begin{array}{r} F(\mathbf{y}) = F(\mathbf{y}') \\ |\mathbf{y}| = |\mathbf{y}'| \end{array}\right\} \implies F(\mathbf{x}\mathbf{y}\mathbf{z}) = F(\mathbf{x}\mathbf{y}'\mathbf{z}).$$

The following results show that preassociativity is a weaker form of associativity, and that B-preassociativity is a weaker form of B-associativity.

Proposition 1 ([2]). *A function $F\colon X^* \to X^*$ is associative if and only if it is preassociative and satisfies $F \circ F = F$.*

Proposition 2 ([6]). *A function $F\colon X^* \to X^*$ is B-associative if and only if it is B-preassociative and satisfies $F(\mathbf{x}) = F(F(\mathbf{x})^{|\mathbf{x}|})$ for all $\mathbf{x} \in X^*$.*

Throughout this note, we focus on the associativity and preassociativity properties, leaving the discussion on the properties of B-associative and B-preassociative functions to the oral presentation.

2 Factorization of preassociative functions

Recall that an equivalence relation θ on X^* is called a *congruence* if it satisfies

$$(\mathbf{x}_1 \theta \mathbf{y}_1 \quad \& \quad \mathbf{x}_2 \theta \mathbf{y}_2) \implies \mathbf{x}_1\mathbf{x}_2 \theta \mathbf{y}_1\mathbf{y}_2.$$

The definition of preassociativity and B-preassociativity can be restated as follows. As usual, for any function $F\colon Z \to Y$, we denote by $\ker(F)$ the equivalence relation defined by $\ker(F) = \{(x, y) \in Z^2 \mid F(x) = F(y)\}$.

Lemma 1. *A variadic function $F\colon X^* \to Y$ is preassociative if and only if $\ker(F)$ is a congruence on X^*.*

If $F\colon Z \to Y$ and if $g\colon Y \to Y'$ is an injective function, then $\ker(F) = \ker(g \circ F)$. Hence, we have the following easy corollary.

Corollary 1 ([3,4]). *Let $F\colon X^* \to Y$ be a variadic function. If F is preassociative and $g\colon \mathrm{ran}(F) \to Y'$ is constant or one-to-one, then $g \circ F$ is preassociative.*

In general, given a preassociative function $F\colon X^* \to Y$, characterizing the maps $g\colon Y \to Y'$ such that $g \circ F$ is preassociative is a difficult task since it amounts to characterizing the congruences above $\ker(F)$ on X^*.

It is easily seen that the only one-to-one associative function $F\colon X^* \to X^*$ is the identity. The next result shows that an associative function which is non-injective is in some sense highly non-injective.

Proposition 3 ([2]). *Let $F\colon X^* \to X^*$ be an associative function different from the identity. Then there is an infinite sequence of associative functions $(F^m\colon X^* \to X^*)_{m \geq 1}$ such that* $\ker(\mathrm{id}) \subsetneq \cdots \subsetneq \ker(F^2) \subsetneq \ker(F^1) \subseteq \ker(F)$.

By carefully choosing g in Corollary 1 we can give the following characterization of preassociative functions (see [2, 4, 6]).

Proposition 4 (Factorization of preassociative functions). *Let $F\colon X^* \to Y$ be a function. The following conditions are equivalent.*

(i) F is preassociative.
(ii) There exists an associative function $H\colon X^* \to X^*$ and a one-to-one function $f\colon \mathrm{ran}(H) \to Y$ such that $F = f \circ H$.

For any variadic function $F\colon X^* \to Y$ and any integer $n \geq 0$, we denote by F_n the *n-ary part* of F, *i.e.*, the restriction $F|_{X^n}$ of F to the set X^n. We also let $X^+ = X^* \setminus \{\varepsilon\}$ and denote the restriction $F|_{X^+}$ of F to X^+ by F^+.

Corollary 2. *Let $F\colon X^* \to Y$ be a standard function. The following conditions are equivalent.*

(i) F is preassociative and satisfies $\mathrm{ran}(F_1) = \mathrm{ran}(F^+)$.
(ii) There exists an associatve ε-standard operation $H\colon X^* \to X \cup \{\varepsilon\}$ and a one-to-one function $f\colon \mathrm{ran}(H^+) \to Y$ such that $F^+ = f \circ H^+$.

Corollary 2 enables us to produce axiomatizations of classes of preassociative functions from known axiomatizations of classes of associative functions. Let us illustrate this observation on an example. Further examples can be found in [5]. Let us recall an axiomatization of the Aczélian semigroups.

Proposition 5 ([1]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. An operation $H\colon \mathbb{I}^2 \to \mathbb{I}$ is continuous, one-to-one in each argument, and associative if and only if there exists a continuous and strictly monotone function $\phi\colon \mathbb{I} \to \mathbb{J}$ such that*

$$H(x, y) = \phi^{-1}(\phi(x) + \phi(y)),$$

where $\mathbb{J}$ is a real interval of the form $]-\infty, b[$, $]-\infty, b]$, $]a, +\infty[$, $[a, +\infty[$ or $\mathbb{R} =]-\infty, +\infty[$ ($b \leq 0 \leq a$). For such an operation H, the interval I is necessarily open at least on one end. Moreover, ϕ can be chosen to be strictly increasing.

Corollary 2 leads to the following characterization result.

Theorem 1 ([5]). *Let $\mathbb{I}$ be a nontrivial real interval, possibly unbounded. A standard function $F\colon \mathbb{I}^* \to \mathbb{R}$ is preassociative and satisfies $\mathrm{ran}(F^+) = \mathrm{ran}(F_1)$, and F_1 and F_2 are continuous and one-to-one in each argument if and only if there exist continuous and strictly monotone functions $\phi\colon \mathbb{I} \to \mathbb{J}$ and $\psi\colon \mathbb{J} \to \mathbb{R}$ such that*

$$F_n(\mathbf{x}) = \psi\left(\sum_{i=1}^{n} \phi(x_i)\right), \qquad n \geq 1,$$

where $\mathbb{J}$ is a real interval of one of the forms $]-\infty, b[$, $]-\infty, b]$, $]a, +\infty$, $[a, +\infty[$ or $\mathbb{R} =]-\infty, \infty[$ ($b \leq 0 \leq a$). For such a function F, we have $\psi : F_1 \circ \phi^{-1}$ and $\mathbb{I}$ is necessarily open at least on one end. Moreover, ϕ can be chosen to be strictly increasing.

3 Preassociativity and transition systems

As illustrated below, preassociative functions $F\colon X^* \to Y$ can be characterized as functions that can be computed through some special transition systems. In this note, a *transition system* is a triple $\mathcal{A} = (Q, q_0, \delta)$ where Q is a set of *states*, $q_0 \in Q$ is an *initial state* and $\delta : Q \times X \to Q$ is a transition map. As usual, the map δ is extended to $Q \times X^*$ by setting for any $q \in Q$,

$$\delta(q, \varepsilon) = q$$

$$\delta(q, \mathbf{x}y) = \delta(\delta(q, \mathbf{x}), y), \quad y \in X, \mathbf{x} \in X^*.$$

We say that a function $F\colon X^* \to Y$ is *right-preassociative* if it satisfies $F(\mathbf{x}) = F(\mathbf{y}) \implies F(\mathbf{xz}) = F(\mathbf{yz})$ for every $\mathbf{xyz} \in X^*$.

Definition 2. *Assume that $F : X^* \to Y$ is an onto right-preassociative function. We define the transitions system $\mathcal{A}(F) = (Y, q_0, \delta)$ by*

$$q_0 = F(\varepsilon) \quad \textit{and} \quad \delta(F(\mathbf{x}), z) = F(\mathbf{x}z).$$

We call $\mathcal{A}(F)$ the transitions system associated *with F.*

The language of transition systems give an elegant way to characterize preassociativity. Indeed, transition systems that arise from preassociative functions can be characterized in the following way. For any transition system $\mathcal{A} = (Q, q_0, \delta)$ and any $q \in Q$, let $L^{\mathcal{A}}(q) = \{\mathbf{x} \in X^* \mid \delta(q_0, \mathbf{x}) = q\}$ and $\mathcal{L}^{\mathcal{A}} = \{L^{\mathcal{A}}(q) \mid q \in Q\}$.

Theorem 2. *Let $\mathcal{A} = (Q, q_0, \delta)$ be a transition system. The following conditions are equivalent.*

(i) There is a preassociative function $F\colon X^* \to Q$ such that $\mathcal{A} = \mathcal{A}(F)$.
(ii) For every $z \in X$, the map defined on $\mathcal{L}^{\mathcal{A}}$ by $L \mapsto zL = \{z\mathbf{x} \mid \mathbf{x} \in L\}$ is valued in $\{2^L \mid L \in \mathcal{L}^{\mathcal{A}}\}$.

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Axiomatic approach to probability of fuzzy events

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Abstract. We discuss the requirements on a fuzzy stochastic system describing probabilities of fuzzy events. We present (and sometimes reject) alternative ways. We show the principal contribution of D. Butnariu and E. P. Klement to this research and we summurize the state-of-the-art and perspectives.

1 Motivation

The study of logic in ancient times started from propositions which were either true or false. Then it was found that there are statements which violate this principle for various reasons; we list four of them:

R1 There are logical paradoxes—sentences which cannot be assigned a meaningful truth value.
R2 The validity of some statements is not known now, but it will be determined by a stochastic experiment whose result cannot be predicted with certainty.
R3 Some properties can hardly be described in two-valued (yes-no) terms even if we have all supporting information. More truth values can describe the "truth degree" satisfactorily.
R4 Some experiments influence the environment irrevertibly and they do not admit repetition under the same conditions. We can evaluate them in yes-no terms, but not all possible tests can be made simultaneously or repeated many times.

Here we deal with a combination of R2 and R3. Problem R4 leads to quantum logic; its combination with the above sources is more complex and less advanced; we refer to a new book [21] for more details on this approach.

The idea of solving problem R3 by the use of more than two truth degrees is rather natural; it was introduced independently by several authors. The most successful (although by far not the first) was the intrduction of *fuzzy sets* by L. A. Zadeh [23].

Fuzzy events can also be subject to stochastic uncertainty; we may make stochastic experiments with fuzzy outcomes. Description of such systems requires to define probability of fuzzy events. This was soon recognized and suggested by L. A. Zadeh [24]; he proposed to define a probability of a fuzzy event A with a membership function μ_A as

$$s(A) = \int \mu_A \, \mathrm{d}P \,, \tag{1}$$

where P is a classical probability measure. However, this formula itself is not satisfactory. To specify its meaning, we have to determine the range of integration (usually the set of all reals), the probability measure P and also the *domain* of s, i.e., the family of all events whose probability is defined. We concentrate on the event structures and on the probability defined on them. In the sequel, we shall discuss various requirements on these objects and we select a "main line" of development in this field. Although the advocated "solution" has already been published in [20], its detailed motivation and alternatives have not been presented till now. We also suggest its modification.

2 Classical probability and its alternatives

The event structure, $\mathcal{E}$, is the set of all propositions that can be tested in an experiment, equipped with logical operations (conjunction $\odot$, disjunction $\oplus$, and negation $'$). A probability is a function $s\colon \mathcal{E} \to [0,1]$ satisfying some axioms. What is now considered a "classical" probability is the following axiomatics by A. N. Kolmogorov:

E1b $\mathcal{E}$ is a Boolean algebra.
E2σ $\mathcal{E}$ is closed under *countable* logical operations.
E3c $\mathcal{E}$ is isomorphic to a family of subsets of some set (universe) with the set-theoretical operations.[1]
P1 $s(0) = 0,\ s(1) = 1,$
P2 $A \odot B = 0 \implies s(A \oplus B) = s(A) + s(B),$
P3 s preserves limits of increasing sequences, i.e., $A_n \nearrow A \implies s(A_n) \nearrow s(A).$

Alternatives to principle E2σ are sometimes considered, mainly the following:

E2f $\mathcal{E}$ is closed under *finite* logical operations.
E2a $\mathcal{E}$ is closed under logical operations *of any arity*.

Requiring only finite operations (E2f) appeared to be a too weak assumption which does not admit a sufficiently strong notion of continuity. Requiring operations of any arity (E2a) appeared to be a too strong assumption; it leads to Banach–Tarski paradox etc.

Principle E3c is not obvious. Although each Boolean algebra has a Stone representation, countable logical operations need not correspond to countable set-theoretical operations. A weaker correspondence follows from the Loomis–Sikorski Theorem:

E3c- $\mathcal{E}$ is a σ-homomorphic image of a family of subsets of some set (universe) with the set-theoretical operations.

The difference seems not to be essential for the theory. Principle E3c makes the theory easier, but some problems (e.g. neglecting possible events with zero probability) arise from this choice and they are often ignored.

The conjunction of principles P2 (finite additivity) and P3 is equivalent to σ-additivity. Without P3, we have only finite additivity, which appears to be insufficient for probability theory. On the other hand, complete addtitivity would be a too restrictive condition, excluding even all continuous ditributions.

[1] The set-theoretical operations correspond to the logical ones, the intersection to the conjuction, the union to the disjunction, the complement to the negation.

3 MV-algebras

Obviously, when generalizing to fuzzy statements, we have to abandon principle E1b and replace it by a weaker condition. A problem is what could be a replacement of a Boolean algebra. One candidate is an MV-algebra [5]. We may consider an axiomatic system based on

E1M $\mathcal{E}$ is an MV-algebra.

and principles E2σ, P1, P2, P3. Due to E2σ, we have in fact a σ-MV-algebra with a σ-additive measure. Unless it is a σ-algebra, it is not representable by sets and E3c can be replaced by

E3L $\mathcal{E}$ is isomorphic to a family of fuzzy subsets of some set (universe) with the Łukasiewicz operations.

The latter requirement is often omitted; the consequences are analogous to the case of Boolean σ-algebras. There is even an analogue of the Loomis–Sikorski Theorem [15]:

E3M– $\mathcal{E}$ is a σ-homomorphic image of a family of fuzzy subsets of some set (universe) with the Łukasiewicz operations.

The theory of σ-additive measures on σ-MV-algebras is developed analogously to the classical probability theory. In fact, each σ-additive measure s on a σ-MV-algebra $\mathcal{E}$ has a unique corresponding σ-additive measure P on the Boolean σ-algebra of *Boolean elements* of $\mathcal{E}$ and (1) holds.

Thus σ-MV-algebras offer a rich and well-developed fuzzification of probability theory. However, it is difficult to justify the choice of Łukasiewicz operations; there are many other fuzzy logical operations which could be considered, too. This is the topic of the following sections.

4 Tribes

More generally, one may want an algebra $\mathcal{E}$ based on a triple of operations—a triangular norm, a triangular conorm, and a fuzzy negation.

E1f *(preliminary vague formulation)* $\mathcal{E}$ is an algebra with the following operations: a triangular norm, a triangular conorm, and a fuzzy negation, forming a de Morgan triple.

Additional conditions can be required and they may restrict the choice of fuzzy logical operations; we postpone the precise formulation. There seems not to be an established name for such an algebra[2]. This was the axiomatic approach of Butnariu and Klement,

[2] There is a well-developed theory of BL-algebras (=algebras of basic logic), initiated by P. Hájek in [7]. It uses a t-norm and its residuum (=residuated implication) as basic connectives. The negation is considered as a derived connective and, in general, there may be no (fuzzy) disjunction dual to the (fuzzy) conjunction represented by the t-norm. This approach is well-motivated by the logic of deduction, which is based on the implication as the principal connective. However, the introduction of probability on BL-algebras suffers numerous problems. We do not deal with this approach here.

successfully developed in their monograph [4][3]. We again need countable logical operations, thus we assume E2σ, and representation by fuzzy sets in a new form:

E3f $\mathcal{E}$ is isomorphic to a family of fuzzy subsets of some set (universe) with the fuzzy operations interpreting those of the algebra considered in E1f.

A non-empty family of fuzzy subsets satisfying E2σ and E3f is called a *tribe* [4]. The choice of the de Morgan triple—a t-norm, a t-conorm, and a fuzzy negation—is free, but the same for all definitions[4]. For the definition of a probability measure, principles P1 and P3 are kept, but P2 appears to be too week, especially for t-norms without zero devisors. Thus it was replaced by the *valuation property*:

P2V $s(A \oplus B) + s(A \odot B) = s(A) + s(B)$.

When the fuzzy logical operations are Łukasiewicz operations, this approach coincides with that of Section 3 with E3L. Then we speak of a *Łukasiewicz tribe*. In contrast to it, other choices of fuzzy operations are considered. In most of their work, Butnariu and Klement concentrated on so-called *Frank t-norms and conorms* which contain Łukasiewicz operations and max-min operations as special cases [6]. For max-min operations, P2V appears to be a very weak condition and it is difficult to find a good substitute of it. Thus max-min operations were also abandoned and Archimedean Frank operations remained in the center of interest. Then one of the main results is that each probability is a convex combinations of one component of the form (1) and a component depending only on the *support* of A, $\operatorname{Supp} A = \{x \mid \mu_A(x) > 0\}$. This is a consequence of P3; in [3], we proposed to strengthen it to

P3+ s preserves limits of monotonic sequences, i.e., $A_n \nearrow A \implies s(A_n) \nearrow s(A)$ and $A_n \searrow A \implies s(A_n) \searrow s(A)$.

After this modification (and for Archimedean Frank operations), only probabilities of the form (1) remain. In the preceding approches, P3 and P3+ were equivalent, thus this change does not modify the older notions. Moreover, for Archimedean Frank operations, a tribe (satisfying E2σ and E3f), it was proved in [4] that E3L is also satisfied, and probabilites satisfying P1, P2V, P3+ satisfy also P2 [3].

This approach was quite successful. However, it is difficult to motivate the choice of the de Morgan triple. One may consider it useful to apply different t-norms and t-conorms (pointwise) in different parts of the universe. This would lead to a direct product of tribes, which need not be a tribe. It is quite desirable to have the class of "fuzzy probability spaces" closed under products. The class of tribes also is not closed under σ-homomorphisms, for the same reason as in Boolean σ-algebras and σ-MV-algebras. Thus the underlying class of algebras in E1f cannot form a variety. With this intension, the following algebraic generalization was suggested.

[3] Similar ideas appeared already in [8, 9].

[4] Up to a change of scale of truth degrees by an increasing bijection, the fuzzy negation may be assumed standard, $\alpha \mapsto 1 - \alpha$, without loss of generality.

5 Common generalization

Trying to find an appropriate underlying variety, we define a *fuzzy algebra*[5]. It is a bounded lattice $\mathcal{E}$ with an antitone involution $'$ and binary operations $\odot, \oplus$. Operations $\odot, \oplus$ satisfy the usual requirements on conjunctions and disjunctions (resp. t-norms and t-conorms): they are commutative, associative, and have neutral elements $0, 1$, respectively. Monotonicity is expressed as distributivity with respect to the lattice operations:

$$(A \wedge C) \odot B = (A \odot B) \wedge (C \odot B)\,,$$
$$(A \vee C) \odot B = (A \odot B) \vee (C \odot B)$$

and similarly for $\oplus$, as a consequence of the De Morgan law $A \oplus B = (A' \odot B')'$. Additionally, we assume the *Kleene condition*[6] $A \wedge A' \leq B \vee B'$.

The proposed new axiomatics uses

E1f *(exact formulation)* $\mathcal{E}$ is a fuzzy algebra.

Principle E2σ is assumed, in the sense that $\mathcal{E}$ is closed under *all* countable operations. Closedness under countable *lattice* operations (used in [20]) suffices; it implies also the closedness under the (monotonic) countable *fuzzy logical* operations. The reverse implication holds only for some fuzzy logical operations, but these seem to include all important cases. Fuzzy set representation E3f is not required. Probability measures are subject to principles P1, P2V, and P3+.

In [20], monotonicity of $\odot$ and $\oplus$ is strengthened to

$$\Big(\bigwedge_{n\in\mathbb{N}} A_n\Big) \odot B = \bigwedge_{n\in\mathbb{N}} (A_n \odot B)\ ,$$
$$\Big(\bigvee_{n\in\mathbb{N}} A_n\Big) \odot B = \bigvee_{n\in\mathbb{N}} (A_n \odot B)\ .$$

This implies also continuity of the t-norms and t-conorms interpreting $\odot, \oplus$, respectively. The appropriateness of this condition can be a point of future discussion.

This is a common generalization of σ-MV-algebras (which need not be represented by fuzzy sets) and tribes (which admit to use other fuzzy operations than the Łukasiewicz ones). The principal new examples covered by this approach are direct products of the above and also products of tribes based on different t-norms and t-conorms.

6 Conclusions

We showed a common generalization of preceding models of probability of fuzzy events. We discussed which features seem necessary and where we have several options; future experience may decide which of them is the most perspective one.

[5] The term *fuzzy algebra* has been already used in several different meanings. We keep it here at least for temporary use because it is a very natural name. In [20], its σ-complete version was introduced and named a *fuzzy σ-algebra*, although this term has been already used by E.P. Klement in [9] for a less general notion.

[6] The Kleene condition does not seem necessary, but it is satisfied in all models based on fuzzy sets. With this condition, $\mathcal{E}$ becomes also a *Kleene algebra*.

The next task is to generalize results (obtained for σ-MV-algebras and tribes) to the new context. In particular, it is highly desirable to know under which conditions the probability measures can be expressed in the integral form (1). Other results, like a generalization of Loomis–Sikorski Theorem etc., could be also of interest from both theoretical and practical points of view.

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Characterization problems and theorems in various classes of means

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Motivated by the celebrated characterization theorem of quasi-arithmetic means (cf. [11]) discovered independently by A. N. Kolmogorov [12], M. Nagumo [15] and B. de Finetti [9] in the thirties of the last century and also by J. Aczél [1] for fixed number of variables in the fourties, it is tempting to search for similar results in various other classes of means, such as the classes of Bajrakterević means [4, 5], Daróczy means [6, 7], Gini means [10], Matkowski means [13] and Stolarsky means [20]. Contrary to the case of quasi-arithmetic means, some of the characterization theorems related to the above classes of means involve properties expressed in terms of functional inequalities instead of functional equations. Several known results from the papers [2, 3, 8, 14, 16–19] will be discussed and compared, but also many challenging open problems will be described.

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Sublinear means

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Abstract. In this paper we concentrate on subadditiveness in the theory of aggregation functions, nonadditive measure theory and related integrals. First we considere the finite case, and then we give some hints on more complex infinite case.

1 Introduction

To point out the importance of subadditive functions we cite M. Kuczma [6]: "*The natural contrapart of the Cauchy equation would be the inequality*

$$f(x+y) \leqslant f(x)+f(y).$$

It seems so the more interesting and astounding that the convex functions and the additive functions share so many properties that such is not the case of additive functions and functions satisfying the previous inequality."

In this paper we concentrate on subadditiveness in the theory of aggregation functions combined with homogenity, nonadditive measure theory and related integrals. First we considere the finite case, and then we give some hints on more complex infinite case. We use the notions from [4].

Definition 1. *A function* $f : \mathbb{R}^n \to [0,\infty]$ *is sublinear if*

$$f(\alpha\mathbf{x}+\beta\mathbf{y}) \leqslant \alpha f(\mathbf{x})+\beta f(\mathbf{y}) \ \ (\mathbf{x},\mathbf{y} \in \mathbb{R}^n, \alpha,\beta \geqslant 0).$$

It is well known that a function $f : \mathbb{R}^n \to [0,\infty]$ is sublinear if and only if it is positively homogeneous and subadittive.

First, we considere the finite case. Let $N = \{1,2,\ldots,n\}$.

Definition 2. *We say that* $\mu : 2^N \longrightarrow [0,\infty]$ *is a submeasure if it satisfies*

(i) $\mu(\emptyset) = 0$.
(ii) (Monotonicity) $E,F \in 2^N$ *and* $E \subset F$ *imply* $\mu(E) \leqslant \mu(F)$.
(iii) (Subadditivity) $E,F \in 2^N$ *and* $\mu(E\cup F) \leqslant \mu(E)+\mu(F)$.

We have for $\mathbf{x} \in \mathbb{R}^n_+$ and $E \subseteq N$, where $E = \{i_1,i_2,\ldots,i_s\}$ and $i_1 \leqslant \cdots \leqslant i_s$, that $M(\mathbf{x})(E) = M(x_{i_1},x_{i_2},\ldots,x_{i_s})$. We introduce the notation $M_i(\mathbf{x}) = M(x_1,x_2,\ldots,x_i)$.

Definition 3. *A mean $M : [0,\infty]^n \to [0,\infty]^{2^N}$ is sublinear if for every $E \subseteq N$*

(i) (Monotonicity) $\mathbf{x} \leqslant \mathbf{y} \Rightarrow M(\mathbf{x})(E) \leqslant M(\mathbf{y})(E)$.

(ii) (Homogenity) $M(\alpha\mathbf{x})(E) = \alpha M(\mathbf{x})(E)$ *for all* $\alpha \geqslant 0$.

(iii) (Subadditivity) $M(\mathbf{x}+\mathbf{y})(E) \leqslant M(\mathbf{x})(E) + M(\mathbf{y})(E)$.

Example 1.
(i) Arithmetic mean is a sublinear mean. More general, the weighgthed sum is also a sublinear mean.
(ii) Geometric mean is a sublinear mean.
(iii) Quasi arithmetical mean, see [4], for f strictly mononotone increasing is a sublinear mean if and only if f is sublinear function.
(iv) The Choquet integral is a sublinear mean if and only if μ is submeasure, see [2, 11, 12].
(v) By [5] every aggregation function A under some conditions can be transformed to an subadditive aggregation function A_*. It is interesting that A_* is the greatest subadditive aggregation function not greater than A. If A is homogeneous, then A_* is also homogeneous, i.e., sublinear function. It is the greatest convex aggregation function dominated by A.
(vi) The convex integral is a sublinear mean, see [8].

Definition 4. *Let μ be a submeasure. Let M be a sublinear mean. Then we define*

$$(MC)\int \mathbf{x}\, d\mu = \int_0^{\max_{i\in N} M_i(\mathbf{x})} \mu(\{i \in N \mid M_i(\mathbf{x}) \geqslant t\})\, dt.$$

The (MC) integral with respect to submeasure μ and $\mathbf{x} \in \mathbb{R}_+^n$ can be written in the following form

$$(MC)\int \mathbf{x}\, d\mu = \sum_{i=1}^{n} (M_{\sigma(i)}(\mathbf{x}) - M_{\sigma(i-1)}(\mathbf{x}))\mu(\{\sigma(i),\ldots,\sigma(n)\}),$$

with a permutation σ such that $M_{\sigma(1)}(\mathbf{x}) \leqslant M_{\sigma(2)}(\mathbf{x}) \leqslant \cdots \leqslant M_{\sigma(n)}(\mathbf{x})$ and the convention $M_{\sigma(0)}(\mathbf{x}) = 0$.

Example 2.
(i) Taking M such that $M_i(\mathbf{x}) = x_i$ for $i = 1,2,\ldots,n$, we obtain the Choquet integral.
(ii) Taking $M(\mathbf{x}) = (C)\int \mathbf{x} d\mu$, for a submeasure μ, we obtain the multilevel Choquet integral, see [9].
(iii) The geometrical mean can not be represented by Choquet integral but as a sublinear mean it can be used in an (MC) integral construction.

Theorem 1. *The (MC) integral is monotone, homogeneous and subadditive.*

Definition 5. *We define for* $\mathbf{x} \in \mathbb{R}_+^n$ $\|\mathbf{x}\|_{L^\infty(N,M)} = \max_{E\in 2^N} M(\mathbf{x})(E)$.

Definition 6. *Let $\mu : 2^N \to [0,\infty]$ be a submeasure and $0 < p < \infty$. Then we define for $\mathbf{x} \in \mathbb{R}^n_+$*

$$\|\mathbf{x}\|_{L^p(N,M)} = \left(\sum_{i=1}^{n} \left(M_{\sigma(i)}(\mathbf{x})^p - M_{\sigma(i-1)}(\mathbf{x})^p \right) \mu(\{\sigma(i), \ldots, \sigma(n)\}) \right)^{1/p},$$

$$\|\mathbf{x}\|_{L^{p,\infty}(N,M)} = \left(\max_{i \in \{1,\ldots,n\}} M_{\sigma(i)}(\mathbf{x})^p \mu(\{\sigma(i), \ldots, \sigma(n)\}) \right)^{1/p},$$

with a permutation σ such that $M_{\sigma(1)}(\mathbf{x})^p \leqslant M_{\sigma(2)}(\mathbf{x})^p \leqslant \cdots \leqslant M_{\sigma(n)}(\mathbf{x})^p$ and the convention $M_{\sigma(0)}(\mathbf{x}) = 0$.

Remark 1. (i) Choquet-like integral [7] and integrals introduced in [10, 13] are related to previous Definition 6.
(ii) The (MC) integral approach can be managed also for Sugeno and Schilkret integrals.

Proposition 1. *Functionals $\|\mathbf{x}\|_{L^\infty(N,M)}, \|\mathbf{x}\|_{L^p(N,M)}$ and $\|\mathbf{x}\|_{L^{p,\infty}(N,M)}$ are norms for $\mathbf{x} \in \mathbb{R}^n_+$.*

We define $L^\infty(N,M)$, $L^p(N,M)$ and $L^{p,\infty}(N,M)$ to be the spaces of elements $\mathbf{x} \in \mathbb{R}^n_+$ endowed with norms $\|\mathbf{x}\|_{L^\infty(N,M)}, \|\mathbf{x}\|_{L^p(N,M)}$ and $\|\mathbf{x}\|_{L^{p,\infty}(N,M)}$, respectively.

Now we investigate the infinite case, which is much more complex, see [1, 3]. Let X be an infinite complete metric space.

Definition 7. *We say that $\mu : 2^X \longrightarrow [0,\infty]$ is an outer measure if $\mu(\emptyset) = 0$, it is monotone and it is countable subadditive, i.e., if $E_1, E_2, \cdots \in 2^X$, then*

$$\mu\left(\bigcup_{j=1}^{\infty} E_i \right) \leqslant \sum_{j=1}^{\infty} \mu(E_j).$$

We introduce now generalization of the sublinear mean on the class $\mathcal{B}(X)$ of all nonnegative Borel measurable functions on X, see [3].

Definition 8. *Let $\mathcal{D}$ be a collection of subsets of X. A function $M : \mathcal{B}(X) \to [0,\infty]^{\mathcal{D}}$ is called size if for every $f, g \in \mathcal{B}(X)$ and every $E \in \mathcal{D}$ the following conditions hold*

(i) (Monotonicity) $f \leqslant g \Rightarrow M(f)(E) \leqslant M(g)(E)$.

(ii) (Homogenity) $M(\alpha f)(E) = \alpha M(f)(E)$ *for all* $\alpha \geqslant 0$.

(iii) (Quasi-Subadditivity) $M(f+g)(E) \leqslant C(M(f)(E) + M(g)(E))$ *for some constant C depending only on M.*

We introduce the following functional for $f \in \mathcal{B}(X)$.

Definition 9. *We define for $f \in \mathcal{B}(X)$ $\|f\|_{L^\infty(X,M)} = \sup_{E \in \mathcal{D}} M(f)(E)$, and denote by $L^\infty(X,M)$ the space of elements $f \in \mathcal{B}(X)$ for which $\sup_{E \in \mathcal{D}} M(f)(E)$ is finite.*

Instead of super level sets $\{x \mid f(x) > \lambda\}$ for a function f a new quantity is introduced and called super level measure, see [3].

Definition 10. *Let $\mu : 2^X \to [0,\infty]$ be an outer measure and $\lambda > 0$. We define for $f \in \mathcal{B}(X)$ the super level measure $\mu(M(f) > \lambda)$ to be the infimum of all values $\mu(F)$, where F runs through all Borel subsets of X such that $\sup_{E\in\mathcal{D}} M(f\mathbf{1}_{X\setminus F})(E) \leqslant \lambda$.*

We introduce the outer space L_p by [3].

Definition 11. *Let $\mu : 2^X \to [0,\infty]$ be an outer measure and $0 < p < \infty$. Then we define for $f \in \mathcal{B}(X)$*

$$\|f\|_{L^p(X,M)} = \left(\int_0^\infty p\lambda^{p-1}\mu(M((f) > \lambda)\,d\lambda\right)^{1/p},$$

$$\|f\|_{L^{p,\infty}(N,M)} = \left(\sup_{\lambda>0} \lambda^p \mu(M(f) > \lambda)\,d\lambda\right)^{1/p},$$

where $\mu(M((f) > \lambda)$ is given in Definition 10. Define $L^p(X,M)$ and $L^{p,\infty}(X,M)$ to be the spaces of elements $f \in \mathcal{B}(X)$ such that the respective quantitives are finite.

Proposition 2. *The functional $\|f\|_{L^p(X,M)}$ for $0 < p \leqslant \infty$ is monotone, homogeneous and quasi-subadditive.*

The L^p theory for outer measure is very useful in the investigations of singular integrals, e.g., Carleson embedding theory and in time frequence analysis, see [1, 3].

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The polytope of ultramodular discrete copulas

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Abstract. We here discuss the polytope of ultramodular discrete copulas, i.e., discrete restrictions of ultramodular copulas. First, we introduce the concept of ultramodularity in a discrete scenario. Then, we describe the polytope of ultramodular discrete copulas through its bounding affine half-spaces.

1 Introduction

Copula functions are largely employed in applied statistics as a flexible tool to describe dependence between random variables (see, e.g., [1, 2, 9]). As a consequence of Sklar's Theorem ([10]), the joint bivariate distribution F_{XY} of two random variables X and Y with univariate margins F_X and F_Y, respectively, can be written as

$$F_{XY}(x, y) = C(F_X(x), F_Y(y)) \qquad (x, y \in R) \tag{1}$$

where C is a bivariate copula uniquely determined on the set ran$(F_X) \times$ ran(F_Y).

Limitations of the usage of Sklar's Theorem in the discrete setting have led to the introduction of *discrete copula*. Such functions, defined as follows, have an interesting statistical meaning and mathematical properties (see, e.g., [5, 7, 8]).

Definition 1. *Let* $I_n := \left\{0, \frac{1}{n}, \ldots, \frac{n-1}{n}, 1\right\}$, $I_m := \left\{0, \frac{1}{m}, \ldots, \frac{m-1}{m}, 1\right\}$ *with* $n, m \in \mathbb{N}$. *A function* $C_{n,m} : I_n \times I_m \to [0, 1]$ *is a* discrete copula *on* $I_n \times I_m$ *if it satisfies:*

(c1) *for all* $i \in \{0, \ldots, n\}$ *and* $j \in \{0, \ldots, m\}$*,*

$$C_{n,m}\left(\tfrac{i}{n}, 0\right) = C_{n,m}\left(0, \tfrac{j}{m}\right) = 0; \quad \textit{and} \quad C_{n,m}\left(\tfrac{i}{n}, 1\right) = \tfrac{i}{n}, \quad C_{n,m}\left(1, \tfrac{j}{m}\right) = \tfrac{j}{m};$$

(c2) *for all* $i \in \{0, \ldots, n-1\}$ *and* $j \in \{0, \ldots, m-1\}$*,*

$$C_{n,m}\left(\tfrac{i}{n}, \tfrac{j}{m}\right) + C_{n,m}\left(\tfrac{i+1}{n}, \tfrac{j+1}{m}\right) \geq C_{n,m}\left(\tfrac{i+1}{n}, \tfrac{j}{m}\right) + C_{n,m}\left(\tfrac{i}{n}, \tfrac{j+1}{m}\right).$$

Note that every restriction of a copula to a discrete set $I_n \times I_m$ is a discrete copula. Furthermore, every discrete copula can be extended to a copula on $[0, 1]^2$ (see [9]).

In the following, we will assume that $n = m$. Then discrete copulas have fascinating mathematical properties as highlighted in [5]; for example, they prove that there exists a one-to-one correspondence between discrete copulas and bistochastic matrices.

In addition, they discuss the role of a special subclass of discrete copulas, so-called *irreducible discrete copulas*, namely discrete copulas with minimal range I_n. This subclass is associated with the class of $\{0,1\}$-valued bistochastic matrices, i.e., the permutation matrices, and also corresponds to the order statistics ([8]). According to [5], the space of all discrete copulas can then be defined as the convex hull of irreducible discrete copulas, i.e., the polytope generated by permutation matrices. This is the well-known Birkhoff polytope ([11]).

In this work, we study a particular subpolytope of the Birkhoff polytope, namely the convex set of discrete ultramodular copulas, i.e., copulas with convex horizontal and vertical sections (see, e.g., [4, 3]). Although the study of certain subpolytopes of the Birkhoff polytope has been of interest per se (see, e.g., [6]), this research can be seen as one of the first efforts in introducing geometric techniques to the domain of discrete copulas.

The paper is organized as follows: first, we present the family of discrete restrictions of ultramodular copulas, i.e., ultramodular discrete copulas. Then, we prove that the space of ultramodular discrete copulas is given as the bounded intersection of $(n-2)^2 + 2(n-1)^2$ (for $n \geq 4$) affine half-spaces and we discuss the defining hyperplanes.

2 The polytope of ultramodular discrete copulas

Consider a function $C : I_n^2 \to [0,1]$, $n \geq 4$, with boundary conditions as in (c1) of Definition 1. C is said to satisfy Property D if all of the following conditions are fulfilled:

(1) $C\Big(\frac{1}{n}, \frac{1}{n}\Big) \geq 0$ and $C\Big(\frac{n-1}{n}, \frac{n-1}{n}\Big) \geq \frac{n-2}{n}$;

(2a) for $j \in \{2, \ldots, n-2\}$,

$$C\Big(\tfrac{1}{n}, \tfrac{j}{n}\Big) + C\Big(\tfrac{2}{n}, \tfrac{j+1}{n}\Big) \geq C\Big(\tfrac{1}{n}, \tfrac{j+1}{n}\Big) + C\Big(\tfrac{2}{n}, \tfrac{j}{n}\Big);$$

(2b) for $i \in \{2, \ldots, n-3\}$ and $j \in \{1, \ldots, n-2\}$,

$$C\Big(\tfrac{i}{n}, \tfrac{j}{n}\Big) + C\Big(\tfrac{i+1}{n}, \tfrac{j+1}{n}\Big) \geq C\Big(\tfrac{i}{n}, \tfrac{j+1}{n}\Big) + C\Big(\tfrac{i+1}{n}, \tfrac{j}{n}\Big);$$

(2c) for $j \in \{1, \ldots, n-3\}$,

$$C\Big(\tfrac{n-2}{n}, \tfrac{j}{n}\Big) + C\Big(\tfrac{n-1}{n}, \tfrac{j+1}{n}\Big) \geq C\Big(\tfrac{n-2}{n}, \tfrac{j+1}{n}\Big) + C\Big(\tfrac{n-1}{n}, \tfrac{j}{n}\Big);$$

(3a) for $i \in \{1, \ldots, n-1\}$ and $j \in \{0, \ldots, n-2\}$,

$$C\Big(\tfrac{i}{n}, \tfrac{j}{n}\Big) + C\Big(\tfrac{i}{n}, \tfrac{j+2}{n}\Big) \geq 2\, C\Big(\tfrac{i}{n}, \tfrac{j+1}{n}\Big);$$

(3b) for $i \in \{0, \ldots, n-2\}$ and $j \in \{1, \ldots, n-1\}$,

$$C\Big(\tfrac{i}{n}, \tfrac{j}{n}\Big) + C\Big(\tfrac{i+2}{n}, \tfrac{j}{n}\Big) \geq 2\, C\Big(\tfrac{i+1}{n}, \tfrac{j}{n}\Big).$$

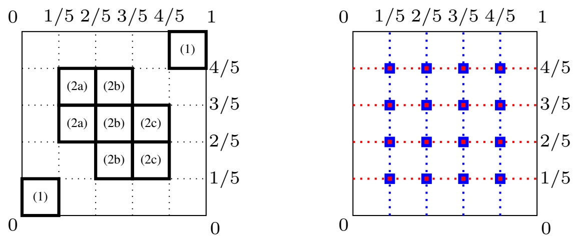

Fig. 1. ($n = 5$) Left side: $(n-2)^2$ squares corresponding to properties (1) and (2). Right side: $(n-1)^2$ middle points of property (3).

To give some intuition for these conditions: conditions (1) and (2) ensure positive measure for $(n-2)^2$ of the n^2 subsquares of I_n^2, while condition (3) corresponds to a discrete version of the convexity property for the horizontal and vertical sections (see Figure 1). In the following result we establish the link between functions C that satisfy Property D and ultramodular copulas.

Theorem 1. *Let $C : I_n^2 \to [0,1]$ be a function that satisfies Property D. Then there exists an ultramodular copula $\hat{C} : [0,1]^2 \to [0,1]$ such that C is the restriction of $\hat{C}$ to the discrete set I_n^2. Conversely, every restriction to I_n^2 of an ultramodular copula defined on $[0,1]^2$ satisfies Property D.*

Theorem 1 asserts that every restriction of an ultramodular copula fulfills Property D. Furthermore, in the following example we prove that in fact all inequalities of Property D are needed in order to obtain a discrete copula C on I_n^2 through restriction of an ultramodular copula.

Example 1. Let $n = 4$ and consider the discrete functions C_1 and C_2 defined on the discrete set $(I_4 \setminus \{0\})^2$ by the following matrices:

$$C_1 = (c_{ij,1}) = \begin{pmatrix} 0 & \frac{1}{12} & \frac{1}{6} & \frac{1}{4} \\ 0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix}, \qquad C_2 = (c_{ij,2}) = \begin{pmatrix} 0 & 0 & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{6} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{pmatrix},$$

and such that they equal zero on $I_4 \times \{0\}$ and $\{0\} \times I_4$. Note that both the functions C_1 and C_2 satisfy statement (c1) in Definition 1. In addition, the functions C_1 and C_2 satisfy all conditions in Property D, except on one subsquare of I_4^2, each (see Figure 2).

We now define two further matrices B_1 and B_2 as functions of C_1 and C_2, respectively, namely through $b_{ij} = n(c_{i+1,j+1}^{(n)} - c_{i+1,j}^{(n)} - c_{i,j+1}^{(n)} + c_{i,j}^{(n)})$. This results in

$$B_1 = (b_{ij,1}) = \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{6} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}, \qquad B_2 = (b_{ij,2}) = \begin{pmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix},$$

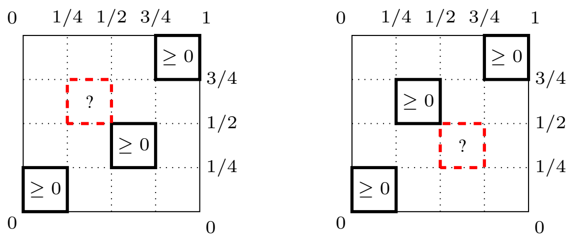

Fig. 2. The black squares correspond to the inequalities assumed true, the red squares to the ones to be checked. Left side: condition satisfied by C_2. Right side: condition satisfied by C_1.

Since B_1 and B_2 have negative entries, C_1 and C_2 fail to be discrete copulas. $\square$

This example shows that all $(n-2)^2 + 2(n-1)^2$ inequalities of Property D are needed to characterize ultramodular discrete copulas. These inequalities define the polytope of ultramodular discrete copulas; the polytope is given by the intersection of the corresponding half-spaces.

Conclusions and open questions

In this work we presented a description of the polytope of discrete ultramodular copulas as the convex set bounded by the $(n-2)^2 + 2(n-1)^2$ (for $n \geq 4$) affine half-spaces corresponding to the inequalities given in Property D. Any polytope can be dually defined as the convex hull of a finite set of points. An interesting open question is related to the vertices of this polytope. We are currently investigating the number of vertices as well as the statistical interpretation the vertices in the classical context of ultramodular copulas.

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Transitivity of dominance and Mulholland inequality

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Abstract. The paper deals with the question of transitivity of the dominance relation on the set of continuous triangular norms. A brief overview of the achieved results is provided. Examples are given showing that the dominance is not transitive neither on the set of continuous triangular norms nor on the sets of strict and nilpotent triangular norms.

A *triangular norm* (a *t-norm*, for short) [1, 7] is a non-decreasing, commutative, and associative binary operation $*\colon\ [0,1]\times[0,1]\to[0,1]$ with neutral element 1. T-norms play an important role, for example, in the framework of the basic logic [4, 5] and the monoidal t-norm based logic [3] which are both prototypical logics of graded truth (fuzzy logics). However, originally the notion of a t-norm has been introduced within the framework of probabilistic metric spaces [8, 18] where they establish the triangular inequality of the probabilistic metrics.

Here we deal exclusively with continuous and continuous Archimedean t-norms. A t-norm is said to be *continuous* if it is continuous as a two-variable real function. A t-norm $*$ is said to be *Archimedean* if, for every $x, y \in\ ]0,1[$, $y \le x$, there is $n \in \mathbb{N}$ such that $x_*^n \le y$. Here, $x_*^0 = 1$ and $x_*^n = x * x_*^{n-1}$. A continuous t-norm is said to be *strict* if its restriction to $]0,1]\times]0,1]$ is strictly increasing in each variable. A continuous t-norm $*$ is said to be *nilpotent* if for every $x \in\ ]0,1[$ there is an $n \in \mathbb{N}$ such that $x_*^n = 0$. A continuous Archimedean t-norm is either strict or nilpotent. An example of a strict and a nilpotent t-norm is the product and the Łukasiewicz t-norm defined, for every $x, y \in [0,1]$, by $x \cdot y$ and $\max\{0, x+y-1\}$, respectively.

Remark 1. By $[0,\infty]$ we denote the set of positive real numbers with the least element 0, the greatest element ∞, and with $x+\infty=\infty$ for any $x \in [0,\infty]$.

A t-norm $*$ is continuous and Archimedean if, and only if, there exist a decreasing bijection $t\colon\ [0,1] \to [0,b]$, $b > 0$, such that $x * y = t^{(-1)}(t(x)+t(y))$ holds for every $x, y \in [0,1]$. Here $t^{(-1)}$ denotes the *pseudo-inverse* [7] of t and it is defined, in this particular case, by $t^{(-1)}(x) = t^{-1}(x)$ if $x \le b$ and by $t^{(-1)}(x) = 0$ otherwise. If $b = \infty$ then $*$ is a strict t-norm, if $b < \infty$ then $*$ is a nilpotent t-norm.

Dominance, in general, is a binary relation on a given set of n-ary operations. In our context, a t-norm $*$ is said to *dominate* a t-norm $\diamond$ (and we denote it by $* \gg \diamond$) if

$$(x \diamond y) * (u \diamond v) \ge (x * u) \diamond (y * v)$$

holds for every $x, y, u, v \in [0,1]$. The motivation to study dominance of t-norms comes from Tardiff who, in his paper from 1984 [20], has recognized that this notion plays a crucial role when constructing Cartesian products of probabilistic metric spaces.

It is easy to check that on the set of t-norms the dominance relation is reflexive and anti-symmetric. However, for a long time it had remained an open problem whether it is also transitive and thus an order [2, Problem 17]. What follows is a list of known partial solutions.

In 1984, Sherwood [19] has proven that the dominance relation is transitive on the class of Schweiser-Sklar t-norms.

In the monograph by Klement, Mesiar, and Pap [7] it is shown that the dominance relation is transitive on the class of Aczél-Alsina t-norms, on the class of Dombi t-norms, and on the class of Yager t-norms.

In 2005, Sarkoci [15] has proven that the dominance relation is transitive on the class of Frank t-norms and on the class of Hamacher t-norms.

In 2005, it has been shown by Saminger-Platz, De Baets, and De Meyer [12] that the dominance relation is transitive on the class of Mayor-Torrens t-norms and on the class of Dubois-Prade t-norms.

In 2009, Saminger-Platz [11] has proven transitivity of the dominance relation on the classes $T^{\mathbf{8}}$, $T^{\mathbf{9}}$, $T^{\mathbf{15}}$, $T^{\mathbf{22}}$, and $T^{\mathbf{34}}$.

In 2011, it has been proven using a symbolic computation system [6] that the dominance relation is transitive on the class of Sugeno-Weber t-norms.

On the other hand, in 2008, Sarkoci has demonstrated that the dominance relation on the set of continuous t-norms is not transitive [14, 16] by the following counter-example. Take the following three t-norms: the t-norm $*$ is an ordinal sum [7] where the only summand is the Łukasiewicz t-norm on the interval $\left[0, \frac{1}{2}\right]$, the t-norm $\diamond$ is an ordinal sum with two summands, both the Łukasiewicz t-norm, on the intervals $\left[0, \frac{1}{2}\right]$ and $\left[\frac{1}{2}, 1\right]$, the t-norm $\odot$ is equal to the Łukasiewicz t-norm. Then we have $* \gg \diamond$ and $\diamond \gg \odot$ but $* \not\gg \odot$.

Further, Sarkoci has given a characterization of the dominance on the class of ordinal sum t-norms that use the Łukasiewicz t-norm as the only summand operation and the class of ordinal sum t-norms that use the product t-norm as the only summand operation [17].

The counter-example by Sarkoci is based on ordinal sum t-norms. Therefore, the question whether the dominance relation is (or is not) transitive on the set of continuous Archimedean t-norms has remained unanswered. As we can see, it has been revealed that for many significant subclasses of continuous Archimedean t-norms the dominance relation is transitive which could lead to a conjecture that this transitivity is valid on the whole class of strict or nilpotent t-norms. We are going to show that this is not the case.

The *Mulholland inequality* has been introduced in 1950 by Mulholland [9] as a generalization of the Minkowski inequality which establishes the triangular inequality of p-norms (also L^p*-norms*).

Definition 1. *Consider an increasing bijection* $f\colon [0,\infty] \to [0,\infty]$. *It is said to* solve the Mulholland inequality *if*

$$f^{-1}\big(f(x+u) + f(y+v)\big) \leq f^{-1}\big(f(x) + f(y)\big) + f^{-1}\big(f(u) + f(v)\big) \tag{1}$$

holds for every $x, y, u, v \in [0, \infty]$.

In 1984, Tardiff [20] has shown that the Mulholland inequality is in a close correspondence with dominance of strict t-norms.

Theorem 1. *Let* $*$ *and* $\diamond$ *be two strict t-norms defined by their additive generators* t_* *and* $t_\diamond$ *as*

$$x * y = t_*^{-1}(t_*(x) + t_*(y)),$$
$$x \diamond y = t_\diamond^{-1}(t_\diamond(x) + t_\diamond(y)).$$

Then $*$ *dominates* $\diamond$ *if, and only if,* $f = t_* \circ t_\diamond^{-1}$ *solves the Mulholland inequality.*

In 2008, this correspondence has been enlarged by Saminger-Platz, De Baets, and De Meyer [13] to the set of all continuous Archimedean t-norms introducing the notion of the *generalized Mulholland inequality*.

Theorem 2. *Let* $*$ *and* $\diamond$ *be two continuous Archimedean t-norms defined by their additive generators* t_* *and* $t_\diamond$ *as*

$$x * y = t_*^{(-1)}(t_*(x) + t_*(y)),$$
$$x \diamond y = t_\diamond^{(-1)}(t_\diamond(x) + t_\diamond(y)).$$

Then $*$ *dominates* $\diamond$ *if, and only if, the function* $f\colon [0,\infty] \to [0,\infty]$ *defined by* $f = t_* \circ t_\diamond^{(-1)}$ *satisfies*

$$f^{(-1)}\big(f(x+u) + f(y+v)\big) \leq f^{(-1)}\big(f(x) + f(y)\big) + f^{(-1)}\big(f(u) + f(v)\big)$$

for every $x, y, u, v \in [0, \infty]$. *Here, the pseudo-inverse of* f *is given by* $f^{(-1)} = t_\diamond \circ t_*^{(-1)}$.

In our recent result [10], we have shown that the set of functions that solve the Mulholland inequality is not closed with respect to their compositions. Namely, both functions $g\colon [0,\infty] \to [0,\infty]$ and $h\colon [0,\infty] \to [0,\infty]$ given, for $x \in [0,\infty]$, by $h(x) = x^2$ and

$$g(x) = \begin{cases} \frac{5}{3}x & \text{if } x \in \left]0,1\right[\,, \\ \frac{7}{3}x - \frac{2}{3} & \text{if } x \in \left[1,2\right[\,, \\ x^2 & \text{if } x \in [2,\infty] \end{cases}$$

solve the Mulholland inequality while $g \circ h$ does not.

This gives us a way how to construct a counter-example that disproves transitivity of dominance on the set of strict t-norms. Indeed, taking an arbitrary decreasing bijection $t_2\colon [0,1] \to [0,\infty]$ and $t_1 = g \circ t_2$, $t_3 = h^{-1} \circ t_2$ we obtain additive generators of three strict t-norms. However, while both $t_1 \circ t_2^{-1}$ and $t_2 \circ t_3^{-1}$ does solve the Mulholland inequality, $t_1 \circ t_3^{-1}$ does not.

Thanks to the result by Saminger-Platz, De Baets, and De Meyer [13], we may proceed analogously in the case of general continuous Archimedean t-norms.

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Fuzzy relation equations in fuzzy functional and topological spaces

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Abstract. In this contribution, we discuss a prominent role of fuzzy relation equations and their systems in determining various fuzzy spaces and notions: spaces with fuzzy partitions, fuzzy topologies and fuzzy functions.

1 Introduction

Let $(L, \leq, \wedge, \vee)$ be a complete frame, i.e., a lattice where arbitrary suprema (joins) and infima (meets) exist and moreover, finite meets distribute over arbitrary joins:

$$\alpha \wedge \{\bigvee_i : i \in I\} = \bigvee_i \{\alpha \wedge \beta_i : i \in I\} \quad \forall \alpha \in L,\ \forall \{\beta_i : i \in I\} \subseteq L,$$

In particular, the top 1_L and the bottom 0_L elements in L exist and $0_L \neq 1_L$.

As a general algebraic structure, we use a cl-monoid $(L, \leq, \wedge, \vee, *)$ extended by the binary operation $\rightarrow$ (residium):

$$\alpha \rightarrow \beta = \bigvee \{\lambda \in L \mid \lambda * \alpha \leq \beta\}.$$

A cl-monoid can be viewed as an integral commutative quantale in the sense of K.I. Rosenthal [9]. Important properties of cl-monoids are collected in [3].

Let X be a set, L a cl-monoid, L^X a family of (L-)fuzzy sets of X. The couple $(L^X, =)$ is called the *ordinary fuzzy space* on X. We assume that basic notions and operations over fuzzy sets are defined in a standard way.

Let X, Y be universal sets. A *(binary) (L-)fuzzy relation* is a fuzzy set of $X \times Y$. If $X = Y$, then a fuzzy set of $X \times X$ is called a (binary) (L-)fuzzy relation on X.

A binary fuzzy relation E on X is called *fuzzy equivalence* on X if for all $x, y, z \in X$, the following holds:

1. $E(x, x) = 1$, *reflexivity,*
2. $E(x, y) = E(y, x)$, *symmetry,*
3. $E(x, y) * E(y, z) \leq E(x, z)$, *$*$-transitivity.*

If fuzzy equivalence E fulfills a stronger version of the first axiom:

1^*. $E(x, y) = 1$ if and only if $x = y$,

then it is called *separated* or a *fuzzy equality* on X.

Let us equip the space X with a fuzzy equivalence E and denote it by (X, E). We will refer to this space as to a *fuzzy space*.

2 Fuzzy Functions

We recall the notion of a fuzzy function and a perfect fuzzy function as they appeared in [4] and in [2].

Definition 1. *Let (X, E) and (Y, F) be fuzzy spaces. A fuzzy function is a binary fuzzy relation ρ on $X \times Y$ that solves the following system of fuzzy relation inequalities:*

FF.1 $(E \circ \rho)(x', y) \leq \rho(x', y)$,
FF.2 $(\rho \circ F)(x, y') \leq \rho(x, y')$,
FF.3 $(\rho * \circ \rho)(y, y') \leq F(y, y')$,

where $\rho * (y, x) = \rho(x, y)$ and $\circ$ is a sup-$*$ *-composition.*
A fuzzy function is called sound *or* perfect*, if it additionally fulfills*

FF.4 *for all $x \in X$, there exists $y \in Y$, such that $\rho(x, y) = 1$.*

A fuzzy function is called *surjective* if

FF.5 for all $y \in Y$, there exists $x \in X$, such that $\rho(x, y) = 1$.

In this contribution, we discuss

- necessary and sufficient conditions of solvability of the system **FF.1**-**FF.3**,
- uniqueness of a solution of this system,
- relationship between a fuzzy function existence and existence of its ordinary core function.

3 Fuzzy Partitions

Let X be a set, L a cl-monoid. According to [1], a $*$-semi-partition of a universe X is a family $\{A_\alpha, \alpha \in \mathscr{I}\}$ of normal fuzzy sets, such that for all $\alpha, \beta \in \mathscr{I}$,

$$\bigvee_{x \in X} (A_\alpha(x) * A_\beta(x)) \leq \bigwedge_{x \in X} (A_\alpha(x) \leftrightarrow A_\beta(x)),$$

is fulfilled.

We show that a family $\{A_\alpha, \alpha \in \mathscr{I}\}$ of normal fuzzy sets constitute a $*$-semi-partition of X if and only if the following system of fuzzy relation equations

$$A_\alpha \circ R = A_\alpha, \alpha \in \mathscr{I} \tag{1}$$

has fuzzy relation $R(x, y) = \bigvee_{\alpha \in \mathscr{I}} (A_i(x) * A_i(y))$ as a solution.

We discuss a characterization of a $*$-semi-partition of X in terms of eigen fuzzy sets of a solution R of (1).

4 Fuzzy Topology

We remind that a Lowen fuzzy topology [8] on a set X is a subset τ of L^X that contains all constant maps and is closed with respect to finite intersections and arbitrary unions. (X, τ) is called a fuzzy topological space, and $A \in \tau$ is called an open fuzzy set. A fuzzy topology τ on X is called a fuzzy Alexandrov topology if the intersection of an arbitrary many open fuzzy sets is still an open fuzzy set.

It is known [7] that a fuzzy Alexandrov topology on X is connected with a family of upper sets of a fuzzy preorder relation on X.

We show that any family $\{A_\alpha, \alpha \in \mathscr{I}\}$ of fuzzy sets of X can be embedded into a fuzzy Alexandrov topology τ on X, if τ is determined by the greatest solution of the following system of fuzzy relation equations

$$A_\alpha \circ R = A_\alpha, \alpha \in \mathscr{I}.$$

We characterize a base and open fuzzy sets of this topology.

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Inequalities on copulas and aggregation functions extending fuzzy measures. An applied point of view

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1 Notation and main purposes

For $m : \{0,1\}^n \to [0,1]$ a *capacity* on the set $[n] \equiv \{1, \ldots, n\}$ and for $C : [0,1]^n \to [0,1]$ an *n-dimensional copula*, we consider the *aggregation function* $A_{m,C} : [0,1]^n \to [0,1]$ defined by

$$A_{m,C}(\mathbf{v}) = \sum_{I \subset [n]} \widehat{m}(I) C(\mathbf{v}_I), \tag{1}$$

where $[n] \equiv \{1, \ldots, n\}$ and $\widehat{m}$ denotes the Mőbius transform of m. The following notation is furthermore used: for $\mathbf{v} \in \mathbb{R}^n$,

$$\mathbf{v}_I := (u_1, \ldots, u_n) \quad \text{where } u_j = \begin{cases} v_j & j \in I, \\ +\infty & \text{otherwise.} \end{cases} \tag{2}$$

Let $F(\mathbf{v})$ be a probability distribution function over $\mathbb{R}^n$, and $\mathbf{V} \equiv (V_1, \ldots, V_n)$ a random vector with joint distribution given by F. Then

$$F(\mathbf{v}_I) = F^{(I)}(v_{j_1}, \ldots, v_{j_{|I|}})$$

will denote the ($|I|$-dimensional marginal) distribution function of the vector $\mathbf{V}_I$, computed on the vector $(v_{j_1}, \ldots, v_{j_{|I|}})$.

Aggregation functions of the form (1) emerge, in a natural way, both in theoretical studies and in different applied fields. In particular, we know (see [3]) that $A_{m,C}(\mathbf{x})$ is an extension of m to $[0,1]^n$, since C has been assumed to be a copula.

Possible meanings of the pair (m, C), and corresponding meanings of $A_{m,C}$, will be briefly discussed in the first part of this talk. Attention will be in particular focussed on the settings of *multi-attribute "target-based" utility functions* and of *multi-state reliability systems,* along the lines that will be sketched below.

A main issue of such a discussion is in that the copulas C are not merely seen as analytical expressions. Actually they are the copulas describing stochastic dependence for random vectors having basic role in the problems at hand. This aspect permits one to interpret some theoretical aspects of aggregation functions under the viewpoint of stochastic dependence. Furthermore it suggests studying some inequalities concerning C and corresponding effects on $A_{m,C}(\mathbf{x})$, once analytical properties (such as *superadditivity*, *subadditivity*, *supermodularity*, ...) have been given for the capacity m. We will devote the second part of the talk to this subject.

In the next sections we provide brief descriptions of the settings, in reliability and utility respectively, that give rise to aggregation functions of the form in (1).

2 Multi-attribute target-based utility functions

Let $[n] \equiv \{1, \ldots, n\}$ be the set of *attributes* and let $\mathbf{y} \equiv (y_1, \ldots, y_n) \in \mathbb{R}^n$ be seen as the vector of possible values taken by the random coordinates of a *prospect* $\mathbf{Y} \equiv (Y_1, \ldots, Y_n)$, in a decision problem under uncertainty.

Let a non-decreasing function $U : \mathbb{R}^n \to \mathbb{R}$ and a deterministic vector $\mathbf{t} \equiv (t_1, \ldots, t_n)$ be fixed. We say that U is a *target-based utility function with deterministic target* $\mathbf{t}$, if $U(\mathbf{y})$ is only a function of those coordinates for which "the target is attained by the prospect". More precisely, we assume the existence of a set function $m : 2^{[n]} \to \mathbb{R}_+$ such that U coincides with the function $U_{m,\mathbf{t}}$ defined as follows

$$U_{m,\mathbf{t}}(\mathbf{y}) = m(Q(\mathbf{t}, \mathbf{y})), \tag{3}$$

where $Q(\mathbf{t}, \mathbf{x}) \subset [n]$ is the subset defined by

$$Q(\mathbf{t}, \mathbf{y}) := \{i \in N | t_i \leq y_i\}.$$

It can be natural to require that the function m, appearing in Eq. (3), is actually a capacity.

It is generally interesting to also consider cases where the target-vector $\mathbf{T}$ is random. Denote by F the joint distribution function of $\mathbf{T}$. We can then attain the following definition

A *multi-attribute target-based utility function*, with *capacity* m and with *target distribution* F, has the form

$$U_{m,F}(\mathbf{y}) = \sum_{I \subseteq N} m(I)\, \mathbb{P}\left(\bigcap_{i \in I}\{T_i \leq y_i\} \cap \bigcap_{i \notin I}\{T_i > y_i\}\right).$$

Notice that, by imposing the special choice

$$m(I) = 0 \text{ for all } I \subset N, m([n]) = 1, \tag{4}$$

one obtains $U_{m,F}(\mathbf{y}) = F(\mathbf{y})$. The position in (4) describes the target-based utility function of a Decision Maker who is satisfied only when all the n targets are achieved.

It is easy to check ([2]) that $U_{m,F}$ can also be written in the equivalent form

$$U_{m,F}(\mathbf{y}) = \sum_{I \subseteq N} \widehat{m}(I) F(\mathbf{y}_I).$$

This statement can be seen as an analogue of several results presented in different settings in the literature.

Now we denote by $G_i(\cdot)$ the (one-dimensional) marginal distributions of F for $i = 1, \ldots, n$ and we assume them to be continuous and strictly increasing. Furthermore we will denote by C the *connecting copula* of F: for $\mathbf{w} \equiv (w_1, \ldots, w_n) \in [0,1]^n$,

$$C(w_1, \ldots, w_n) := F(G_1^{-1}(w_1), \ldots, G_n^{-1}(w_n)). \tag{5}$$

As well-known the copula C describes the structure of stochastic dependence among the coordinates of $\mathbf{T}$.

Using a notation similar to (2), for $\mathbf{w} \in [0,1]^n$ we set

$$\mathbf{w}_I := (v_1, \ldots, v_n) \quad \text{where } \; v_j = \begin{cases} w_j & j \in I, \\ 1 & \text{otherwise.} \end{cases}$$

In this way for the connecting copula $C_F^{(I)}$ of $F^{(I)}$ we can write

$$C_F^{(I)}(w_{j_1}, \ldots, w_{j_{|I|}}) = C(\mathbf{w}_I).$$

We now consider the function $U_{m,F}(G_1^{-1}(w_1), \ldots, G_n^{-1}(w_n))$. The latter is a function of the variables $w_1, \ldots, w_n$, parametrized by the pair (m, F). In view of (5), we see that such a function depends on F only through the connecting copula C and it will be denoted by $\widehat{U}_{m,C}$.

$\widehat{U}_{m,C}$ can be seen as the *aggregation function* of the quantities $w_1, \ldots, w_n$ which, in their turn, can be given the meaning of target-based marginal utilities.

As a corollary of above statements we can obtain that $\widehat{U}_{m,C}$ has the form

$$\widehat{U}_{m,C}(\mathbf{w}) = \sum_{I \subseteq N} \widehat{m}(I) C(\mathbf{w}_I).$$

Thus $\widehat{U}_{m,C}$ is an aggregation function extending m and the copula C has the meaning of the copula describing the form of stochastic dependence among the coordinates of the random target $\mathbf{T}$.

3 Multi-state reliability systems

We consider a multi-state system S made with n binary components. When the *states* $r_1, \ldots, r_n$ of the components of S are known ($\mathbf{r} \equiv (r_1, \ldots, r_n) \in \{0,1\}^n$), the *state* of S is described by a number $y \in [0,1]$, which is a function of the vector $\mathbf{r}$. Namely we assume, for the system S, the existence of a *structure function* $\phi : \{0,1\}^n \to [0,1]$ such that $y = \phi(\mathbf{r})$.

Actually, the structure function ϕ can be seen as a *capacity* on $[n] \equiv \{1, \ldots, n\}$. It is natural, in fact, to set

$$\phi(0,0,\ldots,0) = 0; \phi(1,1,\ldots,1) = 1,$$

and to assume that ϕ is non-decreasing as a function of $(r_1, \ldots, r_n)$.

We consider however the case when the binary vector $\mathbf{R} \equiv (R_1, \ldots, R_n)$ of the state of the components is random and set

$$p_i = \mathbb{P}(R_i = 1).$$

Then we consider the random state of the system. Namely we consider the $[0,1]$-valued random variable $Y = \phi(\mathbf{R})$.

The probability distribution of Y, and then its expected value $\mathbb{E}(Y)$, obviously depend on the marginal "reliabilities" $p_1, \ldots, p_n$. For any fixed structure of stochastic dependence among the components (stochastic independence being just a very special and extreme case), we can consider the function $M : [0,1]^n \to [0,1]$ defined by

$$\mathbb{E}(Y) = M(p_1, \ldots, p_n).$$

The analysis of such an object is very clear when the binary random variables $R_1, \ldots, R_n$ are stochastically independent. Even in the case of dependence, however, it is easily seen that M is an aggregation function extending ϕ. $M(\mathbf{p})$ can be seen as an aggregation of $p_1, \ldots, p_n$ and, in any case, one has that it still takes the form

$$M(\mathbf{p}) = \sum_{I \subset [n]} \widehat{\phi}(I) \Gamma(\mathbf{p}_I),$$

for a suitable aggregation function Γ. We will analyze different aspects of the triple (ϕ, Γ, M). Several related studies have been carried out in different settings, and under different languages, in the field of reliability for the case when S is a binary system, namely when $\phi : \{0,1\}^n \to \{0,1\}$ (see in particular [1], [4] and references cited therein; see also [5]). Differences and similarities among such settings and among corresponding questions will be pointed out in the talk.

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Fuzzy implications satisfying the generalized hypothetical syllogism

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1 Introduction

In the present work, we take up the study of the generalized hypothetical syllogism (GHS), which is also known as chaining syllogism, transitive property etc. In MV logic, (GHS) is defined as follows: Let I be a fuzzy implication on $[0,1]$. Then, I is said to satisfy the generalized hypothetical syllogism (GHS), if for all $a, c \in [0,1]$,

$$I(a,c) = \sup_{b\in[0,1]} \Big(I(a,b) \wedge I(b,c) \Big). \tag{GHS}$$

The (GHS) has been employed in many fields, fuzzy control, decision making, expert systems and especially in approximate reasoning, see for example [3]. The following two facts have motivated us to investigate the fuzzy implications that do satisfy (GHS):

(i) Due to the applicational demand, it is always necessary to have the fuzzy implications that do satisfy (GHS).
(ii) The set of all fuzzy implications satisfying (GHS) forms the set of all idempotent elements of the semigroup $(\mathbb{I}, \overset{*}{\circ})$, where $\mathbb{I}$ is the set of all fuzzy implications and $*$, a t-norm (in our case $* = \min$), see Section 6.4 in [1]. In this way, one can glean algebraic aspects of the set $\mathbb{I}$.

Note that, in the literature only few fuzzy implications that satisfy (GHS) are known, see, for example, Chapter 11 in [3]. However, to the best of the author's knowledge, it is not known the kind of fuzzy implications that have this property so far. Thus in this work, we investigate the fuzzy implications that do satisfy (GHS).

2 Basic Results

Here in this section, we provide some basic necessary conditions required for $I \in \mathbb{I}$ to satisfy (GHS). Let Φ denote the set of all increasing bijections $\varphi\colon [0,1] \to [0,1]$.

Definition 1. *Let $I \in \mathbb{I}$ and $\varphi \in \Phi$. Define $I_\varphi \colon [0,1]^2 \to [0,1]$ by*

$$I_\varphi(x,y) = \varphi^{-1}(I(\varphi(x), \varphi(y))), \qquad x, y \in [0,1].$$

Clearly $I_\varphi \in \mathbb{I}$ for all $\varphi \in \Phi$ and is called the φ- conjugate *of I.*

Lemma 1. *Let $I \in \mathbb{I}$. Then the following statements are equivalent:*

(i) *I satisfies* (GHS).
(ii) *I_φ satisfies* (GHS) *for all $\varphi \in \Phi$.*

Lemma 2. *Let $I \in \mathbb{I}$ satisfy* (GHS). *Then, we have*

$$I(1,y) \wedge I(y,0) = 0, \qquad y \in [0,1]. \tag{1}$$

On the vertical sections $I(1,\cdot)$ and $I(\cdot,0)$:

From (1), it clear that the vertical sections $I(1,\cdot)$ and $I(\cdot,0)$ play an important role in the investigations of (GHS). Hence, we propose the following definition, which will be useful in the sequel.

Definition 2. *Let $I \in \mathbb{I}$. Then define the following real numbers:*

$$\epsilon_1 = \sup\{t \in [0,1] : I(1,t) = 0\},$$
$$\epsilon_2 = \inf\{t \in [0,1] : I(t,0) = 0\}.$$

Since $I \in \mathbb{I}$, from Definition 1.1.1 in [1], it follows that $I(1,0) = 0$ and hence, ϵ_1, ϵ_2 do exist, in general. The following result gives the relation between ϵ_1, ϵ_2 of an $I \in \mathbb{I}$ such that I satisfies (1).

Lemma 3. *Let $I \in \mathbb{I}$. Then the following statements are equivalent:*

(i) *I satisfies* (1).
(ii) $\epsilon_2 \le \epsilon_1$.

Corollary 1. *Let $I \in \mathbb{I}$ satisfy* (GHS). *Then $\epsilon_2 \le \epsilon_1$.*

However, note that the converse of Corollary 1 need not be true always. i.e., $\epsilon_2 \le \epsilon_1$ is only a necessary condition but not sufficient always. Due to the variety of fuzzy implications and the complexity of the functional equation (GHS), it is too much to expect the information of I only knowing the vertical sections $I(1,\cdot)$ and $I(\cdot,0)$. Hence in Section 3, we restrict our investigations of I satisfying (GHS) to some well established families of fuzzy implications, namely, (S,N)-, R- and Yager's f-, g- families of fuzzy implications.

3 Solutions from well known families of fuzzy implications

Here in this section, we investigate the solutions of (GHS) that do come from various families of fuzzy implications. For definitions, properties, characterizations, representations and further details of these families of fuzzy implications, please see [1].

Recall that, a fuzzy implication I is said to satisfy the left neutrality property (NP) if $I(1, y) = y$, for all $y \in [0, 1]$.

Lemma 4. *Let $I \in \mathbb{I}$ satisfy (NP). If I satisfies* (GHS) *then*

$$N_I(x) = N_{\mathbf{D1}}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0. \end{cases}$$

(S,N)-implications:

Theorem 1. *If I be an (S, N)-implication. Then the following statements are true:*

(i) *I satisfies* (GHS).

(ii) $I(x, y) = I_{\mathbf{D}}(x, y) = \begin{cases} 1, & \text{if } x = 0, \\ y, & \text{if } x > 0. \end{cases}$

R-implications:

Theorem 2. *Let I be an R-implication obtained from a left continuous t-norm. Then the following statements are true:*

(i) *I satisfies* (GHS).

(ii) $I(x, y) = I_{\mathbf{GD}}(x, y) = \begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{if } x > y \end{cases}.$

Proof. (i) $\Longrightarrow$ (ii). Let I satisfy (GHS). Then from Theorems 2.5.7 and 2.5.14 in [1], it follows that $T = T_I$ is a t-norm, where T_I is defined as follows:

$$T_I(x, y) = \min\{z | I(x, z) \geq y\}, \qquad x, y \in [0, 1].$$

Claim: $T_I(x, x) = x$, for all $x \in [0, 1]$.
Now, let $x \in [0, 1]$. Then

$$\begin{aligned} T_I(x, x) &= \min\{z \in [0, 1] | I(x, z) \geq x\}, \\ &= \min\{z \geq x | I(x, z) \geq x\} \wedge \min\{z \leq x | I(x, z) \geq x\} \end{aligned}$$

$\underline{\min\{z \geq x | I(x, z) \geq x\}}$**:**

Since I is an R-implication obtained from a left-continuous t-norm, I satisfies the ordering property (OP), viz., $I(x, y) = 1 \Longleftrightarrow x \leq y$. Hence $I(x, z) = 1$ for all $z \geq x$. Thus we have, $\min\{z \geq x | I(x, z) \geq x\} = \min\{z \leq x | 1 \geq x\} = x$.

$\underline{\min\{z \leq x | I(x,z) \geq x\}}$**:**

$$\min\{z \leq x | I(x,z) \geq x\} = \min\{z \leq x | \sup_{y\in[0,1]} \Big(I(x,y) \wedge I(y,z)\Big) \geq x\}$$
$$= \min\Big\{z \leq x | \sup_{y\leq z\leq x} \Big(I(x,y) \wedge I(y,z)\Big) \vee$$
$$\sup_{z\leq y\leq x} \Big(I(x,y) \wedge I(y,z)\Big) \vee \sup_{z\leq x\leq y} \Big(I(x,y) \wedge I(y,z)\Big) \geq x\Big\}$$
$$= \min\Big\{z \leq x | \Big(I(x,z) \vee \sup_{z\leq y\leq x} \Big(I(x,y) \wedge I(y,z)\Big) \vee I(x,z)\Big) \geq x\Big\}$$
$$= \min\{z \leq x | I(x,z) \geq x\} = x.$$

$$\text{Thus, } T_I(x,x) = \min\{z \geq x | I(x,z) \geq x\} \wedge \min\{z \leq x | I(x,z) \geq x\}$$
$$= \min(x,x) = x, \text{ for all } x \in [0,1]$$

and hence, from Proposition 1.9 in [2], it follows that $T = \min$, and $I = I_{\mathbf{GD}}$.
(ii) $\Longrightarrow$ (i). It follows easily.

f-implications:

Lemma 5. *Let I be an f-implication with f-generator h. If I satisfies* (GHS) *then* $h(0) = \infty$.

Theorem 3. *If I is an f-implication then I does not satisfy* (GHS).

Proof. Let I be an f-implication with f-generator h and I satisfies (GHS). Then from Lemma 5, it follows that $h(0) = \infty$. This implies, from Theorem 3.1.7 in [1], that I is continuous except at $(0,0)$. Now, let $0 < x, y, z < 1$. Then,

$$I(x, I(y,z)) = I(x, \sup_{w\in[0,1]} \{I(y,w) \wedge I(w,z)\}), \qquad \lceil\text{from Definition of (GHS)}$$
$$= \sup_{w\in[0,1]} \{I(x, I(y,w) \wedge I(w,z))\}, \qquad \lceil\text{from continuity of } I$$
$$= \sup_{w\in[0,1]} \{I(x, I(y,w)) \wedge I(x, I(w,z))\}, \qquad \lceil\text{from Prop. 7.2.15 in [1]}$$
$$= \sup_{w\in[0,1]} \{I(xy, w) \wedge I(w, I(x,z))\}, \qquad \lceil\text{from (LI) and (EP) of } I$$
$$= I(xy, I(x,z)). \qquad \lceil\text{from Definition of (GHS)}$$

Once again by using (LI) of I w.r.t $T_{\mathbf{P}}$, we get $I(x, I(y,z)) = I(xy, z)$ and hence, we have also $I(xy, I(x,z)) = I(xy, z)$. Then from Lemma 6.25 in [4], it follows that $I(x,z) = z$, which further implies that either $x = 1$ or $z = 1$, a contradiction to the fact that $0 < x, y, z < 1$. This completes the proof.

g-implications:

Theorem 4. *If I is a g-implication then I does not satisfy* (GHS).

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Weakly additive measures of conditionals for events from an MV-algebra, and mean value functions

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Measure-free conditioning works in two steps. In a first step, *conditional events* "a given b" are defined as well-defined elements in terms of the events $a, b \in \mathbb{L}$, a suitable lattice. In a second step, their uncertainty is expressed by suitable *measures of conditional events* (a given b) as extensions of a given measure of the (unconditional) events $a,\ b$.

In the present talk, we start with events from any MV-algebra $\mathbb{L}$ and an additive measure m on $\mathbb{L}$. Then the conditional events as the lattice-intervals

$$(a \parallel b) = [a \wedge b,\ b \to a],\ a, b \in \mathbb{L},$$

are in the Girard algebra $\tilde{\mathbb{L}}$, see Theorem 2.3 from [2] and, additionally, Remarks 6.1 and 6.2 from [4]. The question how to extend the additivity of m has been solved only in the following particular cases. In [1] we have proved that for any Boolean algebra $\mathbb{L}$ there exists a unique extension $\tilde{m}$ on the MV-algebra $\tilde{\mathbb{L}}$ which is additive. For non-Boolean MV-algebras $\mathbb{L}$ it seems that an adequate type of extension of m is a *weakly additive measure* $\tilde{m}$ on $\tilde{\mathbb{L}}$, i.e. where $\tilde{m}$ is additive on all MV-subalgebras of $\tilde{\mathbb{L}}$, see Remark 6.8 from [4]. In [3] we gave a complete characterization for any finite MV-chain $\mathbb{L}$, in [4] we extended this result to any finite MV-algebra $\mathbb{L}$.

The aim of the present talk is to give first results for arbitrary MV-algebras $\mathbb{L}$. On the one hand, we indicate how to find all MV-algebras from $\tilde{\mathbb{L}}$ and, with those, the weakly additive extensions $\tilde{m}$. On the other hand, we show that these $\tilde{m}$ can be represented by a common mean value function M, i.e. as

$$\tilde{m}(a \parallel b) = M(\ m(a \wedge b),\ m(b \to a)\ ),$$

only if $\mathbb{L}$ is either a Boolean algebra or an MV-chain.

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Dependently-Typed Fuzzy Relations

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L-fuzzy relations are relations in which every pair of elements is related up to a certain degree indicated by a membership value from the complete Heyting algebra L. Formally, an L-fuzzy relation R (or L-relation for short) between a set A and a set B is a function $R : A \times B \to L$. They generalize fuzzy relations by replacing the unit interval by an arbitrary complete Heyting algebra and allow, therefore, for incomparable degrees of membership. A suitable abstract theory covering these relations is given by arrow or Goguen categories. These theories have been studied intensively [4–8, 10] including investigations into higher-order fuzziness [11, 12]. In addition to the theoretical studies, these categories have been used to model and specify type-1 and type-2 $\mathcal{L}$-fuzzy controllers [9, 13] as well as L-fuzzy databases [1–3, 15]. One reason for the successful application of these structures in practical examples is that arrow categories allow constructions such as relational sums and products that are essential to model complex input/output types of controllers in particular or data in general. However, certain other constructions such as quotients by partial equivalence relations and relational powers that are useful in more sophisticated applications are not available.

A further generalization of fuzziness is given by moving from the so-called fixed-base to the variable-base case. In the variable-base case relations between different objects may use membership values from different lattices. It was shown in [14] that such an approach requires a certain collection of lattices as basis. Further investigation led to weak arrow categories as a suitable abstract notion for the variable-base case. This approach and its abstract theory is interesting for multiple reasons. First of all, it provides further inside into the relationship between the different approaches to fuzziness. Second, it provides the required foundation for an internal version of higher-order fuzziness, i.e., a theory where type-1 and type-2 fuzzy relations come from the same category rather than from two different categories. Last but not least, it can serve as the underlying theory to model fuzzy controllers that use different lattices for membership within different components. Unfortunately, we will show in this presentation that weak arrow categories may provide even fewer constructions than arrow categories. In particular, they normally do not provide relational sums which are essential in practical applications.

In this presentation we are interested in so-called dependently typed fuzzy relations. In such a relation each pair may use a different lattice of membership values. After defining the concrete Dedekind category with cut operations of dependently typed fuzzy relations over a basis of complete Heyting algebras we will show that this category has all desired constructions.

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