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Abstracts

Set Functions in Games and Decision

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Abstracts

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LINZ 2019

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SET FUNCTIONS IN GAMES AND DECISION

ABSTRACTS

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Since their inception in 1979 the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2019 will be the 38th seminar carrying on this tradition and is devoted to the theme "Set Functions in Games and Decisions". The goal of the seminar is to present and to discuss recent advances in set functions and non-classical measure theory, and their applications in operations research, coalitional game theory and decision theory.

A large amount of interesting contributions were submitted for possible presentation at LINZ 2019 and subsequently reviewed by PC members. This volume contains the abstracts of the accepted contributions. These regular contributions are complemented by four invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

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Contents

Harsanyi power solutions for cooperative games on voting structures

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Abstract. This paper deals with Harsanyi power solutions for cooperative games in which partial cooperation is based on specific union stable systems given by the winning coalitions derived from a voting game. This framework allows for analyzing new and real situations in which there exists a feedback between the economic influence of each coalition of agents and its political power. We provide an axiomatic characterization of the Harsanyi power solutions on the subclass of union stable systems arisen from the winning coalitions from a voting game when the influence is determined by a power index. In particular, we establish comparable axiomatizations, in this context, when considering the Shapley-Shubik power index, the Banzhaf index and the equal division value which reduces to the Myerson value on union stable systems.

1 Introduction

For a firm, it is one thing to have the ability to produce, but it is useless unless the firm is allowed to produce. In other words, the economic power of a firm emerges only if it is accompanied by political or legal power. As an example, Google has a huge worldwide economic power, but cannot exert it in China where its government currently prevents Google's search service to operate without censure. As a more counterexample, let us consider the social cost problems first suggested by [7] in which are involved a set of victims and a set of polluters. The activity of the latter create damage that affect the victims. In order to iron those conflicts and to solve the problem of social cost, a negotiation will take place within a coalition of polluters and victims with the objective to sign a binding agreement about how much activity the polluter will be able to undertake. Now, the permission granted to each coalition which wants to sign such binding agreements about the level of activity of the polluter is interpreted as the ability for the coalition to control the decision of a committee that assigns these rights (see [13] for details).

In this article, we provide a model based on cooperative game theory in order to apprehend such situations. We enrich the classical model of a (economic) cooperative game — a set of agents and a characteristic function specifying the economic power of each coalition of agents — with a voting game on the same set of agents. This voting game is modeled by a nonempty set of winning coalitions with the usual monotonicity property: each superset of a winning coalition is also a winning coalition. In this framework, an allocation rule specifies a utility for each agent for participating in each pair of economic and voting games. The latter two structures are likely to influence each other in the design of an allocation rule. On the one hand, as suggested above, the sharing of economic resources can depend on the political power of coalitions of agents. On the other hand, the measure of political power might be impacted by the economic power of coalitions, for example because of their ability to incur lobbying expenses.

Here, we explore the first of the two types of influence. For classical cooperative games a class of well-known allocation rules is the class of Harsanyi solutions introduced by [22] and then studied by [9], among others. Each Harsanyi solution distributes the Harsanyi dividend of each coalition among its members in proportion to exogenously given weights. For voting games, two well-known power indices are provided by [19] and [5]. The Shapley-Shubik index measures the likelihood that an agent is decisive if the agents are called upon to vote one by one in favor of a decision. The Banzhaf index measures the proportion of coalitions for which a given agent is pivotal (i.e. a winning coalition that is not winning anymore without this agent). We combine both types of allocation rules in order to study a specific class of Harsanyi (power) solutions in which the Harsanyi dividend of each winning coalition is shared among its members in proportion to their relative political power as measured by an arbitrary power index σ in the voting subgame induced by the coalition. The idea to combine economic and political power also appears in [16] where the weights on an asymmetric Nash bargaining solution are specified by the Shapley-Shubik index of a voting game.

We characterize the Harsanyi power solution induced by the power index σ. In addition to the classical axioms of Efficiency and Additivity, we introduce three other axioms. The first one is a variant of the Null agent out axiom [8], which requires that an allocation rule is not sensitive to the removal of a null agent in the economic game. The second one requires that if all winning coalitions enjoy a null worth in the economic game, then all agents should be treated equally. The third one is inspired by the axioms of σ-point unanimity [3] and Communication ability [6]. If only the grand coalition has a non null worth in the economic game, the axiom imposes that the agents are paid in proportion to the power index σ.

Our contribution possesses some similarities with the literature on games played on combinatorial structures. The closest article is perhaps [3], where the exogenous structure is a union stable system, *i.e.* a set of feasible coalitions such that the union of two intersecting coalitions is also feasible. The authors provide a similar characterization of a class of Harsanyi power solutions. There are, however, two major differences with our work. Firstly, while the set of winning coalitions in a voting game is a union stable system, the converse is not true. As a consequence, some of the axioms invoked by [3] cannot be reused in our case as, for instance, the Inessential support property. Secondly, any set of connected coalitions on a graph is a union stable system, which implies that games played on union stable systems can be seen as a generalization of communication graph games introduced in [18]. To the contrary, the voting structure that we consider cannot be assimilated to a graph structure. In fact, the set of winning coalitions of a voting game does not always correspond to the set of connected coalitions of a graph on the player set as pointed by [20]. This allows us to replace the classical power indices on graph (such as the degree of each node) by voting power indices.

Another advantage of our contribution is that our characterization is still valid on the larger class of games played on union stable systems subject to a minor adaptation of the axiom of Efficiency. We also single out the relevant Harsanyi power solutions obtained by using, in the voting game, the Shapley-Shubik power index, the Banzhaf index and the Equal division index, respectively. The latter one coincides with the Myerson value for games played on union stable systems introduced by [2]. Our characterization provides a comparable axiomatization of these three Harsanyi power solutions.

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References

1. Algaba, E., Bilbao, J. M., Borm, P., López, J., 2000. The position value for union stable systems. Mathematical Methods of Operations Research 52, 221–236.
2. Algaba, E., Bilbao, J. M., Borm, P., López, J. J., 2001. The Myerson value for union stable structures. Mathematical Methods of Operations Research 54, 359–371.
3. Algaba, E., Bilbao, J. M., van den Brink, R., 2015. Harsanyi power solutions for games on union stable systems. Annals of Operations Research 225, 27–44.
4. Algaba, E., van den Brink, R., Dietz, C., 2018. Network structures with hierarchy and communication. Journal of Optimization Theory and Applications 179, 265–282.
5. Banzhaf, J. F., 1965. Weighted voting doesn’t work: a mathematical analysis. Rutgers Law Review 19, 317–343.
6. Borm, P., Owen, G., Tijs, S., 1992. On the position value for communication situations. SIAM Journal on Discrete Mathematics 5, 305–320.
7. Coase, R., 1960. The problem of social cost. Journal of Law and Economics 3, 1–44.
8. Derks, J., Haller, H. H., 1999. Null players out? Linear values for games with variable supports. International Game Theory Review 1, 301–314.
9. Derks, J., Haller, H. H., Peters, H., 2000. The selectope for cooperative TU-games. International Journal of Game Theory 29, 23–38.
10. Dubey, P., 1975. On the uniqueness of the Shapley value. International Journal of Game Theory 4, 131–140.
11. Dubey, P., Shapley, L. S., 1979. Mathematical properties of the Banzhaf power index. Mathematics of Operations Research 4, 99–131.
12. Feltkamp, V., 1995. Alternative axiomatic characterizations of the Shapley and Banzhaf values. International Journal of Game Theory 24, 179–186.
13. Gonzalez, S., Marciano, A., Solal, P., 2018. The social cost problem, rights and the (non)empty core, forthcoming in Journal of Public Economic Theory.
14. Harsanyi, J. C., 1959. A bargaining model for cooperative n-person games. In: Tucker, A. W., Luce, R. D. (Eds.), Contributions to the Theory of Games IV. Princeton University Press, pp. 325–355.
15. Laruelle, A., Valenciano, F., 2001. Shapley-Shubik and Banzhaf indices revisited. Mathematics of Operations Research 26, 89–104.

16. Laruelle, A., Valenciano, F., 2007. Bargaining in committees as an extension of Nash's bargaining theory. Journal of Economic Theory 132, 291–305.
17. Meessen, R., 1988. Communication games (in Dutch). Master's thesis, Department of Mathematics, University of Nijmegen, The Netherlands.
18. Myerson, R. B., 1977. Graphs and cooperation in games. Mathematics of Operations Research 2, 225–229.
19. Shapley, L. S., Shubik, M., 1954. A method for evaluating the distribution of power in a committee system. The American Political Science Review 48, 787–792.
20. van den Brink, R., 2012. On hierarchies and communication. Social Choice and Welfare 39, 721–735.
21. van den Brink, R., van der Laan, G., Pruzhansky, V., 2011. Harsanyi power solutions for graph-restricted games. International Journal of Game Theory 40, 87–110.
22. Vasil'ev, V. A., 1982. On a class of operators in a space of regular set functions (in Russian). Optimizacija 28, 102–111.
23. Vasil'ev, V. A., 2003. Extreme points of the Weber polytope (in Russian). Discretnyi Analiz i Issledonaviye Operatsyi 10, 17–55.

What to learn and what not to learn – a mathematical view on ethical learning

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Many learning algorithms are struggling with large data sets, and miss information present in the data simply for computational reasons. A larger and mostly hidden problem is that many algorithms learn (unintentionally and unnoticed) triggering patterns that are not supported by the data. Using such classifiers, we may jump to conclusions that are unjustifiable based on our existing data sets. Both errors imply potential problems: missing important triggers and/or using unsupported ones. This brings up both ethical and legal questions. In this talk we demonstrate these issues with an example. We propose a mathematically sound notion of a "justifiable" classifier AND learning algorithm. Furthermore, on the positive side, we show some results about the existence of learning algorithms that always produce a "justifiable" classifier.

Winning coalitions in plurality voting democracies

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The crucial and immediate question after parliamentary elections have taken place concerns the winner(s) of the election. Sometimes, even more than one party can call itself a winner of the election. In the present paper we study if it is possible to assign weights to political parties that indicates who is the winner of an election.

We define a cooperative game that assigns the value one to coalitions of parties if it is a winning coalition, and zero to a coalition of parties that is not winning (i.e., losing). Typically, whether a coalition is winning depends on the way how players outside the coalition are organized. A coalition that is negotiating to form a government might be winning if the other parties are not organized. For instance, if some other parties form a coalition, it might be attractive for one of the negotiating parties to stop negotiations and start negotiating the formation of a government with the 'new coalition'.

We model situations with such externalities in two different ways. The first way of modeling uses standard simple games (defined on a fixed and finite player set N) in which whether a coalition is winning does not depend on the coalition formation of players outside the coalition. Nevertheless, when being paired with coalition structures, these games provide some information on the extent to which a winning coalition is 'stronger' than the rest of the players. We define rudimentary plurality games (rp-games) by requiring at least one coalition in each coalition structure to be winning in the standard simple game. This extension of the model of a standard simple game allows us to examine if such games single-support or block-support plurality voting democracies in the sense of assignment of weights to players such that the win of a coalition $S \subseteq N$ in the simple game implies the sum of its players' weights to exceed the weight of every single player in $N \setminus S$ or the sum of players' weights in $N \setminus S$, respectively. We show that every rp-game is single-supportive and every decisive rp-game is block-supportive.

The second way in which externalities can be modeled is by a direct consideration of simple games in partition function form. In other words, we assign a worth of either one or zero to each pair of a coalition and a partition containing that coalition. We call such games plurality games if there is at least one coalition which wins in a partition. In this case, winning does not necessarily mean that the coalition has a majority and can pass a bill, but simply that it is the strongest in a given coalitional configuration as represented by a partition. So, a party that does not have the majority, but is considered as the winner of the election before any negotiation to form a government has taken place, has worth one in the discrete partition (i.e., the partition into singletons). It simply means that the party, or coalition of parties, can do something (for instance, take the initiative to form a government), but this does not necessarily mean that it can pass a bill.

Within the framework of plurality games, we study the notion of a game precisely supporting a plurality voting democracy. Is it possible to assign weights to the players such that a coalition being winning in a partition implies that the sum of its players' weights is maximal over all coalitions in the partition? We show that this is always possible for small decisive games, that is, for games with at most four players and where for each partition there is exactly one winning coalition in it. Games with more than five players turn out also to be precisely supportive when it is always a largest coalition which wins in a partition (majority games). Allowing in a minimal way for players in a plurality game to be non-symmetric results in the definition of almost symmetric games. We show that decisive and almost symmetric plurality games are also precisely supportive.

Measure systems and measure spaces

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A “flat” system $(X, A, \models)$ is well known. It posits a system comprising a set X of objects, a family A of “predicates”, and a “satisfaction” relation $\models$ which documents which objects satisfy which predicates. A system is flat if there is no structure on A or there are no conditions on $\models$ respecting the structure(s) on A. Morphisms between systems track the changes both of objects and predicates consistent with a distinctive adjointness condition involving the satisfaction relations. A “first” example: any set of objects X, together with its powerset $\wp(X)$ and the associated membership relation $\in$, comprises a flat system $(X, \wp(X), \in)$. Given a system $(X, A, \models)$, a predicate $a \in A$ determines the set of all objects which satisfy a, called its “extent” and denoted $ext(a)$. It is the case that the set X, together with family $Ext(A)$ of all the extents, forms an extent space which is also “flat”. The category **FlatSys** of all flat systems, with morphisms as described above, is an essentially algebraic category, and its associated category **FlatSp**, of all flat spaces and appropriate morphisms, is a topological construct.

It is natural in the setting of computer science to have and use non-flat systems $(X, A, \models)$ in which certain structural properties are assumed for the predicates and corresponding structural properties are also assumed of the satisfaction relation. Two examples of such types of systems are the following:

1. The family A of predicates is assumed a frame—a complete lattice in which the first infinite distributive law holds—and the satisfaction relation $\models$ possesses arbitrary join and finite meet “interchange” laws. These types of systems $(X, A, \models)$ are called “topological” systems, and the associated extent spaces are topological spaces. As in the flat case, the category **TopSys** is an essentially algebraic category, and, of course, its associated category **Top** is a topological construct.
2. The family $\mathfrak{M}$ of predicates is assumed a σ-algebra and the satisfaction relation $\models$ possesses countable join and negation “interchange” laws. These systems $(X, \mathfrak{M}, \models)$ are called “measure” systems, and the associated extent spaces are measure spaces. Again, as in the flat case, the category **MeasSys** is an essentially algebraic category, and its associated category **Meas** is a topological construct.

Topological systems and various related notions of many-valued topological systems have had deep and extensive developments over the past 10 years; for example, topological systems are fundamentally related to “variable-basis” many-valued topological spaces. However, there has been no development of measure systems. This talk presents research which attempts to fill that gap and explores related topics such as probabilistic systems.

Characterizations of idempotent n-ary uninorms

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Abstract. In this paper we provide a characterization of the class of idempotent n-ary uninorms on a given chain. When the chain is finite, we also provide an axiomatic characterization of the latter class by means of four conditions only: associativity, quasitriviality, symmetry, and nondecreasing monotonicity. In particular, we show that associativity can be replaced with bisymmetry in this axiomatization.

1 Introduction

Let X be a nonempty set and let $n \geq 2$ be an integer. Binary aggregation functions have been extensively investigated since the last decades due to their usefulness in merging data (see, e.g. [5] and the references therein). Among these functions, uninorms play an important role in fuzzy logic. Meanwhile, the study of n-ary uninorms also raised some interest (see, e.g. [6]).

In this paper, which is a shorter version of [4], we investigate the class of idempotent n-ary uninorms $F\colon X^n \to X$ on a chain $(X, \leq)$ (Definition 3). We provide in Section 2 a characterization of these operations and show that they are nothing other than idempotent binary uninorms (Proposition 1). We also provide a description of these operations as well as an alternative axiomatization when the chain is finite (Theorem 1). In Section 3 we investigate some subclasses of bisymmetric n-ary operations and derive an equivalence involving associativity and bisymmetry. More precisely, we show that if an n-ary operation is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric (Proposition 3). This observation enables us to replace associativity with bisymmetry in our axiomatization (Corollary 1).

We use the following notation throughout. A chain $(X, \leq)$ will simply be denoted by X if no confusion may arise. For any chain X and any $x, y \in X$ we use the symbols $x \wedge y$ and $x \vee y$ to represent $\min\{x, y\}$ and $\max\{x, y\}$, respectively. For any integer $k \geq 0$, we set $[k] = \{1, \ldots, k\}$. Finally, for any integer $k \geq 0$ and any $x \in X$, we set $k \cdot x = x, \ldots, x$ (k times). For instance, we have $F(3 \cdot x, 2 \cdot y, 0 \cdot z) = F(x, x, x, y, y)$.

Definition 1. *An operation $F\colon X^n \to X$ is said to be*

- idempotent *if $F(n \cdot x) = x$ for all $x \in X$;*

– quasitrivial *(or* conservative*) if* $F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$ *for all* $x_1,\ldots,x_n \in X$*;*
– symmetric *if* $F(x_1,\ldots,x_n)$ *is invariant under any permutation of* $x_1,\ldots,x_n$*;*
– associative *if*

$$F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) \\ = F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1})$$

for all $x_1,\ldots,x_{2n-1} \in X$ *and all* $i \in [n-1]$*;*
– bisymmetric *if*

$$F(F(\mathbf{r}_1),\ldots,F(\mathbf{r}_n)) \;=\; F(F(\mathbf{c}_1),\ldots,F(\mathbf{c}_n))$$

for all $n \times n$ *matrices* $[\mathbf{c}_1 \;\cdots\; \mathbf{c}_n] = [\mathbf{r}_1 \;\cdots\; \mathbf{r}_n]^T \in X^{n\times n}$.

If $(X,\leq)$ *is a chain, then* $F\colon X^n \to X$ *is said to be*

– nondecreasing *(for* $\leq$*) if* $F(x_1,\ldots,x_n) \leq F(x'_1,\ldots,x'_n)$ *whenever* $x_i \leq x'_i$ *for all* $i \in [n]$.

Definition 2. *Let* $F\colon X^n \to X$ *be an operation. An element* $e \in X$ *is said to be a* neutral element *of* F *if*

$$F((i-1)\cdot e, x, (n-i)\cdot e) \;=\; x$$

for all $x \in X$ *and all* $i \in [n]$.

2 First characterization

In this section we provide a characterization of the n-ary operations on the chain X that are associative, quasitrivial, symmetric, and nondecreasing. We will also show that in the case where the chain is finite these operations are nothing other than n-ary idempotent uninorms.

Recall that a *uninorm* on a chain X is a binary operation $U\colon X^2 \to X$ that is associative, symmetric, nondecreasing, and has a neutral element (see [3, 7]). It is not difficult to see that any idempotent uninorm is quasitrivial.

The concept of uninorm can be easily extended to n-ary operations as follows.

Definition 3 (see [6]).
An n-ary uninorm *is an operation* $F\colon X^n \to X$ *that is associative, symmetric, nondecreasing, and has a neutral element.*

The next proposition provides a characterization of idempotent n-ary uninorms.

Proposition 1. *Let* X *be a chain and let* $F\colon X^n \to X$ *be an operation. Then* F *is an idempotent* n*-ary uninorm if and only if there exists an idempotent uninorm* $U\colon X^2 \to X$ *such that*

$$F(x_1,\ldots,x_n) \;=\; U(\textstyle\bigwedge_{i=1}^n x_i\,, \bigvee_{i=1}^n x_i), \qquad x_1,\ldots,x_n \in X.$$

In this case, the uninorm U *is uniquely defined as* $U(x,y) = F((n-1)\cdot x, y)$.

We now introduce the concept of single-peaked linear ordering which first appeared for finite chains in social choice theory (see Black [1, 2]).

Definition 4. *Let $(X, \leq)$ and $(X, \preceq)$ be chains. We say that the linear ordering $\preceq$ is* single-peaked for $\leq$ *if for any $a, b, c \in X$ such that $a < b < c$ we have $b \prec a$ or $b \prec c$.*

The following theorem provides a characterization of the class of associative, quasitrivial, symmetric, and nondecreasing operations $F\colon X^n \to X$. We observe that it generalizes Proposition 1 since the latter class does not require the existence of a neutral element. In particular, it provides a new axiomatization as well as a description of idempotent n-ary uninorms when the chain X is finite.

Theorem 1. *Let $F\colon X^n \to X$ be an operation. The following assertions are equivalent.*

(i) F is associative, quasitrivial, symmetric, and nondecreasing.
(ii) There exists a quasitrivial, symmetric, and nondecreasing operation $G\colon X^2 \to X$ such that

$$F(x_1, \ldots, x_n) \; = \; G(\textstyle\bigwedge_{i=1}^n x_i \, , \bigvee_{i=1}^n x_i), \qquad x_1, \ldots, x_n \in X.$$

(iii) There exists a linear ordering $\preceq$ on X that is single-peaked for $\leq$ and such that F is the maximum operation on $(X, \preceq)$, i.e.,

$$F(x_1, \ldots, x_n) \; = \; x_1 \curlyvee \cdots \curlyvee x_n \, , \qquad x_1, \ldots, x_n \in X.$$

If $X = [k]$ for some integer $k \geq 1$, then any of the assertions (i)–(iii) above is equivalent to the following one.

(iv) F is an idempotent n-ary uninorm.

3 Second characterization

In this section we investigate bisymmetric n-ary operations and derive an equivalence involving associativity and bisymmetry. More precisely, if an n-ary operation is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric. In particular this latter observation enables us to replace associativity with bisymmetry in Theorem 1.

Definition 5. *We say that a function $F\colon X^n \to X$ is* ultrabisymmetric *if*

$$F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) \; = \; F(F(\mathbf{r}'_1), \ldots, F(\mathbf{r}'_n))$$

for all $n \times n$ matrices $[\mathbf{r}_1 \; \cdots \; \mathbf{r}_n]^T, [\mathbf{r}'_1 \; \cdots \; \mathbf{r}'_n]^T \in X^{n \times n}$, where $[\mathbf{r}'_1 \; \cdots \; \mathbf{r}'_n]^T$ is obtained from $[\mathbf{r}_1 \; \cdots \; \mathbf{r}_n]^T$ by exchanging two entries.

Proposition 2. *Let $F\colon X^n \to X$ be an operation. If F is ultrabisymmetric, then it is bisymmetric. The converse holds whenever F is symmetric.*

Proposition 3. *Let $F\colon X^n \to X$ be an operation. Then the following assertions hold.*

(a) If F is quasitrivial and ultrabisymmetric, then it is associative and symmetric.
(b) If F is associative and symmetric, then it is ultrabisymmetric.
(c) If F is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric.

From Proposition 3(c) we immediately derive the following corollary, which is an important but surprising result.

Corollary 1. *In Theorem 1 we can replace associativity with bisymmetry.*

References

1. D. Black. On the rationale of group decision-making. *J Polit Economy*, 56(1):23–34, 1948
2. D. Black. *The theory of committees and elections*. Kluwer Academic Publishers, Dordrecht, 1987.
3. B. De Baets. Idempotent uninorms. *Eur. J. Oper. Res.*, 118:631–642, 1999.
4. J. Devillet, G. Kiss, and J.-L. Marichal. Characterizations of quasitrivial symmetric nondecreasing associative operations. *Semigroup Forum*. In press. arXiv: 1705.00719.
5. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation functions*. Encyclopedia of Mathematics and its Applications 127. Cambridge University Press, Cambridge, UK, 2009.
6. G. Kiss and G. Somlai. A characterization of n-associative, monotone, idempotent functions on an interval that have neutral elements. *Semigroup Forum*, 96(3):438–451, 2018.
7. R. R. Yager and A. Rybalov. Uninorm aggregation operators. *Fuzzy Sets and Systems*, 80:111–120, 1996.

The ν-additive measure as an alternative to the λ-additive measure

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Abstract. In this study, the λ-additive measure (Sugeno λ-measure) is revisited and put into a new light by introducing the so-called ν-additive measure, which may be viewed as an alternatively parameterized λ-additive measure. The motivation for introducing the ν-additive measure lies in the fact that its parameter $\nu \in (0,1)$ has an important semantic meaning as it is the fix point of the complement operation. Here, by utilizing the ν-additive measure, some well-known results concerning the λ-additive measure are put into a new light and rephrased in more advantageous forms. Next, novel proofs are presented in which the so-called basic probability assignment (mass function) is not utilized to demonstrate how the ν-additive (λ-additive) measure is connected with the belief-, probability- and plausibility measures. Here, it is also demonstrated how a ν-additive measure (or a λ-additive measure) can be transformed to a probability measure and vice versa. Moreover, it is pointed out how the ν-additive measures are connected with rough sets, multi-attribute utility functions and with certain operators of fuzzy logic.

1 The λ-additive measure and the ν-additive measure

Relaxing the additivity property of the probability measure, the λ-additive measures were proposed by Sugeno in 1974 [5].

Definition 1. *The function $Q_\lambda : \mathcal{P}(X) \to [0,1]$ is a λ-additive measure (Sugeno λ-measure) on the finite set X, iff Q_λ satisfies the following requirements:*

(1) $Q_\lambda(X) = 1$
(2) for any $A, B \in \mathcal{P}(X)$ and $A \cap B = \emptyset$,

$$Q_\lambda(A \cup B) = Q_\lambda(A) + Q_\lambda(B) + \lambda Q_\lambda(A) Q_\lambda(B),$$

where $\lambda \in (-1, \infty)$.

1.1 The λ-additive complement and the Dombi form of negation

The following proposition concerning the λ-additive measure of a complement set can be proven.

Proposition 1. *If X is a finite set and Q_λ is a λ-additive measure on X, then for any $A \in \mathcal{P}(X)$ the Q_λ measure of the complement set $\overline{A} = X \setminus A$ is*

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)}. \tag{1}$$

Now, let us assume that $0 \leq Q(A) < 1$. Then, Eq. (1) can be written as

$$Q_\lambda(\overline{A}) = \frac{1 - Q_\lambda(A)}{1 + \lambda Q_\lambda(A)} = \frac{1}{1 + (1+\lambda)\frac{Q_\lambda(A)}{1-Q_\lambda(A)}}. \tag{2}$$

In continuous-valued logic, the Dombi form of negation with the neutral value $\nu \in (0, 1)$ is given by the operator $n_\nu : [0, 1] \to [0, 1]$ as follows:

$$n_\nu(x) = \begin{cases} \frac{1}{1+\left(\frac{1-\nu}{\nu}\right)^2 \frac{x}{1-x}} & \text{if } x \in [0,1) \\ 0 & \text{if } x = 1, \end{cases} \tag{3}$$

where $x \in [0, 1]$ is a continuous-valued logic variable [1]. Note that the Dombi form of negation is the unique Sugeno's negation [6] with the fix point $\nu \in (0, 1)$. Also, for $Q_\lambda(A) \in [0, 1)$, the formula of λ-additive measure of $Q_\lambda(\overline{A})$ in Eq. (2) is the same as the formula of the Dombi form of negation in Eq. (3) with $x = Q_\lambda(A)$ and $\left(\frac{1-\nu}{\nu}\right)^2 = 1 + \lambda$. Following this line of thinking, here, we will introduce the ν-additive measure and state some of its properties.

Definition 2. *The function $Q_\nu : \mathcal{P}(X) \to [0, 1]$ is a ν-additive measure on the finite set X, iff Q_ν satisfies the following requirements:*

(1) $Q_\nu(X) = 1$
(2) for any $A, B \in \mathcal{P}(X)$ and $A \cap B = \emptyset$,

$$Q_\nu(A \cup B) = Q_\nu(A) + Q_\nu(B) + \left(\left(\frac{1-\nu}{\nu}\right)^2 - 1\right) Q_\nu(A)Q_\nu(B),$$

where $\nu \in (0, 1)$.

Note that if X is an infinite set, then the continuity of function Q_ν is also required. We will utilize the concept of dual pair of ν-additive measures.

Definition 3. *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X. Then, Q_{ν_1} and Q_{ν_2} are said to be a dual pair of ν-additive measures iff $Q_{\nu_1}(A)+Q_{\nu_2}(\overline{A}) = 1$ holds for any $A \in \mathcal{P}(X)$.*

We have proven the following properties of the ν-additive measure.

Proposition 2. *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X. Then, Q_{ν_1} and Q_{ν_2} are a dual pair of ν-additive measures if and only if $\nu_1 + \nu_2 = 1$.*

Corollary 1. *Let Q_{ν_1} and Q_{ν_2} be a dual pair of ν-additive measures on the finite set X. Then, $\nu_1 \in [1/2, 1)$ if and only if $\nu_2 \in (0, 1/2]$.*

It should be mentioned here that one of the λ parameters of a dual pair of λ-additive measures is always in the unbounded interval $[0, \infty)$. At the same time, the ν parameters of a dual pair of ν-additive measures are both in a bounded interval; namely, one of them is in the interval $(0, 1/2]$ and the other one is in the interval $[1/2, 1)$.

2 Connections with belief, probability, plausibility, rough sets, multiattribute utility functions and fuzzy operators

2.1 Connections with belief, probability and plausibility

Here, we will discuss how the ν-additive (λ-additive) measure is connected with the belief-, probability- and plausibility measures. It should be added that these connections are demonstrated via novel proofs in which the so-called basic probability assignment (mass function) is not utilized.

Proposition 3. *Let X be a finite set and let Q_λ be a λ-additive measure on X. Then, on set X, Q_λ is a*

(1) plausibility measure if and only if $-1 < \lambda \leq 0$
(2) probability measure if and only if $\lambda = 0$
(3) belief measure if and only if $\lambda \geq 0$.

Proposition 4. *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X. Then Q_{ν_1} and Q_{ν_2} are a dual pair of belief- and plausibility measures on X if and only if $\nu_1 + \nu_2 = 1$.*

Proposition 5. *Let Σ be a σ-algebra over the set X and let Q_ν and P_ν be two continuous functions on the space (X, Σ) such that*

$$P_\nu(A) = \frac{1}{2} \frac{\ln\left(1 + \left(\left(\frac{1-\nu}{\nu}\right)^2 - 1\right) Q_\nu(A)\right)}{\ln\left(\frac{1-\nu}{\nu}\right)}$$

holds for any $A \in \Sigma$, $\nu \in (0,1)$, $\nu \neq 1/2$. Then, P_ν is a probability measure on (X, Σ) if and only if Q_ν is a ν-additive measure on (X, Σ).

2.2 Connections with rough sets

Dual pairs of ν-additive measures are strongly associated with the lower- and upper approximation pairs of rough sets. Utilizing the results of Skowron [3, 4] and Yao and Lingras [7], we have proven the following properties of the ν-additive measure.

Proposition 6. *Let Q_{ν_1} and Q_{ν_2} be two ν-additive measures on the finite set X, and let $R \subseteq X \times X$ be a binary equivalence relation on X. Furthermore, let $(\underline{R}(A), \overline{R}(A))$ be the rough set of $A \in \mathcal{P}(X)$ with respect to the approximation space (X, R) and let the functions $\underline{q}, \overline{q} : \mathcal{P}(X) \to [0,1]$ be given by $\underline{q}(A) = \frac{|\underline{R}(A)|}{|X|}$, $\overline{q}(A) = \frac{|\overline{R}(A)|}{|X|}$, where $\underline{R}(A)$ and $\overline{R}(A)$ are the lower- and upper approximations of A, respectively, for any $A \in \mathcal{P}(X)$. Then, if the equations $Q_{\nu_1}(A) = \underline{q}(A)$, $Q_{\nu_2}(A) = \overline{q}(A)$ hold for any $A \in \mathcal{P}(X)$, then Q_{ν_1} and Q_{ν_2} are a dual pair of ν-additive measures on X with $\nu_1 \in (0, 1/2]$, $\nu_2 \in [1/2, 1)$.*

Proposition 7. *If Q_{ν_1} and Q_{ν_2} are a dual pair of ν-additive measures on the finite set X with $\nu_1 \in (0, 1/2]$, $\nu_2 \in [1/2, 1)$ and m is a basic probability assignment that satisfies the conditions:*

(1) the set of focal elements of m is a partition of X
(2) $m(A^*) = |A^*|/|X|$ for every focal element A^* of m
(3) $m(A^*) = \sum_{B \subseteq A^*} (-1)^{|A^* \setminus B|} Q_{\nu_1}(B)$ for any $A^* \in \mathcal{P}(X)$,

then there exists an equivalence relation R on the set X, such that the equations

$$Q_{\nu_1}(A) = \underline{q}(A),\ Q_{\nu_2}(A) = \overline{q}(A)$$

hold for any $A \in \mathcal{P}(X)$, where $(\underline{R}(A), \overline{R}(A))$ is the rough set of A with respect to the approximation space (X, R), $\underline{q}, \overline{q} : \mathcal{P}(X) \to [0, 1]$ are given as

$$\underline{q}(A) = \frac{|\underline{R}(A)|}{|X|},\ \overline{q}(A) = \frac{|\overline{R}(A)|}{|X|},$$

and $\underline{R}(A)$ and $\overline{R}(A)$ are the lower- and upper approximations of A, respectively.

2.3 Connections with multi-attribute utility functions and with fuzzy operators

(1) There are interesting formal connections between the λ-additive measures and the multi-attribute utility functions. Namely,
  (a) if $\lambda = 0$, then the λ-additive measure of the union of n pairwise disjoint sets is computed in the same way as the multi-attribute utility of n additive independent attributes
  (b) if $\lambda > -1$ and $\lambda \neq 0$, then the λ-additive measure of the union of n pairwise disjoint sets is computed in the same way as the multi-attribute utility of n mutually utility independent attributes.

(2) There is an interesting formal connection between the λ-additive measure and certain operators of continuous-valued logic. Namely, if $\lambda > -1$ and $\lambda \neq 0$, then the computation method of λ-additive measure of union of n pairwise disjoint sets is identical with that of the generator function of the Dombi operator [2] at the value of the generalized Dombi operation [1] over n continuous-valued logic variables.

References

1. Dombi, J.: Towards a general class of operators for fuzzy systems. IEEE Transactions on Fuzzy Systems 16(2), 477–484 (2008)
2. Dombi, J.: A general class of fuzzy operators, the de morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators. Fuzzy sets and systems 8(2), 149–163 (1982)
3. Skowron, A.: The relationship between the rough set theory and evidence theory. Bulletin of Polish academy of science: Mathematics 37, 87–90 (1989)
4. Skowron, A.: The rough sets theory and evidence theory. Fundam. Inf. 13(3), 245–262 (Oct 1990)
5. Sugeno, M.: Theory of fuzzy integrals and its applications. Ph.D. thesis, Tokyo Institute of Technology, Tokyo, Japan (1974)
6. Sugeno, M.: Fuzzy measures and fuzzy integrals-a survey. In: Dubois, D., Prade, H., Yager, R.R. (eds.) Readings in Fuzzy Sets for Intelligent Systems, pp. 251 – 257. Morgan Kaufmann (1993)
7. Yao, Y., Lingras, P.: Interpretations of belief functions in the theory of rough sets. Information Sciences 104(1), 81 – 106 (1998)

Integral representations of a coherent upper conditional prevision with respect to its associated Hausdorff outer measure

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Abstract. Sufficient conditions are given to assure that different integral representations of a coherent upper conditional prevision with respect to its associated Hausdorff outer measure coincide. The given coherent conditional prevision is proven to be linear on the class of continuous random variables.

1 Instroduction

Let Ω be a non empty set and let **B** be a partition of Ω. In the subjective probabilistic approach coherent probability is defined on an arbitrary class of sets and any coherent probability can be extended to a larger domain. So in this framework no measurability condition is required for random variables. In the sequel a bounded random variable is a function $X : \Omega \to \Re$ such that $|X| \leq M$ for some real constant M and $L(\Omega)$ is the class of all bounded random variables defined on Ω; for every $B \in \mathbf{B}$ denote by $X|B$ the restriction of X to B and by $\sup(X|B)$ the supremum of valus that X assumes on B. Let $L(B)$ be the class of all bounded random variables $X|B$. Denote by I_A the indicator function of any event $A \in \wp(B)$, i.e. $I_A(\omega) = 1$ if $\omega \in A$ and $I_A(\omega) = 0$ if $\omega \in A^c$. For every $B \in \mathbf{B}$ coherent upper conditional previsions $\overline{P}(\cdot|B)$ are functionals defined on $L(B)$ [20].

Definition 1. *Coherent upper conditional previsions are functionals $\overline{P}(\cdot|B)$ defined on $L(B)$, such that the following axioms of coherence hold for every X and Y in $L(B)$ and every strictly positive constant λ:*

1) $\overline{P}(X|B) \leq \sup(X|B)$*;*
2) $\overline{P}(\lambda X|B) = \lambda\overline{P}(X|B)$ *(positive homogeneity);*
3) $\overline{P}(X + Y|B) \leq \overline{P}(X|B) + \overline{P}(Y|B)$ *(subadditivity);*
4) $\overline{P}(I_B|B) = 1$.

Suppose that $\overline{P}(X|B)$ is a coherent upper conditional prevision on $L(B)$. Then its conjugate coherent lower conditional prevision is defined by $\underline{P}(X|B) = -\overline{P}(-X|B)$. Let K be a linear space contained in $L(B)$; if for every X belonging to K we have $P(X|B) = \underline{P}(X|B) = \overline{P}(X|B)$ then $P(X|B)$ is called a coherent *linear* conditional prevision [3],[13],[18] and it is a linear, positive and positively homogenous functional on K [20, Corollary 2.8.5].

The unconditional coherent upper prevision $\overline{P} = \overline{P}(\cdot|\Omega)$ is obtained as a particular case when the conditioning event is Ω. Coherent upper conditional probabilities are obtained when only 0-1 valued random variables are considered.

Definition 2. *Given a partition **B** and a random variable $X \in L(\Omega)$, a coherent upper conditional prevision $\overline{P}(X|\boldsymbol{B})$ is a random variable on Ω equal to $\overline{P}(X|B)$ if $\omega \in B$.*

Definition 3. *A random variable $X \in L(\Omega)$ is **B**-measurable or measurable with respect to the partition **B** it it is constant on the sets of the partition.*

The necessity to propose a new tool to define coherent upper conditional previsions arises because they cannot be obtained as extensions of linear expectations defined, by the Radon-Nikodym derivative, in the axiomatic approach [1]; it occurs because one of the defining properties of the Radon-Nikodym derivative, that is to be measurable with respect to the σ-field of the conditioning events, contradicts the necessary condition for the coherence [9, Theorem 1] $\overline{P}(X|\mathbf{B}) = X$ for every **B**-measurable random variable.

2 Coherent upper conditional prevision defiend by its associated Haussdorff outer measure

A model of coherent upper conditional prevision and probability, based on Hausdorff outer measures has been introduced in a metric space (Ω, d) [5], [6], [7], [8], [9], [10], [12] and its applications have been investigated [11].

For the definition of Hausdorff outer measure and its basic properties see [19] and [15]. Let (Ω, d) be a metric space and let **B** be partition of Ω.

Let $\delta > 0$ and let s be a non-negative number. The *diameter* of a non empty set U of Ω is defined as $|U| = \sup\{d(x, y) : x, y \in U\}$ and if a subset A of Ω is such that $A \subseteq \bigcup_i U_i$ and $0 < |U_i| \leq \delta$ for each i, the class $\{U_i\}$ is called a δ-cover of A.

The *Hausdorff s-dimensional outer measure* of A, denoted by $h^s(A)$, is defined on $\wp(\Omega)$, the class of all subsets of Ω, as

$$h^s(A) = \lim_{\delta \to 0} \inf \textstyle\sum_{i=1}^{+\infty} |U_i|^s .$$

where the infimum is over all δ-covers $\{U_i\}$.

A subset A of Ω is called *measurable* with respect to the outer measure h^s if it decomposes every subset of Ω additively, that is if $h^s(E) = h^s(A \cap E) + h^s(E - A)$ for all sets $E \subseteq \Omega$.

Hausdorff s-dimensional outer measures are submodular, continuous from below and their restriction on the Borel σ-field is countably additive.

The *Hausdorff dimension* of a set A, $dim_H(A)$, is defined as the unique value, such that

$$h^s(A) = +\infty \text{ if } 0 \leq s < dim_H(A),$$
$$h^s(A) = 0 \text{ if } dim_H(A) < s < +\infty.$$

For every $B \in \mathbf{B}$ denote by s the Hausdorff dimension of B and let h^s be the Hausdorff s-dimensional Hausdorff outer measure associated to the coherent upper conditional prevision. For every bounded random variable X a coherent upper conditional prevision $\overline{P}(X|B)$ is defined by the Choquet integral with respect to its associated Hausdorff outer measure if the conditioning event has positive and finite Hausdorff outer measure in its Hausdorff dimension. Otherwise if the conditioning event has Hausdorff outer measure in its Hausdorff dimension equal to zero or infinity it is defined by a 0-1 valued finitely, but not countably, additive probability.

Theorem 1. *Let (Ω, d) be a metric space and let **B** be a partition of Ω. For every $B \in \boldsymbol{B}$ denote by s the Hausdorff dimension of the conditioning event B and by h^s the Hausdorff s-dimensional outer measure. Let m be a 0-1 valued finitely additive, but not countably additive, probability on $\wp(B)$. Then for each $B \in \boldsymbol{B}$ the functional $\overline{P}(X|B)$ defined on $L(B)$ by*

$$\overline{P}(X|B) = \frac{1}{h^s(B)} \int_B X dh^s \text{ if } 0 < h^s(B) < +\infty$$

and by

$$\overline{P}(X|B) = m_B \text{ if } h^s(B) \in \{0, +\infty\}$$

is a coherent upper conditional prevision.

If $B \in \mathbf{B}$ is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s and X is the indicator function of a set A by Theorem 1 we obtain that the fuzzy measure μ_B^* defined for every $A \in \wp(B)$ by $\mu_B^*(A) = \frac{h^s(AB)}{h^s(B)}$ is a coherent upper conditional probability, which is submodular, continuous from below and such that its restriction to the σ-field of all μ_B^* measurable sets is a Borel regular countably additive probability. The coherent upper unconditional probability $\overline{P} = \mu_\Omega^*$ defined on $\wp(\Omega)$ is obtained for B equal to Ω.

3 Integral representations

The following results give sufficient conditions which assure that different integral representations of a coherent upper conditional previsions coincide and conditions such that a coherent conditional prevision defined with respect to its associated Hausdorff outer measure is linear. The Choquet integral [4], the concave integral [16] and the pan integral agree with the Lebesgue integral in the case where the monotone set function is countably additive.

In the next theorem it is proven that, if B is a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s, the Choquet integral, the concave integral and the pan-integral with respect to the coherent upper conditional probability μ_B^* coincide with the Lebesgue integral on the class of all S-measurable random variables.

Theorem 2. *Let B be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s and let $\mu_B^*(A) = \frac{h^s(A \cap B)}{h^s(B)}$. Let S be the class of all h^s-measurable sets. Then the Choquet integral, the concave integral and the pan-integral with respect to μ_B^* are equal to the Lebesgue integral on the class of all S-measurable random variables.*

In the next theorem it is proven that on the class of continuous random variables the coherent upper conditional prevision defined in Theorem 1 is linear.

Theorem 3. *Let B be a set with positive and finite Hausdorff outer measure in its Hausdorff dimension s and let $K \subset L(B)$ be the class of continuous random variables. Then $\underline{P}(X|B) = \overline{P}(X|B)$ for every $X \in K$.*

References

1. Billingsley, P. 1986. *Probability and Measure*. New York: Jhon Wiley & Sons.
2. Choquet, G. 1953. "Theory of capacities." *Annales de l'institut Fourier* 5: 131–295.
3. de Finetti, B. 1974. *Theory of Probability*. London: Wiley.
4. Denneberg, D. 1994. *Non-additive Measure and Integral*. Kluwer Academic Publishers.
5. Doria, S. 2007. "Probabilistic Independence With Respect to Upper and Lower Conditional Probabilities Assigned by Hausdorff Outer and Inner Measures." *International Journal of Approximate Reasoning* 46: 617–635.
6. Doria, S. 2008. "Convergences of Random Variables With Respect to CoherentUpper Probabilities Defined by Hausdorff Outer Measures." *Advances in Soft Computing* 48: 281–288.
7. Doria, S. 2010. "Coherent Upper Conditional Previsions and Their Integral Representation With Respect to Hausdorff Outer Measures." *In Combining Soft Computing and Statistical Methods in Data Analysis*, edited by C. Borgelt et al. 209-216. Berlin, Heidelberg: Springer.
8. Doria, S. 2011 "Coherent Upper and Lower Conditional Previsions Defined by Hausdorff Outer and Inner Measures." *In Modeling, Design, and Simulation Systems with Uncertainties*, edited by A. Rauh and E. Auer 175–195. Berlin, Heidelberg: Springer.
9. Doria, S. 2012. "Characterization of a Coherent Upper Conditional Prevision as the Choquet Integral with Respect to Its Associated Hausdorff Outer Measure." *Annals of Operations Research* 195: 33–48.
10. Doria, S. 2014. "Symmetric Coherent Upper Conditional Prevision by the Choquet Integral with Respect to Hausdorff Outer Measure." *Annals of Operations Research* 229: 377-396.
11. Doria, S. 2015. "Coherent Conditional Measures of Risk Defined by the Choquet Integral With Respect to Hausdorff Outer Measure and Stochastic Independence in Risk Management." *International Journal of Approximate Reasoning* 65: 1–10.
12. Doria, S. 2016. "On the Disintegration Property of a Coherent Upper Conditional Prevision by the Choquet Integral With Respect to Its Associated Hausdorff Outer Measure." *Annals of Operations Research* 256: 253-269.
13. Dubins, L.E. 1975. "Finitely Additive Conditional Probabilities, Conglomerability and Disintegrations." *The Annals of Probability* 3: 89–99.
14. Even, Y. and Lehrer, E. 2014. "Decomposition-integral: Unifying Choquet and Concave Integrals." *Econ. Theory* 56: 33–58.
15. Falconer, K.J. 1986. *The Geometry of Fractals Sets*. Cambridge: Cambridge University Press.
16. Lehrer, E. 2009. "A New Integral for Capacities." *Economic Theory* 39: 157-176.
17. Mesiar R., and Mesiarová A. 2008. "Fuzzy Integrals and Linearity." *International Journal of Approximate Reasoning* 47: 352–358.
18. Regazzini, E. 1987. "De Finetti's Coherence and Statistical Inference." *The Annals of Statistics* 15: 845–864.
19. Rogers, C. A. 1970. *Hausdorff Measures*. Cambridge: Cambridge University Press.
20. Walley, P. 1991. *Statistical Reasoning with Imprecise Probabilities*. London: Chapman and Hall.

Double Sugeno integrals that commute

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In decision problems involving two dimensions (like several agents and several criteria) the properties of expected utility ensure that the result of a multicriteria multiperson evaluation does not depend on the order with which the aggregations of local evaluations are performed (agents first, criteria next, or the converse). We say that the aggregations on each dimension commute. Ben Amor, Essghaier and Fargier have shown [2] that this property holds when using pessimistic possibilistic integrals on each dimension, or optimistic ones, while it fails when using a pessimistic possibilistic integral on one dimension and an optimistic one on the other. This paper studies and completely solves this problem when Sugeno integrals are used in place of possibilistic integrals, indicating that there are capacities other than possibility and necessity measures that ensure commutation of Sugeno integrals. In connection with this problem, we study two-dimensional capacities that can be reconstructed from their projections, and their link to the commutation problem.

References

1. M. Behrisch, M. Couceiro, K. A. Kearnes, E. Lehtonen, A. Szendrei, Commuting polynomial operations of distributive lattices, Order 29(2): 245-269, 2012.
2. N. Ben Amor, F. Essghaier, H. Fargier: Egalitarian Collective Decision Making under Qualitative Possibilistic Uncertainty: Principles and Characterization. Proc. AAAI 2015: 3482-3488, 2015.
3. D.Dubois, H. Fargier, A. Rico: Sugeno Integrals and the Commutation Problem. Proc. MDAI 2018: LNCS 11144, Springer, 48-63.
4. D. Dubois, H. Prade. R. Sabbadin. Decision theoretic foundations of qualitative possibility theory. European Journal of Operational Research, 128: 459-478, 2001.
5. R. Halas, R. Mesiar, J. Pocs, V. Torra. A note on discrete Sugeno integrals: composition and associativity. Fuzzy Sets and Systems, to appear, 2018.
6. J. Harsanyi (1955). Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. Journal of Political Economy 63: 309-321, 1955
7. J.-L. Marichal. On Sugeno integrals as an aggregation function. Fuzzy Sets and Systems, 114(3): 347-365, 2000.
8. Y. Narukawa, V. Torra: Multidimensional generalized fuzzy integral. Fuzzy Sets and Systems 160(6): 802-815, 2009.
9. M. Sugeno. Fuzzy measures and fuzzy integrals: a survey. In: Fuzzy Automata and Decision Processes, (M. M. Gupta, G. N. Saridis, and B. R. Gaines, eds.), North-Holland, 89-102, 1977.

Contact and influence between objects in a system-of-systems

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Abstract. In this paper we describe relations between objects in a general system-of-systems model based on powers of attributed expressions and values described qualitatively and spatially.

1 Introduction

Industrial applications in particular in production and production plants are generally described as hierarchical systems-of-systems with objects being "part-of" and in "contact to" to each other. These objects roughly build upon elements divided, respectively, into types components, people and activities, like typically seen in the DSM model [4]. Relational models involved can be equipped with various information structure enrichments so as to enable monitoring of subsystem conditions as alternatives to fault trees [5].

In the finite case, Boolean algebras (BAs) are powerset algebras, so in the infinite case BAs are intuitively 'generalized power algebras' rather than algebras of structured elements. However, powers are obviously more than just embracing subsets of points, in particular if we deal with sets of values and expressions defined by underlying signatures. Such 'sets of terms' are elements in objects provided by set functors in form of monad compositions of general powerset functors φ (extendable to monads $\boldsymbol{\varphi}$) and the term monad $\mathbf{T}_\Sigma$ over some signature Σ [3]. In this case, a system-of-systems is not just to be seen as based on sets-of-sets, or sets-of-sets-of-sets, etc., so as to deal with functor composition $\varphi \circ \varphi \circ \ldots$, but more as based on compositions of compositions of powersets and terms $\boldsymbol{\varphi}\mathbf{T}_\Sigma \circ \boldsymbol{\varphi}\mathbf{T}_\Sigma \circ \ldots$, where swapper transformation $\sigma : \mathsf{T}_\Sigma \circ \varphi \to \varphi \circ \mathsf{T}_\Sigma$ come into play.

In the presentation we will provide concrete examples showing how these structures can be used to model systems, their physical layout, and the influence of subsystems on each other. In particular, we will demonstrate how failure resp. wear and tear propagates and influences the overall behavior of the system and its subsystems.

2 Mathematical Structures

We will be using a fixed complete Heyting algebra $L = \langle L, +, \cdot, 0, 1\rangle$ as the domain of membership values for L-fuzzy sets and relations. The induced order on L will be denoted by $\leq$. An L-fuzzy subset M of a set A is a function $M : A \to L$, and a binary L-fuzzy relation R on A is an L-fuzzy subset of $A \times A$, i.e., a function $R : A \times A \to L$.

2.1 L-Fuzzy Contact Relations

Qualitative spatial reasoning (QSR) is an alternative approach for reasoning about spatial entities by dealing only with qualitative features of those entities. In particular, mereotopology is a mathematical theory that combines the two aspects of being "part-of" and being in "contact to". The first aspect is usually modeled by a Boolean algebra B, and the second aspect by a binary contact relation among regions resulting in the well-established theory of Boolean contact algebras. For further information we refer to [1, 2, 6]. We will be using a fuzzy version of BCAs by replacing the contact relation with an L-fuzzy relation. From application point of view as related to a system-of-systems, such a B can be of the form $\mathsf{P} \circ \mathsf{T}_{\Sigma} X$, where P is the powerset functor, and Σ contains operators typically representing measurement devices [4].

Definition 1. *Let $B = \langle B, +, \cdot, ^*, 0, 1\rangle$ be a Boolean algebra. A binary L-fuzzy relation C on B is called a* contact relation *if it satisfies*

(C0) $C(a, b) \neq 0$ *implies* $a, b \neq 0$.
(C1) $a \neq 0$ *implies* $C(a, a) = 1$.
(C2) $C(a, b) = C(b, a)$, *i.e., C is symmetric.*
(C3) $b \leq c$ *implies* $C(a, b) \leq C(a, c)$.
(C4) $C(a, b + c) \leq C(a, b) + C(a, c)$.

The pair $\langle B, C\rangle$ is called an L-fuzzy Boolean contact algebra (FBCA). The first axiom requires that the empty region is not contact (degree $\neq 0$) to any region. The second axiom states that contact is reflexive for non-empty regions. (C2) requires C to be symmetric. (C3) states that the degree of being in contact increases if we enlarge one of the two regions, and, finally, the last axiom requires that if a region a is in contact with a certain degree d to a region made out of two parts b and c, then the join of the degrees of a being in contact to b resp. c must be at least as big as d.

The proof of the following lemma is an easy exercise and, therefore, omitted.

Lemma 1. *Let $\langle B, C\rangle$ be a FBCA. Then we have:*

1. $C(a, 0) = C(0, a) = 0$.
2. $0 \neq a \leq b$ *implies* $C(a, b) = 1$.
3. $a \neq 0$ *implies* $C(a, 1) = C(1, a) = 1$.
4. $C(a, b + c) = C(a, b) + C(a, c)$.

Occasionally, an additional axiom might be useful

(C7) $C(a, a^*) = 1$.

Please note that we kept the number of the axiom above consistent with the numbering in the classical case. This axiom requires that every region is in full contact (degree = 1) with its complement.

2.2 *L*-Fuzzy Influence Relations

In addition to the topological aspect of being in contact we are interested in the influence of a defect in one part of a system on another part. We model this aspect again by a binary L-fuzzy relation I. The intuition behind this relation is as follows. If we have $I(a, b) = d$, then a defect in a influences the function of b by the degree d.

Definition 2. *Let B be a Boolean algebra. A binary L-fuzzy relation I on B is called an* influence relation *if it satisfies*

(I0) $I(a, b) \neq 0$ *implies* $a, b \neq 0$.
(I1) $a \neq 0$ *implies* $I(a, a) = 1$.
(I2) $I(a, b) \cdot I(b, c) \leq I(a, c)$,
(I3) $I(a, b + c) \leq I(a, b) + I(a, c)$.
(I4) $I(b, a) + I(c, a) \leq I(b + c, a)$.

The pair $\langle B, I\rangle$ is called an L-fuzzy Boolean influence algebra (FBIA). As above the first two axiom require that the empty region does not influence or is influenced by any other region and that every non-empty region influence itself fully. ($I3$) states that influence is a transitive relation in the L-fuzzy sense. Please note that influence is not necessarily symmetric. For example, a defect in a wheel bearing might influence the tire but not necessarily vice versa. (I3) is similar to (C4). In particular, (I3) says that if a influences a region $b + c$ by a certain degree d, then d is smaller or equal the combined degree of a influencing b resp. c. However, the situation becomes different if we consider the influence of a region $b + c$ on any other region. (I4) requires that the influence that $b + c$ has on a is at least as big as the sum of the individual influence. Please note that we do not have a counterpart of (C3), i.e., monotonicity, in the second parameter of the influence relation. For example, we may obtain a larger region or system by duplicating the functionality of a subsystem, i.e., we consider region a and $a + a'$ where a and a' provide the same services. Obviously $a \leq a + a'$ but a total failure in a does not imply a total failure in $a + a'$, i.e., we do not have $I(a, a + a') = 1$.

Besides some derived properties the following lemma shows that we do have monotonicity in the first parameter of I.

Lemma 2. *Let $\langle B, I\rangle$ be a FBIA. Then we have:*

1. $I(a, 0) = I(0, a) = 0$.

2. $b \leq c$ *implies* $I(b,a) \leq I(c,a)$.
3. $0 \neq a \leq b$ *implies* $I(b,a) = 1$.
4. $a \neq 0$ *implies* $I(1,a) = 1$.

The final structure of this section combines contact and influence.

Definition 3. *Let* B *be a Boolean algebra,* C *be an* L*-fuzzy contact relation on* B*, and* I *be an* L*-fuzzy influence relation on* B*. Then the structure* $\langle B, C, I\rangle$ *is called an* L*-fuzzy Boolean contact influence algebra (FBCIA) iff*

(IC) $I(a,b) \leq \sum_{c \in B} C(a,c) \cdot I(a,c) \cdot C(c,b) \cdot I(c,b)$.

The axiom (IC) says that the influence of a on b can be computed by tracking the passage of influence along regions that are in contact with a and b.

The following lemma shows that the transitivity of I implies equality in (IC).

Lemma 3. *Let* $\langle B, C, I\rangle$ *be a FBCIA. Then we have:*

$$I(a,b) = \sum_{c \in B} C(a,c) \cdot I(a,c) \cdot C(c,b) \cdot I(c,b).$$

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References

1. Düntsch, I., Winter, M.: Algebraization and Representation of Mereotopological Structures. JoRMiCS 1, pp. 161-180 (2004).
2. Düntsch, I., Winter, M.: A Representation Theorem for Boolean Contact Algebras. TCS 347(3), pp. 498-512 (2005).
3. Eklund, P., Galán, M.A., Ojeda-Aciego, M., Valverde, A.: Set functors and generalised terms. Proc. IPMU 2000, 8th Information Processing and Management of Uncertainty in Knowledge-Based Systems Conference, vol. III, 2000, 1595-1599.
4. Eppinger, S.D., Browning, T.R.: Engineering Systems: Design Matrix Methods and Applications: MIT Press (2012).
5. Johansson, M., Eklund, P., Kortelainen, J., Winter, M.: The algebra, logic and topology of System-of-Systems. Proceedings of the 20th International DSM Conference, Trieste (Italy) Oct 15-17 (2018).
6. Stell, J.G., Worboys, M.F.: The algebraic structure of sets of regions. In Conf. on Spatial Inf. Theory (COSIT-97), LNCS 1329, pp. 163-174 (1997).

Combinatorial structure of 2-additive measures polytope

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Abstract. In this communication we study the polytope of 2-additive measures. We obtain its combinatorial structure, namely the adjacency structure and the structure of 2-dimensional faces, 3-dimensional faces and so on. From this information, we build a triangulation of this polytope satisfying that all simplices in the triangulation have the same volume. This allows a very simple and appealing way to generate points in a random way in this polytope. Moreover, it permits us to find the centroid of this polytope.

1 Introduction and main results

Consider a finite set X of n elements, $X = \{x_1, ..., x_n\}$. Elements of X are criteria in the field of Multicriteria Decision Making, players in Cooperative Game Theory, and so on. We will denote subsets of X by $A, B, ...$. In order to simplify notation, we will often use $i_1 i_2 \cdots i_n$ for denoting the set $\{i_1, i_2, \cdots, i_n\}$ specially for singletons and pairs. We also define $\binom{X}{k}$ to be the set of all k-element subsets of X. Remember that a **fuzzy measure** [3], [1], [7] is a set function $\mu : \mathcal{P}(X) \to [0, 1]$ satisfying $\mu(\emptyset) = 0, \mu(X) = 1$ and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$. Fuzzy measures, together with Choquet integral [1], have been proved to be a powerful tool applying to many different fields, as Decision Making, Game Theory and Imprecise Probabilities among many others (see e.g. [5] and the references therein). The reason of this success relies on the fact that fuzzy measures are able to model situations that probability measures can not.

On the other hand, this wealth in terms of interpretation is paid with an increment of the complexity needed to define a fuzzy measure. To cope with this problem, several alternatives have arisen; one of them is to add additional constraints to the definition, thus defining several subfamilies. Among the many subfamilies appearing in the literature, perhaps the most successful subfamily is the subfamily of k-additive measures [4], and inside this subfamily, the most appealing case is the case of 2-additive measures.

The concept of k-additivity is based on the Möbius transform, an alternative representation of fuzzy measures. The **Möbius transform** of μ is defined by $m(A) := \sum_{B \subseteq A} (-1)^{|A \setminus B|} \mu(B), \forall A \subseteq X$. The Möbius transform gives a measure of the importance of a coalition by itself, without taking account of its different parts. In this sense, note that it could be difficult for an expert to assess values to interactions of many palyers and interpret what these interactions mean. Then, it makes sense to restrict the range of meaningful interactions to coalitions of a reduced number of criteria. This translates in the condition $m(A) = 0$ if $|A| > k$.

Definition 1. *[4] A fuzzy measure μ is said to be k-**additive** if its Möbius transform vanishes for any $A \subseteq X$ such that $|A| > k$ and there exists at least one subset A of exactly k elements such that $m(A) \neq 0$.*

Specially appealing is the 2-additive case, that allows to model interactions between two criteria, that are the most important interactions, while keeping a reduced complexity. We will denote by $\mathcal{FM}^2(X)$ the set of all fuzzy measures being *at most* 2-additive.

2 Combinatorial structure of $\mathcal{FM}^2(X)$

It can be easily seen that $\mathcal{FM}^2(X)$ is a convex polyhedron, i.e. a polytope [2]. In this section we tackle the problem of obtaining its combinatorial structure. First, the vertices of $\mathcal{FM}^2(X)$ have been obtained in [6] and are given in next proposition.

Proposition 1. *The set of vertices of $\mathcal{FM}^2(X)$ are given by the $\{0,1\}$-valued fuzzy measures in $\mathcal{FM}^2(X)$, i.e. u_i, u_{ij}, μ_{ij}, that are defined by*

$$u_A(B) := \begin{cases} 1 & \text{if A} \subseteq \text{B} \\ 0 & \text{otherwise} \end{cases}, \quad \mu_{ij}(B) := \begin{cases} 1 & \text{if i} \in \text{B or j} \in \text{B} \\ 0 & \text{otherwise} \end{cases}$$

Then, $\mathcal{FM}^2(X)$ has n^2 vertices.

Next theorem gives a complete description of the combinatorial structure of $\mathcal{FM}^2(X)$. For this, we use the fact that a face can be characterized by the family of vertices inside the face, because the face is given by the convex hull of these vertices. Then, it suffices to characterize whether a collection of vertices determine a face.

Theorem 1. ***Combinatorial structure of** $\mathcal{FM}^2(X)$.*
Let $\mathcal{C}$ be a collection of vertices of $\mathcal{FM}^2(X)$. Then, the following are equivalent:

$i)$ $\text{Conv}(\mathcal{C})$ *is a face of* $\mathcal{FM}^2(X)$.
$ii)$ $u_i, u_j \in \mathcal{C} \Leftrightarrow u_{ij}, \mu_{ij} \in \mathcal{C}$.

From this theorem, the adjacency structure of $\mathcal{FM}^2(X)$ can be derived in an easy and fast way. For this, it suffices to remark that two vertices are adjacent if and only if its convex hull if a 1-dimensional face (an edge) of the polytope.

Corollary 1. *Let μ_1 and μ_2 be two different vertices of $\mathcal{FM}^2(X)$. Then, μ_1 and μ_2 are adjacent vertices in $\mathcal{FM}^2(X)$ except if $\mu_1 = u_i$, $\mu_2 = u_j$ or $\mu_1 = u_{ij}, \mu_2 = \mu_{ij}$.*

Let us denote by $\mathcal{F}_\mathcal{C}$ the face whose vertices are $\mathcal{C}$. Now, we can describe any k-dimensional face of this polytope. For this, we have to define consecutive pyramids.

For a convex polytope $\mathcal{P}$ and a non-collinear point $\boldsymbol{x}$, called **apex**, we define a pyramid with base $\mathcal{P}$ and apex $\boldsymbol{x}$, denoted by pyr$(\mathcal{P}, \boldsymbol{x})$, to be the polytope which has as vertices the vertices of $\mathcal{P}$ and $\boldsymbol{x}$. For a pyramid, observe that the apex is adjacent to every vertex in $\mathcal{P}$. Now, if we consider $\boldsymbol{y} \notin$ aff(pyr$(\mathcal{P}, \boldsymbol{x})$), then $\boldsymbol{y}$ is an apex for pyr$(\mathcal{P}, \boldsymbol{x})$, and we can define a new pyramid pyr(pyr$(\mathcal{P}, \boldsymbol{x}), \boldsymbol{y})$, denoted cpyr$(\mathcal{P}, \{\boldsymbol{x}, \boldsymbol{y}\})$. Iterating this process, we can define a **consecutive pyramid** with apexes $\mathcal{A} = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r\}$, denoted by cpyr$(\mathcal{P}, \mathcal{A})$.

Theorem 2. *Let $\mathcal{F}_{\mathcal{C}}$ be a face of $\mathcal{FM}^2(X)$ and let us consider the following sets*

$$\mathcal{U} := \{i \in X : u_i \in \mathcal{C}\},\ \mathcal{V} := \{ij \in \binom{X}{2} : |\{u_{ij}, \mu_{ij}\} \cap \mathcal{C}| = 1\}.$$

Then, the following holds:

i) If $|\mathcal{U}| \leq 1$, then $\mathcal{F}_{\mathcal{C}}$ is a simplicial face of dimension $|\mathcal{C}| - 1$.
ii) If $|\mathcal{U}| > 1$, then $\mathcal{F}_{\mathcal{C}}$ is a non-simplicial face of dimension $\binom{|\mathcal{U}|}{2} + |\mathcal{U}| + |\mathcal{V}| - 1$. If $\mathcal{V} = \emptyset$, then $\mathcal{F}_{\mathcal{C}} = \mathcal{FM}^2(\mathcal{U})$. Otherwise, $\mathcal{F}_{\mathcal{C}} = cpyr(\mathcal{FM}^2(\mathcal{U}), \mathcal{V})$.

3 A random procedure for generating points in $\mathcal{FM}^2(X)$

The results developed before can be applied to derive a procedure for generating random points uniformly distributed in $\mathcal{FM}^2(X)$ based on triangulation methods. The triangulation method is based on the decomposition of the polytope into simplices such that any pair of simplices intersects in a (possibly empty) common face. Once the decomposition is obtained, we assign to each simplex a probability proportional to its volume. These probabilities are used for selecting one of the simplices. Finally, for the chosen simplex, a random point in it is generated. The subjacent idea behind the triangulation method is that generating points in a simplex is very easy and fast, as it is enough to sample a point in the simplex $X_1 \leq X_2 \leq \cdots \leq X_{\binom{n}{2}+n}$ and then do the affine transformation $Y = C \cdot X + c_0$ that sends $X_1 \leq X_2 \leq \cdots \leq X_{\binom{n}{2}+n}$ to the desired simplex.

The difficult step in the triangulation method is to split the polytope in a suitable way into simplices. For obtaining a triangulation in our case, we will apply that

$$\frac{1}{2}u_i + \frac{1}{2}u_j = \frac{1}{2}u_{ij} + \frac{1}{2}\mu_{ij}.$$

Then, the following can be shown.

Lemma 1. *Given $\mu \in \mathcal{FM}^2(X)$, it is possible to write μ as a unique convex combination of vertices of $\mathcal{FM}^2(X)$ in a way such that either u_{ij} or either μ_{ij} has null coefficient, for all pairs $ij \in \binom{X}{2}$.*

Hence, we obtain the following algorithm.

SAMPLING ALGORITHM FOR 2-ADDITIVE MEASURES $\mathcal{F}M^2(X)$

1. Choose randomly between u_{ij} and μ_{ij} for any pair of elements $ij \in \binom{X}{2}$.
2. Select a random point in the selected simplex.

Let us consider a partition $\{\mathcal{A}^-, \mathcal{A}^+\}$ of $\binom{X}{2}$. Set $\mathcal{A}^-$ denotes the pairs where u_{ij} is selected and $\mathcal{A}^+$ the set of pairs where μ_{ij} is chosen. There are $2^{\binom{n}{2}}$ possible choices for $\{\mathcal{A}^-, \mathcal{A}^+\}$.By Lemma 1, this produces a partition of $\mathcal{FM}^2(X)$. Next theorem shows that indeed each element of this partition is a simplex.

Theorem 3. *Let Δ be the collection of all the polytopes $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ where $\{\mathcal{A}^-,\mathcal{A}^+\}$ is any possible partition of $\binom{X}{2}$. Then, Δ is a triangulation of $\mathcal{FM}^2(X)$.*

Next step to show that the algorithm works is to prove that all these simplices share the same volume. Thus, all of them have the same probability in the triangulation method and it suffices to select one of them.

Proposition 2. *Given $\{\mathcal{A}^-,\mathcal{A}^+\}$, then each simplex $\mathcal{FM}^2_{\mathcal{A}^-,\mathcal{A}^+}(X)$ has the same volume in $\mathbb{R}^{\binom{n}{2}+n-1}$.*

As a corollary, we obtain the centroid of this polytope. For finding the centroid, we use that the centroid of a simplex is the arithmetic mean of the vertices. Thus, as all the simplices obtained in the triangulation share the same volume, it suffices to find the centroids of all of them and compute the arithmetic mean. The following can be shown.

Proposition 3. *The centroid of $\mathcal{FM}^2(X)$ is given by $\overline{\mu}$ given by*

$$\overline{\mu}(B) = \frac{|B|}{n}.$$

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References

1. G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, (5):131–295, 1953.
2. E. F. Combarro and P. Miranda. On the polytope of non-additive measures. *Fuzzy Sets and Systems*, 159(16):2145–2162, 2008.
3. D. Denneberg. *Non-additive measures and integral*. Kluwer Academic, Dordrecht (The Netherlands), 1994.
4. M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, (92):167–189, 1997.
5. M. Grabisch, editor. *Set functions, games and capacities in Decision Making*, volume 46 of *Theory and Decision Library*. Springer, 2016.
6. P. Miranda, E.F. Combarro, and P. Gil. Extreme points of some families of non-additive measures. *European Journal of Operational Research*, 33(10):3046–3066, 2006.
7. M. Sugeno. *Theory of fuzzy integrals and its applications*. PhD thesis, Tokyo Institute of Technology, 1974.

Axiomatic foundations of a unifying core

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Abstract. We provide an axiomatic characterization of the core of games in effectiveness form. We point out that the core, whenever it applies to appropriate classes of these games, coincides with a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, pairwise stability, and stable matchings, among others. Our characterization of the core invokes the axioms of restricted non-emptiness, coalitional unanimity, and Maskin invariance together with a principle of independence of irrelevant states, and uses in its proof a holdover property echoing the conventional ancestor property. Taking special cases of this general characterization of the core, we derive new characterizations of the previously mentioned stability concepts.

1 Introduction

Many theorists in economics and political science have been occupied in studying a wide variety of stability concepts in social choice and game theory for a century or more. Generally speaking, these stability concepts are mainly founded on the idea that given some prevailing state, individuals possess some blocking power to oppose that state and exercise it when they have an interest to do so. A stable state is understood to be a state for which no individual or group of individuals has the power to change the status quo by choosing a more desirable situation. This arises, for example, in a general equilibrium of markets where economic agents on both the demand and supply sides do not have any incentive to alter their consumption or production decisions at the given market price. In the same vein, elections in political systems rely on voting rules (quorum, majority, etc.) that allow some coalitions of voters to impose their chosen candidate on the entire society. In like manner, equilibrium concepts for non-cooperative games (Nash equilibrium, subgame perfect equilibrium, etc.) recommend a state robust to deviations in strategy in both static and dynamic settings. Likewise, many solution concepts for coalitional games (core, stable set, etc.) stress cooperative agreements on utility allocation that no coalition would contest.

In this article, we consider the general framework of games in effectiveness form (henceforth e-form games), first introduced by [11], which encompasses a vast range of contexts, including voting problems, normal form games, network problems, and matching models, among others. The canonical e-form game has the following features. A set N of players is equipped with preferences over a set A of states. Players are mutually aware of each other's preferences, can form coalitions, and sign binding agreements to oppose a given state. In addition, the blocking power distribution among coalitions is described by an "effectiveness function"; given a prevailing state a in A, coalition S is effective if it can force all players to move from state a to some state in B. Such a function specifies for every coalition S of players and subset $B \subseteq A$ of states whether or not S is effective for B. Without going into details, this way of defining the effectivity of coalitions is similar to the "inducement correspondence" introduced by [5] and is more general than the notions of "effectivity function" and "effectiveness relation" respectively proposed by [7] and [4]. The effectiveness function also corresponds to a special case of the "local effectivity function" suggested by [1].

Since players can behave cooperatively to oppose a given state, the solution concept we consider here is a version of the core of e-form games [11]. A state a is core-stable if there are no coalition S of players and a subset B of states for which S is effective for B at a and in which every player in S strictly prefers every state in B to a. The most remarkable feature of the core is the fact that a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, pairwise stability, and stable matchings, among others, coincide with the core applied to some classes of e-form games by means of an appropriate effectiveness function. More precisely, by fixing what constitutes the blocking power of coalitions, we can express these stability concepts in terms of the core for a suitable class of e-form games.

Despite the diversity of existing stability concepts such as those just mentioned, very little is known about the properties that unify them. To address this issue, we propose to axiomatically characterize the core on a vast range of classes of e-form games. Formally, the core is a correspondence that associates each e-form game with a (possibly empty) subset of core-stable states. Perhaps unexpectedly, the core is characterized on a wide range of classes of e-form games by a set of four axioms which are reasonably weak and intuitive. "Restricted non-emptiness" requires that when the core is non-empty, a solution to contain at least one state. "Coalitional unanimity" establishes that if a state a is selected for an e-form game, then a must belong to any unanimously best set of states B for players in some coalition S effective for B at a. If a state a is selected for an e-form game, then "Maskin invariance" asserts that it is also selected in an e-form game where a has (weakly) improved in the preference rankings of all players. "Independence of irrelevant states" specifies that if a state is selected for an e-form game, then it is still selected when non-selected states are removed from the game. This latter axiom is in line with other principles of independence widely used in characterizations of game-theoretic solutions [8, 2, 3, 12, 13]. For some classes of e-form games such as, for example, those derived from network problems, states cannot be removed without withdrawing players associated with them. Our principle of independence permits withdrawing such players when necessary.

We first show that if a solution is coalitionally unanimous and Maskin invariant, then it is a subsolution of the core. Then, we prove that if a subsolution of the core is non-empty and satisfies independence of irrelevant states, then it is the core, provided the class of e-form games satisfies a new property, called the holdover property, which plays a key role in the proof of this statement. This property echoes the conventional "ancestor property" and specifies that, given a state a in the core of an e-form game, it is always possible to introduce additional states (and their associated players when necessary) in such a way that the core of the new augmented e-form game only contains state a. This methodology constitutes an alternative to the use of the so-called bracing lemma, which is a typical consistency result for many game-theoretic models [14]. The complementarity of these two approaches is highlighted in the paper.

Using the building blocks leading up to our axiomatic characterization of the core, we provide new axiomatic characterizations of the Condorcet winner correspondence, the Nash equilibrium correspondence, and the pairwise stability correspondence. This mainly consists in reformulating our general axioms for specific classes of e-form games underlying these stability concepts and showing that these classes satisfy the holdover property. The Condorcet winner has recently been characterized by [6] with axioms different from ours. As far as we know, the pairwise stability correspondence has never been characterized axiomatically before. Our characterization of the Nash equilibrium correspondence is compared with the existing ones proposed by [9] and [10], allowing two other axiomatic characterizations to be established. By invoking a consistency principle instead of an independence principle and applying a bracing lemma in the framework of e-form games, we provide a second axiomatic characterization of the core and apply it in the context of the stable matchings.

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References

1. Abdou, J. and H. Keiding (2003). On necessary and sufficient conditions for solvability of game forms. *Mathematical Social Sciences 46*(3), 243–260.
2. Arrow, K. J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy 58*(4), 328–346.
3. Chernoff, H. (1954). Rational selection of decision functions. *Econometrica 22*(4), 422–443.
4. Chew, M. S.-Y. (1994). Farsighted coalitional stability. *Journal of Economic Theory 63*(2), 299–325.
5. Greenberg, J. (1990). *The Theory of Social Situations: An Alternative Game-Theoretic Approach*. Cambridge University Press.
6. Horan, S., M. J. Osborne, and M. R. Sanver (2018). Positively responsive collective choice rules and majority rule: a generalization of May's theorem to many alternatives. *International Economic Review* .
7. Moulin, H. and B. Peleg (1982). Cores of effectivity functions and implementation theory. *Journal of Mathematical Economics 10*(1), 115–145.
8. Nash, J. F. (1950). The bargaining problem. *Econometrica 18*(2), 155–162.
9. Peleg, B. and S. Tijs (1996). The consistency principle for games in strategic form. *International Journal of Game Theory 25*(1), 13–34.

10. Ray, I. (2000). Consistency, dummy and independence of irrelevant strategies for strategic games. *Social Choice and Welfare 17*(2), 217–222.
11. Rosenthal, R. W. (1972). Cooperative Games in Effectiveness Form. *Journal of Economic Theory 5*(1), 88–101.
12. Sen, A. (1969). Quasi-transitivity, rational choice and collective decisions. *The Review of Economic Studies 36*(3), 381–393.
13. Sen, A. (1993). Internal consistency of choice. *Econometrica 61*(3), 495–521.
14. Thomson, W. (2011). Consistency and its converse: an introduction. *Review of Economic Design 15*(4), 257–291.

Size-based super level measures and the Choquet integral

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Abstract. In the contribution we concentrate on horizontal approach to integration with level measures as a crucial point in the definitions of such family of integrals. We discuss more general concept of level measures based on sizes, recently introduced in [1] and we present its relation to standard level measures. The potential application of this concept lies for instance in decision making processes.

Many well-known integrals are defined through the so-called (super) level measures $\mu(\{x \in X : f(x) > \alpha\})$, which is shortly abbreviated by $\mu(f > \alpha)$. Y. Do and C. Thiele [1] introduced a new generalized concept of level measure $\mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha)$ based on the notion of a size s that provides a new generalization of level measure concept and integrals, as well. From another point of view the subadditivity property of size was the motivation for a generalization of sublinear means, see [3, Definition 3], and led to disscussion about the subadditiveness of aggregation functions, nonadditive measures and integrals.

Let X be a topological space, $\mathbf{E}_{\mathrm{B}}$ the σ-algebra of Borel subsets of X and $\mathscr{B}(X)$ the set of all complex-valued Borel-measurable functions on X. Couple $(X, \mathbf{E}_{\mathrm{B}})$ will be called Borel space.

Definition 1. *Let $(X, \mathbf{E}_{\mathrm{B}})$ be a Borel space. A size is a map*

$$\mathsf{s} : \mathscr{B}(X) \to [0, +\infty]^{\mathbf{E}_{\mathrm{B}}}$$

such that for any $f, g \in \mathscr{B}(X)$ and $E \in \mathbf{E}_{\mathrm{B}}$ it holds

(i) *if $|f| \leq |g|$, then $\mathsf{s}(f)(E) \leq \mathsf{s}(g)(E)$;*
(ii) *$\mathsf{s}(\lambda f)(E) = |\lambda|\, \mathsf{s}(f)(E)$ for each $\lambda \in \mathbb{C}$;*
(iii) *$\mathsf{s}(f+g)(E) \leq C_{\mathsf{s}}\big(\mathsf{s}(f)(E) + \mathsf{s}(g)(E)\big)$ for some fixed $C_{\mathsf{s}} \geq 1$ depending only on s.*

Modifying the standard super level sets $\{x \in X : f(x) > \alpha\}$ for a function f, Y. Do and C. Thiele [1] get a new quantity called super level measure. In the following we present its definition. We are considering a subcollection $\mathbf{E} \subset \mathbf{E}_{\mathrm{B}}$ and suppose μ is a monotone set function on $\mathbf{E}_{\mathrm{B}}$, i.e. $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$ with the condition $m(\emptyset) = 0$ and $\mu(A) \leq \mu(B)$ for all $A, B \in \mathbf{E}_{\mathrm{B}}, A \subseteq B$. The triple $(X, \mathbf{E}, \mathsf{s})$ will be called a sub-Borel size space.

Definition 2. Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space. The quantity

$$\mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha) := \inf\left\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}, \ \sup_{b\in\mathbf{E}} \mathsf{s}(f\mathbf{1}_{X\setminus a})(b) \leq \alpha\right\}, \quad \alpha > 0,$$

is called a *super level measure* of $f \in \mathscr{B}(X)$ with respect to monotone measure μ on X, size s and subcollection $\mathbf{E}$.

In [2], there is an example of a sub-Borel size space $(X, \mathbf{E}, \mathsf{s})$ and a monotone measure μ on X with corresponding super level measure $\mu(s(f)\langle\mathbf{E}\rangle > \alpha)$ different from any standard super level measure $\nu(\{x \in X : \ g(x) > \alpha\})$. However, we shall show that this is not the case when one may consider measures on a new space Y. In fact, a hyperspace $\mathbf{E}_{\mathrm{B}}$ takes the role of a new space Y. Thus, a monotone measure μ on X is transformed to a monotone measure m_μ on $\mathbf{E}_{\mathrm{B}}$, which is defined for any $F \subseteq \mathbf{E}_{\mathrm{B}}$ by

$$m_\mu(F) := \inf\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}} \setminus F\}.$$

Furthermore, a transformation of a size s and a function $f \in \mathscr{B}(X)$ with respect to a collection $\mathbf{E}$ is a function $\mathbf{t}_f : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$ defined for any $a \in \mathbf{E}_{\mathrm{B}}$ by

$$\mathbf{t}_f(a) := \sup_{b\in\mathbf{E}} \mathsf{s}(f\mathbf{1}_{X\setminus a})(b).$$

Theorem 1 (Representation theorem). *Let $(X, \mathbf{E}, \mathsf{s})$ be a sub-Borel size space, μ being a monotone measure on X. Then for every $f \in \mathscr{B}(X)$ we have*

$$\mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha) = m_\mu(\{a \in \mathbf{E}_{\mathrm{B}} : \ \mathbf{t}_f(a) > \alpha\}).$$

Following our previous work [2] we will study non-additive integrals based on super level measures that were introduced recently in [1]. Especially, we focus on the Choquet integral based on super level measures given by the formula

$$\mathbf{I}_{\mathrm{Ch}}(\mu, \mathsf{s}, \mathbf{E}, f) := \int_0^\infty \mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha)\,\mathrm{d}\alpha.$$

Under some conditions (see [2, Proposition 5.4]), the previous formula can be rewritten as the clasical Choquet integral with respect to standard level measure. As the classical Choquet integral on discrete set (together with others well-known integrals) it is useful in decision making processes, one can find the interpretation of our more general concept in this area. The benefit of size-based integrals lies in the fact that they take into account more interactions between elements on basic set. This is related to the fact that while level measure may achieve at most $card(X)$ different values, super level measure can achieve up to $2^{card(X)}$ values.

Infimum is computed with respect to the set $\{\mu(a) : \ a \in \mathbf{E}_{\mathrm{B}}\}$, i.e., $\inf \emptyset = \max\{\mu(a) : a \in \mathbf{E}_{\mathrm{B}}\}$ if it exists, otherwise $\inf \emptyset = +\infty$. In discrete space, minimum of empty set is infimum.

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References

1. Do, Y., Thiele, C.: L^p theory for outer measures and two themes of Lennart Carleson united. *Bull. Amer. Math. Sci.* **52**(2) (2015), 249–296.
2. Halčinová, L., Hutník, O., Kiseľák, J., Šupina, J.: Beyond the scope of super level measures, Fuzzy Sets and Systems, https://doi.org/10.1016/j.fss.2018.03.007.
3. Pap, E.: Sublinear means. In: 36th Linz Seminar on Fuzzy Sets Theory: Functional Equations and Inequalities, Linz, 2 - 6 February 2016, 75-88.

The computation of suprema of fuzzy sets and its application to stochastic programming

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Let X be the set of *decision variables*, $X \xrightarrow{f} [0,1]$ be a map called *fuzzy subset* of X, and let $X \xrightarrow{z} [0,+\infty]$ be an *object function*. We ask the following

Question. Find the *supremum* of the *restriction of z to f* — i.e. the *join* of the image of f under z.

Problems of this type arise in *stochastic linear programming* where f is given by the *probability distribution of a random set* determined by stochastic restrictions (e.g. uncertainty in agriculture expressed by weather conditions).

A *solution* can be obtained provided we understand the nonnegative extended real line $[0,+\infty]$ as a *right $\mathfrak{Q}$-module* w.r.t. a unital quantale on $[0,1]$ — i.e. $\mathfrak{Q} = ([0,1], *, e)$ (cf. [1]).

Let $[0,1]^{op}$ be the real unit interval provided with the dual order. Since

$$[0,+\infty] \xrightarrow{\exp(-\_)} [0,1]^{op}$$

is an order isomorphism, on $[0,1]^{op}$ we consider the *right action* given by the *right implication* $\searrow$ on $[0,1]$. Then $[0,+\infty]$ is a right $\mathfrak{Q}$-module w.r.t. the *right action* $\boxdot$ defined by:

$$x \boxdot \alpha = -\ln\bigl(\alpha \searrow \exp(-x)\bigr), \qquad x \in [0,+\infty],\ \alpha \in [0,1].$$

Hence the *supremum of z restricted to f* has the form:

$$\sup_{x\in X} z(x) \boxdot f(x)$$

and is attained at some decision variable x_0 under certain continuity assumptions.

Special Cases (a) If we consider the Łukasiewicz arithmetic conjunction $*$ on $[0,1]$, then the supremum has the form:

$$\sup_{x\in X} -\ln\bigl(\min\bigl(1-f(x)+\exp(-z(x)),1\bigr)\bigr).$$

(b) If we consider $* = \min$ on $[0,1]$, then the supremum has the form

$$\sup\{z(x) \mid x \in X \text{ and } -\ln(f(x)) < z(x)\}.$$

References

1. P. Eklund, J. Gutiérrez García, U. Höhle, J. Kortelainen, Semigroups in Complete Lattices: Quantales, Modules and Related Topics, Developments in Mathematics, vol. 54, Springer International Publishing 2018.

Several integrals related to scientific impact problem

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Abstract. We describe new functionals which deserve to be called an integral and relate them with the measuring of scientific productivity and impact. The first one is based on an iteration of Sugeno integral and provides a generalization of upper and lower 2-h-index introduced in [8]. The second integral is a variant of Sugeno integral with respect to super level measure – a recent concept introduced and studied in [4].

1 Introduction

The idea of applying citation analysis in scientific quality control is quite old (in fact, going back to 1920's). Citations reflect the intensiveness of information use and may be conceived as manifestations of papers' recognition among the scientific community. The need for assessment, ranking, or just indication of prominent individual authors appear in many contexts, e.g. in research policy, funding, and scientometrics. Such a process classically bases on a proper aggregation of the citations number received by author's publications. Thus, it uses some kind of combination of citations to obtain a single numeric value which is representative (in some sense) for the whole input.

The most popular citation index is the Hirsch index [5] taking into account the quality of individual papers as well as their number. The resulting *h-index* is a symmetric, integer-valued function monotonic with respect to each aggregated variable, and also with respect to the length of the input vector. More precisely, let

$$\mathcal{S} = \{(x_1, \dots, x_n);\ n \in \mathbb{N}, x_i \in \mathbb{N}_0, x_1 \geq x_2 \geq \dots x_n\}.$$

In scientometrics, the value x_i usually represents the number of citations of the ith most cited paper of a scientist represented by the citation sequence $\mathbf{x}$. The h-index is a function $\mathsf{H} : \mathcal{S} \to \mathbb{R}$ such that

$$\mathsf{H}(x_1, \dots, x_n) = \begin{cases} \max\{h = 1, \dots, n; x_h \geq h\}, & x_1 \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Equivalently, $\mathsf{H}(x_1, \dots, x_n) = \max\{\min\{x_1, 1\}, \dots, \min\{x_n, n\}\}$. Many properties of such aggregation function is already known. Torra and Narukawa in [9] showed that the h-index is a special case of the discrete Sugeno integral with respect to the counting measure $\mu_{\#}$.

Despite the great success of h-index in various contexts where there is a need to combine quality and quantity of agents represented by non-negative numeric lists into a single number, h-index has some unpleasant defects. For instance, it does not differentiate between papers with no citations and non-existing papers, also h-index is insensitive to a large number of papers with relatively small number of citations, etc., see e.g. [8]. Several approaches towards how to compensate some of the defects of the h-index without neglecting its "spirit" are known in the literature.

In this contribution we aim to describe the following two new approaches based on the integral representation of h-index (as the discrete Sugeno integral):

(i) motivated by the work [8] (as well as other works on indices) we extend the upper 2-h-index and lower 2-h-index defined therein to the general case of upper n-Sugeno integral and lower n-Sugeno integral, respectively;
(ii) motivated by the research started in [4] we discuss the size-based Sugeno integral defined via the super level measure concept.

Both approaches are briefly described in what follows. First, we introduce some notation: $(\Omega, \mathcal{A})$ is a measurable space with $\mathcal{A}$ being a σ-algebra of subsets of a non-empty set Ω. The class of all $\mathcal{A}$-measurable functions $f\colon \Omega \to Y$, where $Y = [0, \bar{y}]$ for $0 < \bar{y} \leqslant +\infty$, is denoted by $\mathcal{F}_{(\Omega,Y)}$. Usually, we take $\bar{y} = 1$ or $\bar{y} = +\infty$. A *monotone measure* on $\mathcal{A}$ is a nondecreasing set function $\mu\colon \mathcal{A} \to [0, +\infty]$, i.e., $\mu(A) \leqslant \mu(B)$ whenever $A \subset B$ with $\mu(\emptyset) = 0$ and $\mu(\Omega) > 0$. The range of μ we write as $\mu(\mathcal{A})$. and the class of all monotone measures on $(\Omega, \mathcal{A})$ we denote by $\mathcal{M}_{(\Omega,\mathcal{A})}$. Hereafter, $a \wedge b = \min(a, b)$ and $a \vee b = \max(a, b)$.

2 Iterated Sugeno integrals

We say that $\circ\colon Y \times Y \to Y$ is an *admissible fusion map* if it is nondecreasing in each variable and $0 \circ a \leqslant a$ for all $a \in Y$. The most important examples are: the addition, multiplication, conjunctive and disjunctive averaging aggregation, semicopula, copula, t-norm, t-semiconorm and fuzzy conjunction. The iteration process is then defined as follows.

Definition 1. *Let $\circ$ be an admissible fusion map, $(\mu, f) \in \mathcal{M}_{(\Omega,\mathcal{A})} \times \mathcal{F}_{(\Omega,Y)}$, and $n \geqslant 1$. The upper n-Sugeno integral is defined using the recurrence*

$$\mathrm{Su}^{\circ}_{n+1}(\mu, f) := \sup_{t \in Y} \big\{ (t \circ \mathrm{Su}^{\circ}_{n}(\mu, f)) \wedge \mu(\{f \geqslant t\}) \big\},$$

with the initial condition $\mathrm{Su}^{\circ}_{1}(\mu, f) := \mathrm{Su}(\mu, f)$, the standard Sugeno integral of f wrt μ.

One can show that the upper n-Sugeno integral has several properties of the Sugeno integral. We study the basic properties and provide sufficient and necessary conditions for validity of some of them (e.g. minitive/maxitive comonotonicity, subadditivity, etc.).

A natural counterpart is the lower n-Sugeno integral defined as follows. We say that $\star\colon [0, +\infty] \times Y \to [0, +\infty]$ is a *link map* if it is nondecreasing in each variable and $0 \star a \leqslant a$ for all $a \in Y$.

Definition 2. *Let $\star$ be a link map and $(\mu, f) \in (\mathcal{M}_{(\Omega,\mathcal{A})}, \mathcal{F}_{(\Omega,Y)})$ and $n \geqslant 1$. The lower n-Sugeno integral is defined by*

$$\mathrm{Su}_{\star}^{n}(\mu, f) := \sup_{t \in Y} \big\{ t \wedge \big(\mu(\{f \geqslant t\}) \star \mathrm{Su}_{\star}^{n-1}(\mu, f)\big) \big\},$$

where $\mathrm{Su}_{\star}^{1}(\mu, f) := \mathrm{Su}(\mu, f)$.

Again, the lower n-Sugeno integral possesses some basic properties of the Sugeno integral. Interestingly, we can use them to compute certain pseudo-decomposition integrals, however, we do not include these results here. We demonstrate that the upper/lower 2-Sugeno integrals generalize some known scientometric indices:

(i) Evidently, $\mathrm{Su}_{2}^{+}(\cdot)$ is the upper 2-h-index

$$\mathsf{H}_2^u = \max_k \big\{ (k + \mathsf{H}) \wedge x_k \big\},$$

if $Y = [0, +\infty]$ and $\mu = \mu_{\#}$ is the counting measure. Here, H is the h-index of the citation sequence $\mathbf{x} \in \mathcal{S}$. Clearly, $\mathrm{Su}_{\star}^{2}(f)$ is equal to the lower 2-h-index, cf. [8],

$$\mathsf{H}_2^l = \max_k \big\{ k \wedge (x_k + \mathsf{H}) \big\}.$$

(ii) We assume $a \circ b = \lambda a$, where $Y = [0, +\infty]$ and $\lambda > 0$ is a fixed constant. Then the upper 2-Sugeno integral is equal to h_λ-index introduced by Van Eck [10].

(iii) The map $a \circ_\gamma b = a^\gamma$ for $Y = [0, 1]$ or $Y = [0, +\infty]$ leads to the index proposed by Lehmann et al [7] for $\gamma > 0$ and provide the $h(2)$-index defined by Kosmulski [6] for $\gamma = 2$ in the form

$$\mathrm{Su}_2^{\circ_\gamma}(a \mathbf{1}_A) = a^p \wedge \mu(A).$$

(iv) Functional defined $\mathrm{Su}_{+}^{\infty}(f) := \sup_n \mathrm{Su}_{+}^{n}(f)$ gives the number of publications with at least one citation.

3 Super level measure-based Sugeno integrals

Further extension of scientometric indices can be done using the concept of integrals based on super level measures. In this concept the integrated function is modified by a certain mapping called a *size*. For that reason we consider the Borel algebra $\mathbf{E}_{\mathrm{B}}$ of sets of Ω, the set $\mathcal{B}(\Omega)$ of all real-valued Borel-measurable functions on Ω and a collection $\mathbf{E}$ of subsets of X.

Definition 3. *A size is a map $\mathsf{s} : \mathcal{B}(\Omega) \to [0, +\infty]^{\mathbf{E}_{\mathrm{B}}}$ such that for any $f, g \in \mathcal{B}(\Omega)$ and $E \in \mathbf{E}_{\mathrm{B}}$ it holds*

(i) if $|f| \leq |g|$, then $\mathsf{s}(f)(E) \leq \mathsf{s}(g)(E)$;
(ii) $\mathsf{s}(\lambda f)(E) = |\lambda|\, \mathsf{s}(f)(E)$ for each $\lambda \in \mathbb{R}$;
(iii) $\mathsf{s}(f + g)(E) \leq C_{\mathsf{s}}\big(\mathsf{s}(f)(E) + \mathsf{s}(g)(E)\big)$ for some $C_{\mathsf{s}} \geq 1$ depending only on s.

Thus, the size may be viewed as a kind of an aggregation operator. The most commonly used sizes are the supremum size $\mathsf{s}_\infty(f)(E) = \sup_{x\in E}|f(x)|$ and L^p-based size

$$\mathsf{s}^{(\mathrm{L})}_{\Lambda,p}(f)(E) := \left(\frac{1}{\Lambda(E)}\int_E |f(x)|^p \,\mathrm{d}\Lambda(x)\right)^{\frac{1}{p}}, \quad p > 0,$$

with Λ being the Lebesgue measure. However, non-additive integrals can also be used as sizes, see [4]. The main ingredient of the theory is the super level measure concept (as a generalization of level measure).

Definition 4. *The quantity*

$$\mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha) := \inf\left\{\mu(F):\ F \in \mathbf{E}_{\mathrm{B}},\ \sup_{E\in\mathbf{E}} \mathsf{s}(f\mathbf{1}_{\Omega\setminus F})(E) \le \alpha\right\}, \quad \alpha > 0,$$

is called a super level measure of $f \in \mathcal{B}(\Omega)$ *with respect to a monotone measure* $\mu : \mathbf{E}_{\mathrm{B}} \to [0, +\infty]$*, a size* s *and a collection* $\mathbf{E}$*.*

Super level measure can coincide with the measure of the upper level set. For instance, this is the situation of s_∞ as well as $\mathsf{s}^{(\mathrm{L})}_{\Lambda,p}$. In general, they differ, e.g. for the size generated by non-additive integrals of Shilkret and Choquet. The size-based Sugeno integral is then defined in a standard way, i.e.,

$$\mathbf{I}_{\mathsf{Su}}(\mu, \mathsf{s}, f, \mathbf{E}) := \sup_{\alpha>0}\ \alpha \wedge \mu(\mathsf{s}(f)\langle\mathbf{E}\rangle > \alpha).$$

Considering various sizes and collections we can get interesting functionals. We study certain concrete cases and their interpretation.

A computation of super level measure (the most important part in integral computation) is a challenging problem. However, it seems possible to provide a simple algorithm for super level measure computation in some specific situations, for instance on a discrete universe X, see the recent paper [2]. Since scientometric indices are exactly this case, we mention a modification of the above approach.

Example 1. Let $\Omega = \{1, \ldots, n\}$ with the counting measure $\mu_\#$. For a vector $\mathbf{x} = (x_1, \ldots, x_n)$ consider a size-transform $\mathbf{X}$ of $\mathbf{x}$ of the form $X_i = \mathsf{s}(\mathbf{x}\mathbf{1}_{E_i})(\Omega)$ with $i = 1, 2, \ldots, n$ and collection $\mathbf{E} = \{E_i;\, i = 1, 2, \ldots, n\}$ with $E_i = \{1, 2, \ldots, i\}$. Then the size-based Sugeno integral takes the form

$$\mathbf{I}_{\mathsf{Su}}(\mu_\#, \mathsf{s}, \mathbf{x}, \mathbf{E}) = \sup_{\alpha>0} \alpha \wedge \mu_\#(\{i \in X; X_i \ge \alpha\}),$$

which corresponds to the standard Sugeno integral of $\mathbf{X}$ wrt $\mu_\#$. Regarding a scientometric interpretation, size and collection can bring a new information (requirement), e.g. weighting the citation input by quality of journals in which the corresponding citation appears, etc. We discuss these issues in our presentation.

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References

1. Boczek, M., Hovana, A., Hutník, O., Kaluszka, M.: New monotone measure-based integrals inspired by scientific impact problem. Preprint (2018).
2. Borzová, J., Halčinová, L., Šupina, J.: Size-based super level measures on discrete space. IPMU tba...
3. Gagolewski, M., Mesiar, R.: Monotone measures and universal integrals in a uniform framework for the scientific impact assessment problem. *Inform. Sci.* **263** (2014), 166–174.
4. Halčinová, L., Hutník, O., Kiseľák, J., Šupina, J.: Beyond the scope of super level measures. *Fuzzy Sets and Systems* (2018), https://doi.org/10.1016/j.fss.2018.03.007.
5. Hirsch, J.E.: An index to quantify an individual's scientific research output. *Proc. Nat. Acad. Sci. USA* **102** (2005), 16569–16572.
6. Kosmulski, M.: A new Hirsch-index saves time and works equally well as the origin h-index. *ISSSI Newsletter* **2** (2006), 4–6.
7. Lehmann, S., Jackson, A., Lautrup, B.: A quantitative analysis of indicators of scientific performance. *Scientometrics* **76** (2008), 369–390.
8. Mesiar, R., Gagolewski, M.: H-index and other Sugeno integrals: some defects and their compensation. *IEEE Trans. Fuzzy Systems* **24** (2016), 1668–1672.
9. Torra, V., Narukawa, Y.: The h-index and the number of citations: Two fuzzy integrals. *IEEE Trans. Fuzzy Systems* **16**(3) (2008), 795-–797.
10. Van Eck, N.J., Waltman, L.: Generalizing the h-and g-indices. *J. Informetr.* **2** (2008), 263–271.

The facts about strict and weak C-universal integrals

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Abstrakt We focus on the specific class of universal integrals based on copulas, which resembles mean value of continuous random variable. We point out the important property of copulas, which has essential effect on integration. We illustrate it on examples.

1 Introduction

Copulas are mathematical objects that fully capture the dependence structure among random variables. For comprehensive summary see [1]. They can be helpful in the filed of multicriterial decision making. Therefore, from current constructions of universal integrals based on copulas[3], see [3,4], we will focus on the following specific class, the form of which resembles mean value of continuous random variable ξ with support $[0,1]$

$$\mathbb{E}[X] = \int_0^1 t f_\xi(t)\,\mathrm{d}t = \int_0^1 P(\xi \geq t)\,\mathrm{d}t.$$

Similarly, the mean value of an integrable function $g : [0,1] \to [0,1]$ is given by

$$\mathrm{E}[g] = \int_0^1 \lambda(g \geq t)\,\mathrm{d}t = \int_0^1 g(t)\,\mathrm{d}\lambda(t),$$

where $\lambda(g \geq t) := \lambda(\{x \in [0,1] : g(x) \geq t\})$ and λ denotes Lebesgue measure.

In further, we assume that $X \neq \varnothing$ is a set (a universe), $\mathcal{A}$ is a σ-algebra in X and $m : \mathcal{A} \to [0,1]$ is a monotone measure (non-additive, in general) such that

[3] Original definition of a bivariate copula in probabilistic terms is stated as follows: $C : [0,1]^2 \to [0,1]$ is a 2-dimensional copula if C is a joint cdf of a 2-dimensional random vector on $[0,1]^2$ with uniform marginals. Notice that in analytic terms it can be equivalently defined by

(i) $C(0,x) = C(x,0) = 0$, (groundedness);
(ii) $C(1,x) = C(x,1) = x$, (normalized marginals - neutral element);
(iii) $(\forall a,b,c,d \in [0,1])(a \leq b,\, c \leq d)\quad C(b,d) - C(a,d) - C(b,c) + C(a,c) \geq 0$, (2-increasingness or rectangle inequality).

$m(\emptyset) = 0,\ m(X) = 1$, and $f : X \to [0,1]$, representing the input to be aggregated, is $\mathcal{A}$-measurable function. Previous ideas force us to define distribution function $h_{m,f} : [0,1] \to [0,1]$ of a $\mathcal{A}$-measurable function f, as $h_{m,f}(t) := m(\{x \in X : f(x) \geq t\})$. Obviously, $h_{m,f}$ is non-increasing (therefore Borel measurable) and satisfies $h_{m,f}(0) = 1$. We define also sets

$$\varDelta^-_{h_{m,f}} := \{(t,y) \in [0,1]^2; y < h_{m,f}(t)\}$$

and

$$\varDelta^+_{h_{m,f}} := \{(t,y) \in [0,1]^2; y \leq h_{m,f}(t)\}.$$

Very important fact for our purposes is a bijection[4] between the space of copulas and space of specific[5] probability measures. Its immediate consequence is that the two probability measures are different (and, therefore, incomparable) whenever associated copulas are different.

Since independence copula $\varPi(x,y) = xy$ generates two-dimensional Lebesgue measure $P_\varPi = \lambda_2$, one has

$$\mathrm{E}[g] = P_\varPi\left(\varDelta^+_{h_{\lambda,g}}\right) = \lambda_2\left(\varDelta^+_{h_{\lambda,g}}\right) = \iint_{\varDelta^+_{h_{\lambda,g}}} \mathrm{d}\lambda_2,$$

i.e., the Lebesgue measure of the hypograph of the map $h_{\lambda,g}$. Motivated by this observation we may consider $\mathrm{E}_{C,m}[f] = P_C\left(\varDelta^+_{h_{m,f}}\right)$ with arbitrary copula C and measure m, whereas clearly $\varDelta^+_{h_{m,f}}$ is Borel measurable for all m and f.

Definition 1. *Let C be a bivariate copula. Mappings $\boldsymbol{K}^-_C, \boldsymbol{K}^+_C$ given by*

$$\boldsymbol{K}^-_C(m,f) := P_C\left(\varDelta^-_{h_{m,f}}\right),$$
$$\boldsymbol{K}^+_C(m,f) := P_C\left(\varDelta^+_{h_{m,f}}\right),$$

are called a strict and weak C-universal integral on $[0,1]$.

This approach was already given by [2] as generalized fuzzy integral. Notice that by choosing adequate copulas C, we obtain some well-known types of integrals. Here E is arbitrary Borel subset of $[0,1]^2$. For copula $\varPi$ one obtains Choquet integral $\int_0^1 h_{m,f}(t)\,\mathrm{d}t$. Moreover, for minimum copula $M(x,y) = \min(x,y)$ we have $P_M(E) = \lambda(\{x \in [0,1] : (x,x) \in E\})$, which thanks to monotonicity of $h_{m,f}$ implies Sugeno integral $\sup_{t\in[0,1]} \min(t, h_{m,f}(t))$. If C equals the Lukasiewicz copula $W(x,y) = \max(x+y-1,0)$ it reduces to opposite-Sugeno integral with $P_W(E) = \lambda(\{x \in [0,1] : (x, 1-x) \in E\})$.

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[4] Every bivariate copula C induces a doubly stochastic measure P_C on the measurable space $([0,1]^2, \mathcal{B}([0,1]^2))$ defined on the rectangles $R :=]u_1, v_1[\times]u_2, v_2[$ contained in $[0,1]^2$, by $P_C(R) := V_C(R) = C(v_1,v_2) - C(u_1,v_2) - C(v_1,u_2) + C(u_1,u_2)$. Conversely, to every doubly stochastic measure μ on $([0,1]^2, \mathcal{B}([0,1]^2))$ there corresponds a unique bivariate copula C defined by $C(u,v) := \mu(]0,u[\times]0,v[)$.

[5] A measure μ on $([0,1]^2, \mathcal{B}([0,1]^2))$ is said to be *doubly stochastic* if $\mu([0,1] \times A) = \mu(A \times [0,1]) = \lambda(A)$ for each $A \in \mathcal{B}([0,1])$. In other words, the image measure of μ under any projection equals the Lebesgue measure on the Borel sets of $[0,1]$.

2 Basic properties

Clearly we have $\mathbf{K}_C^-(m,f) = \iint_{\Delta^-_{h_{m,f}}} \mathrm{d}C$ and also $\mathbf{K}_C^+(m,f) = \iint_{\Delta^+_{h_{m,f}}} \mathrm{d}C$. However, this formulas can not be used for direct computation very often. This is because not all copulas are absolutely continuous[6]. If a copula $C : [0,1]^2 \to [0,1]$ is absolutely continuous, then there exists a density function (suitable integrable function) $c : [0,1]^2 \to [0,+\infty)$ such that $C(x,y) = \int_0^x \int_0^y c(u,v)\,\mathrm{d}u\mathrm{d}v$, where $c(u,v) = \frac{\partial^2}{\partial u \partial v} C(u,v)$ holds for almost all $(u,v) \in [0,1]^2$. One can conclude that for absolutely continuous copulas $\mathbf{K}_C^{\pm}(m,f)$ equals

$$\int_0^1 \frac{\partial C}{\partial t}(t,y)|_{y=h_{m,f}(t)}\,\mathrm{d}t, \tag{1}$$

i.e. equals conditional probability $\Pr(Y \le h_{m,f}(X)|X = x)$. We see that for absolutely continuous copula (1) is uniquely defined. Natural question is, when the integrals $\mathbf{K}_C^{\pm}(m,f)$ equals (1), i.e. equals integral over conditional probability? It can not be true in general. Notice that for Borel measurable function h it can happen that function $\frac{\partial C}{\partial t}(\cdot, h_{m,f}(\cdot))$, (as a function of one variable $\cdot$) changes on the set which is not of measure zero in $[0,1]$. This means that (1) can return more than one value. Partial answer was given by [4, Proposition 4], where sufficient and necessary condition says that the support of singular part of copula consists only graphs of monotone increasing functions.

Example 1. For Lukasiewicz copula W and fixed $f(x) = 1-x,\ x \in [0,1]$ it is not possible to use formula (1) directly. One can show that $\mathbf{K}_C^+(m,f)1 \neq 0 = \mathbf{K}_C^-(m,f)$.

The key role in previous example is that the support[7] of the singular part contributes considerably to the integration because of the intersection with the graph of distribution function $h_{m,f}$. Another example, see [1], is copula $Ci(u,v) = \min\left(u, v, \frac{u^2+v^2}{2}\right)$ is singular and its support consists of the two quarter circles in $[0,1]^2$ (each with radius 1, centered at $(1,0)$ and $(0,1)$), see Figure 1.

Naturally if this is not true, then one can expect that both integrals coincide. This illustrates next example.

Example 2. For copula Ci we have

$$\mathbf{K}_{Ci}^+(m,f) = \mathbf{K}_{Ci}^-(m,f) = \int_\Omega 1\,\mathrm{d}t + \int_V t\,\mathrm{d}t = \lambda(\Omega) + \lambda_2(V_2),$$

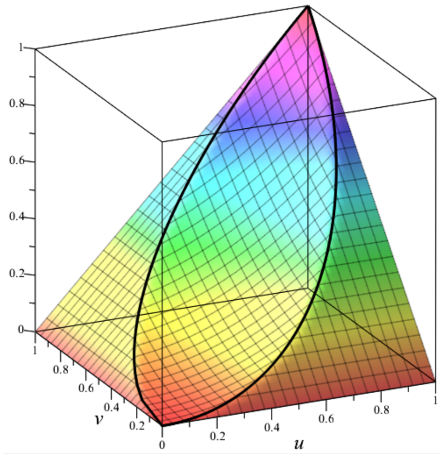
where

$$\Omega = \left\{ t \in [0,1] : h_{m,f}(t) > \sqrt{t(2-t)} \right\},$$

$$V = \left\{ t \in [0,1] : \sqrt{t(2-t)} < h_{m,f}(t) < 1 - \sqrt{1-t^2} \right\}$$

[6] A copula C will be called *absolutely continuous (respectively, singular)* if so is the measure P_C induced by C with respect to λ_2, i.e., if $P_C = \mu_C^{ac}$ (respectively, $P_C = \mu_C^s$).

[7] The support of copula C is the support (or spectrum)of the doubly stochastic measure P_C induced by C on $([0,1]^2, \mathcal{B}([0,1]^2))$, i.e., the largest (closed) subset of $[0,1]^2$ for which every open neighbourhood of every point of the set has positive measure.



Obr. 1. Copula Ci with singular support.

and $V_2 = \left\{(t,u) \in [0,1]^2 : t \in V, 0 \le u \le t\right\}$. E.g. for $f(t) = 1 - t$ and $m = \lambda$ we have $\mathbf{K}^{+}_{Ci}(\lambda, 1-t) = \left(1 - \frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right) = \frac{1}{2}$ and the same result holds for $\mathbf{K}^{+}_{Ci}(\lambda, 1-t)$.

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Literatúra

1. Durante, F., Sempi, C.: *Principles of Copula Theory*, Chapman and Hall/CRC, 978-1-4398-8442-3, 2015
2. Imaoka, H.: On a subjective evaluation model by a generalized fuzzy integral. Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 5, 5 (1997), 517-529.
3. Klement, E.P., Spizzichino, F., Mesiar, R., Stupňanová, A.: Copula-based universal integrals. In: Proceedings 2013 IEEE International Conference on Fuzzy Systems, Hyderabad, pp. 1065-1069. (2013)
4. Klement, E.P., Mesiar, R., Spizzichino, F., Stupňanová, A.: Universal integrals based on copulas. Preprint (2013).

Asymmetry of copulas arising from shock models

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Abstract. We study the asymmetry (or nonexchangeability) of copulas with emphasis on copulas arising from shock models. The main contribution is the *maximal asymmetry function* for a family of copulas. We compute this function for the major families of shock-based copulas, i.e. Marshall, maxmin and reflected maxmin (RMM for short) copulas and also for some other important families. We also give the statistical interpretation of shocks in a given model at which the maximal asymmetry measure bound is attained.

A very important class of (bivariate) copulas for applications are those arising from shock models: Marshall copulas, maxmin copulas, and reflected maxmin copulas (RMM for short). These copulas have a long history starting with [24] and [23] and going up to [17], say, where an extensive overview of these methods is given together with an appropriate bibliography. A comprehensive list of references of concrete applications of shock-based copulas would be too long to present here, so let us limit ourselves to four of them, relatively recent ones and in quite different fields: [20, 1, 7, 13]. Note that our investigations are not only of a theoretical interest, but also of a practical impact in the construction of statistical models (in a parametric as well as non-parametric context). When choosing the right copula for the data at hand a key point is to distinguish the family that describes the phenomenon behind the data at the best. In this respect, a better choice of the copulas could be obtained when the information about the non-symmetry of the data (measured, for instance, in a nonparametric way by means of the empirical copula) is also taken into account (cf. [6, Section 5] where this point is further discussed).

One of our main contributions is a systematic application of the *maximal asymmetry function* on a given family of copulas. This function is computable for all the families that we are considering and it is helpful not only in determining the sharp bound of measures of asymmetry for a given family, but also in the detailed analysis of shocks in a given model at which the bound is attained. This brings us to another important contribution of ours, the statistical interpretation of shocks in these models. We believe this approach is helpful to practitioners in search of the copulas that will fit their data the best and simultaneously the models they are hoping for. So, when one has measured the asymmetry of the data at hand and established a statistical hypothesis on the studied phenomenon, one can compare using our approach whether the pattern of occurrences of the shocks given the asymmetry allows for the desired conclusion.

Exchangeability is an important concept in probability theory extending the notion of independence. Investigations in this direction were initiated in the 1930's simultaneously with the search for a general axiomatic approach to probability and started with a famous result of de Finetti (cf. [3, 4, 15]) later extended by Hewitt and Savage [12]. A recent result in this area in connection with copulas is given by Mai and Scherer [21], where an interested reader may also find an excellent overview of the subject together with extensive bibliography.

However, our aim is not so much to study exchangeability as the lack of it, a subject that had attracted little attention up to the point when Klement and Mesiar [16] and Nelsen [26] noticed it only a dozen of years ago.

Many classical copulas are symmetric, (sometimes also called *exchangeable* due to reasons given above): Archimedean and meta-elliptical copulas are prime examples. Observe that two of the most important copulas, the *Fréchet-Hoeffding lower bound*, respectively *upper bound*, $W(u,v) = \max\{0, u+v-1\}$, respectively $M(u,v) = \min\{u,v\}$ (being so called due to the fact that $W(u,v) \leq C(u,v) \leq M(u,v)$ for every copula C and all $u, v \in [0,1]$) are symmetric. Also, the independence of two random variables is being modeled via the *product copula* $\Pi(u,v) = uv$ which is also symmetric. So, in view of the classical exchangeability results one might vaguely think of more asymmetric copulas as modeling more dependent relations among random variables.

Copula $C(u,v)$ is called *positive quadrant dependent (PQD for short)* if $\Pi(u,v) \leq C(u,v)$ for all $u, v \in [0,1]$, and it is *negative quadrant dependent (NQD for short)* if $C(u,v) \leq \Pi(u,v)$ for all $u, v \in [0,1]$. We denote by $\mathcal{C}$, respectively $\mathcal{P}$, respectively $\mathcal{N}$, the set of all copulas, respectively PQD copulas, respectively NQD copulas. For any $C \in \mathcal{C}$ we denote by C^t the copula defined by $C^t(u,v) = C(v,u)$ for all $u, v \in [0,1]$ (so that C is symmetric if and only if $C = C^t$). The *maximal asymmetry function* for any particular family of copulas is defined as the point-wise supremum of all possible differences of $|C - C^t|$ when C runs through the given family. Klement and Mesiar [16] were using this notion only on the family $\mathcal{C}$. We refer to monographs [10, 14, 22, 25] for further details on copulas.

In practice dependence is often asymmetric, as data collected from the real world may exhibit. This necessitates developing asymmetric copulas that can model such data and it also urges the study of various measures of asymmetry that may help the practitioners to decide about which copulas to choose in their models according to the data. This line of study of copulas was started by Klement and Mesiar [16] and Nelson [26], as already mentioned. These papers started a vivid interest in the subject. De Baets, De Meyer, and Mesiar [2] present an asymmetric version of semilinear copulas as an asymmetric version of the previously introduced symmetric version of this family. Durante, Klement, Sempi, and Úbeda-Flores [5] introduce a measure of asymmetry μ in general and μ_p for $p \in [1, \infty]$ in particular. It is shown in [16, 26, 11] that $\mu_\infty(C) \leq \frac{1}{3}$ for $C \in \mathcal{C}$ and that the bound is attained so that $\frac{1}{3}$ is the sharp bound of asymmetry measure μ_∞ of the set $\mathcal{C}$. Similarly, the sharp bound of asymmetry measure μ_∞ for the set $\mathcal{P}$ was given in [2] and the sharp bound of asymmetry measure μ_∞ for the set $\mathcal{N}$ was given in [9].

One of our important contributions is the actual computation of the sharp bounds of asymmetry measure μ_∞ for the classes of copulas arising from shock models. Our results show that the maxmin copulas do not seem to be more asymmetric than the Marshall copulas since their maximal asymmetry is equal and equals $\frac{4}{27}$ (≈ 0.148). This is (slightly) less than the maximal asymmetry $3-2\sqrt{2}$ (≈ 0.172) of the family PQD copulas to which they both belong. The maximal asymmetry of the RMM copulas equals $3-2\sqrt{2}$. This is slightly less than $\sqrt{5}-2$ (≈ 0.236), the maximal asymmetry of the family of NQD copulas to which it belongs.

We follow Klement and Mesiar [16] and define function $d^*_{\mathcal{F}} : [0,1]\times[0,1] \to [0,\infty)$ that we call the maximal asymmetry function of the family of copulas $\mathcal{F}$. Its value at a fixed point $(x,y) \in [0,1]\times[0,1]$ is given by

$$d^*_{\mathcal{F}}(x,y) = \sup_{C\in\mathcal{F}} \{|C(x,y) - C(y,x)|\}.$$

Klement and Mesiar showed that $d^*_{\mathcal{C}}(x,y) = \min\{x, y, 1-x, 1-y, |x-y|\}$. We show that the maximal asymmetry function of the family $\mathcal{P}$ of PQD copulas is equal to $d^*_{\mathcal{P}}(x,y) = \min\{x(1-y), (1-x)y, |x-y|\}$.

Furthermore, the family of copulas $P_\lambda(x,y) = \max\{M(x, y-\lambda), xy\}$ for $\lambda \in [0,1]$ is such that

$$d^*_{\mathcal{P}}(x,y) = |P_\lambda(x,y) - P_\lambda(y,x)|$$

for all $x, y \in [0,1]$ with $|x-y| = \lambda$.

Maximal asymmetry function of the family $\mathcal{N}$ of NQD copulas is equal to $d^*_{\mathcal{N}}(x,y) = \min\{xy, (1-x)(1-y), |x-y|\}$. Furthermore, the family of copulas $N_\lambda(x,y) = \max\{W(x,y), \min\{y-\lambda, xy\}\}$ for $\lambda \in \left[0, \frac{1}{2}\left(3-\sqrt{5}\right)\right]$ is such that

$$d^*_{\mathcal{N}}(x,y) = |N_\lambda(x,y) - N_\lambda(y,x)|$$

for all $x, y \in [0,1]$ with $|x+y-1| = \mu$, where $\mu = \frac{1-3\lambda+\lambda^2}{1-\lambda}$ and $\mu \in [0,1]$.

We give similar results also for the three families of copulas that appear in shock models.

References

1. U. Cherubini, S. Mulinacci, *Systemic risk with exchangeable contagion: application to the European banking system*, ArXiv e-prints, 2015.
2. B. De Baets, H. De Meyer, and R. Mesiar. *Asymmetric semilinear copulas.* Kybernetika **43** (2007), no. 2, 221–233.
3. B. De Finetti. *Funzione caratteristica di un fenomeno aleatorio.* Atti della R. Academia Nazionale dei Lincei, Serie 6. Memorie, Classe di Scienze Fisiche, Mathematice e Naturale, **4**, (1931), 251–299.
4. B. De Finetti. *La prévision: ses lois logiques, ses sources subjectives*. Ann. Inst. H. Poincaré **7** (1937), 1–68.
5. F. Durante, E. P. Klement, C. Sempi, and M. Úbeda-Flores. *Measures of non-exchangeability for bivariate random vectors.* Statist. Papers **51** (2010), no. 3, 687–699.

6. F. Durante, A. Kolesarova, R. Mesiar, and C. Sempi. *Semilinear copulas.* Fuzzy Sets and Systems **159** (2008), 63–76.
7. F. Durante and O. Okhrin. *Estimation procedures for exchangeable Marshall copulas with hydrological application.* Stoch. Environ. Res. Risk Assess., **29** (2015), 205–226.
8. F. Durante, M. Omladič, L. Oražem, and N. Ružić. *Shock models with dependence and asymmetric linkages.* Fuzzy Sets and Systems **323** (2017), 152–168.
9. F. Durante and P. L. Papini. *Non-exchangeability of negatively dependent random variables.* Metrika **71** (2010), no. 2, 139–149.
10. F. Durante and C. Sempi. Principles of Copula Theory. CRC/Chapman & Hall, Boca Raton, 2015.
11. C. Genest and J. G. Nešlehová. *Assessing and modeling asymmetry in bivariate continuous data.* Copulae in mathematical and quantitative finance, Lect. Notes Stat. **213** (2013), 91–114.
12. E. Hewitt, and L.J. Savage. *Symmetric measures on Cartesian products.* Transactions of the American Mathematical Society, **80**, (1955), 470–501.
13. T. E. Huillet, *Stochastic species abundance models involving special copulas*, Phisica A, **490** (2018) 77–91.
14. H. Joe. Dependence Modeling with Copulas. Chapman & Hall/CRC, London, 2014.
15. J.F.C. Kingman. *Uses of exchangeability*. Ann. Probab. **7** (1978) 187–197.
16. E. P. Klement and R. Mesiar. *How non-symmetric can a copula be?* Comment. Math. Univ. Carolin. **47** (2006), no. 1, 141–148.
17. D. Kokol Bukovšek, T. Košir, B. Mojškerc, and M. Omladič. *Asymmetric linkages: Maxmin vs. Reflected Maxmin Copulas.* Submitted. Available at arXiv:1808.07737
18. D. Kokol Bukovšek, T. Košir, B. Mojškerc, and M. Omladič. *Asymmetry of copulas arising from shock models.* Submitted. Available at arXiv:1808.09698
19. T. Košir and M. Omladič. *Reflected maxmin copulas and modelling quadrant subindependence.* Submitted. Available at arXiv:1808.07646
20. F. Lindskog, A. J. McNeil, *Common Poisson Shock Models: Applications to insurance and credit risk modelling*, ASTIN Bulletin, **33**, No. 2, (2003), 209–238.
21. J.-F. Mai and M. Scherer. *H-extendible copulas.* Journal of Multivariate Analysis **110** (2012), 151–160.
22. J.-F. Mai and M. Scherer. Simulating Copulas. 2nd edition, World Scientific, Singapore, 2017.
23. A. W. Marshall. *Copulas, marginals and joint distributions.* Distribution Functions with fixed marginals and related topics, Institute of Mathematical Statistics, Lecture Notes-Monograph Series **28** (1996), 213-222.
24. A. W. Marshall, I. Olkin, *A multivariate exponential distributions*, J. Amer. Stat. Assoc., **62**, (1967), 30–44.
25. R. B. Nelsen. An introduction to copulas. *Second edition.* Springer Series in Statistics. Springer, New York, 2006.
26. R. B. Nelsen. *Extremes of nonexchangeability.* Statist. Papers **48** (2007), no. 2, 329–336.
27. M. Omladič and N. Ružić. *Shock models with recovery option via the maxmin copulas.* Fuzzy Sets and Systems **284** (2016), 113–1289.

Set-valued games: allocation in multi-agenda disputes and decisions

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1 Introduction

Typically a (cooperative) *game* measures the 'worth' of any subgroup of individuals. This value can be interpreted as the productivity of the subgroup under consideration when investing time and effort in a particular enterprise, its social or political power, etc. Often times though there are multiple enterprises that could be invested in while the resources, say time, are constrained. In such scenarios there are many different time allocations to each of the different projects, each of which resulting in a different outcome. It seems that if one wishes to model such tradeoffs without pre-committing to a specific time allocation, and thereby to a specific level of production in each project, one needs to consider a more general notion of a game than the classical ones.

In this paper we introduce and study the concept of *set-valued games*: each subgroup of individuals is associated with a *set of real valued vectors*. The set of vectors associated with a subgroup is *all* production possibilities across the different enterprises. Note that this approach does not take a stand on the aggregation of (or preferences over) payoffs across the different agendas as in existing work on multi-agenda disputes. We seek to study the primitive, as opposed to the reduced form, and wish to consider a model that is robust to the aggregation process. As in the classical theory, we address the issue of allocation. The appropriate notion of the core of a set-valued game is defined and analyzed.

For the sake of clarity of our results, consider a grand group of N individuals that have to invest in k enterprises. A set-valued game (henceforth SVG) is a function $\mathbf{v}$ that associates a subset $\mathbf{v}(S) \subseteq \mathbb{R}_+^k$ for every subgroup (or coalition) S of individuals out of N. Under the classical notion of a game, $k = 1$ and $\mathbf{v}(S)$ is a singleton. The interpretation behind an SVG $\mathbf{v}$ is the following. Every $x \in \mathbb{R}_+^k$ is a vector of production levels for each of the k projects. A coalition can produce the bundle $x = (x_1, ..., x_k)$ if by distributing their limited resources (say one unit of time) across the different projects, they can accomplish x_ℓ of project ℓ, for every $\ell = 1, ..., k$. If $x \in \mathbb{R}_+^k$ is indeed produceable by coalition S (in one time unit), then $x \in \mathbf{v}(S)$. A member of the *core* of an SVG is a payoff for each player for each of the k enterprises, that is feasible (that is, in

$\mathbf{v}(N)$), and such that no coalition can deviate and do better by itself in every one of the k enterprises.[3]

We start by paying specific attention to a special class of SVGs called *multi-game based*. An SVG in this class is a convex span of k (classical) games. Formally, letting v_ℓ be the ℓth game in the base of the SVG $\mathbf{v}$, the value of a coalition S is all vectors for which the ℓth production component is $\alpha_\ell v_\ell(S)$ (where the α_ℓ's are non-negative and sum to 1). This corresponds to production output being linear in effort. For this class of games, it is interesting to see the relation between our solution concept and the cores of the individual games (which are the base of the SVG), and between our notion of a core and existing approaches to allocation in multi-agenda disputes, in which the payoffs across agendas are aggregated uniformly.

We provide the following results. First, we show that it is possible that while the cores of all base games, and of the sum of these games, are all empty, the core of the SVG itself is not empty. We further show that if the cores of all base games are non empty, then the convex span of these cores are a subset of the core of the SVG. In addition, this containment could be strict. In particular, we prove that whenever the cores of the base games all consist of a single allocation, then there is always a possibility of logrolling (e.g., the trading of favors such as vote trading by legislative members) among several individuals that is not possible when resorting to previous approaches These results indicate that the approach and solution concept presented here are conceptually different than the existing approaches, and allow for more cooperation.

We then provide a non Bondareva-Shapley result: an appropriate definition of balancedness, that serves as a characterization of core non-emptiness in the classical cooperative games setup, is a sufficient condition for the core of an SVG being non-empty, but it is not necessary. An example is provided in which an SVG has a non-empty core while it is not balanced. We point to the fact that a characterization in the current setup can not rely on convexity and duality considerations as in Bondareva-Shapley. A characterization where duality technique can be applied is provided at the end.

2 Set-Valued Games

2.1 The definition

A *set-valued game* is a generalization of the notion of a game. For some $k \in \mathbb{N}$, a set-valued game associates to each coalition a *subset of vectors in* $\mathbb{R}^k_+$.

Definition 1. *A* set-valued game (SVG) *over a collection N of players, is a function $\mathbf{v} : 2^N \to 2^{\mathbb{R}^k_+}$ defined over all subsets of N, and satisfies:*

1. $\mathbf{v}(\emptyset) = \{0\}$*;*
2. Closedness. *For every $S \subseteq N$, $\mathbf{v}(S)$ is a closed set;*
3. Comprehensiveness. *If $x \in \mathbf{v}(S)$ and $y \in \mathbb{R}^k_+$ is such that*[4] *$y \leq x$, then $y \in \mathbf{v}(S)$.*

[3] Note that this is consistent with the approach that no particular aggregation of the payoffs across games is considered.

[4] $y \leq x$ means that $y_j \leq x_j$ for every $j = 1, ..., k$.

2.2 Set-valued and classical games

The values that an SVG takes are subsets of $\mathbb{R}^k_+$. A (classical) game, on the other hand, is defined, like an SVG on subsets of N, but takes numbers as values. However, there is a natural connection between the two notions. A game v is related to SVG $\mathbf{v}$ that takes values in $\mathbb{R}_+$, that is, $k = 1$. In this case the values that $\mathbf{v}$ takes are closed intervals whose left side in 0. The game associated with such $\mathbf{v}$ is defined as $v(S) = \max \mathbf{v}(S)$ for every $S \subseteq N$. And vise versa: if v is a game then the SVG associated with it is the one defined as $\mathbf{v}(S) = [0, v(S)]$ for every $S \subseteq N$.

3 The Core of an SVG

Let $(x^i)_{i\in N}$, where $x^i \in \mathbb{R}^k, i \in N$ and let $S \subseteq N$ Define, $x(S) = \sum_{i\in S} x^i$.

Definition 2. *The core of an SVG* $\mathbf{v}$ *is defined as,*

$$CORE(\mathbf{v}) := \Bigg\{ (x^i)_{i\in N};\ \text{(a) for every } i \in N, x^i \in \mathbb{R}^k_+; \\ \text{(b)} \sum_{i\in N} x^i \in \mathbf{v}(N);\ \text{and} \\ \text{(c)}\forall S \subseteq N, x(S) \in \ \text{if } y \in \mathbf{v}(S) \text{ and } y \geq x(S),\ \text{then } y = x(S) \Bigg\}.$$

When $(x^i)_{i\in N}$ is in the core of $\mathbf{v}$, member i of N would get the share x^i, which is a vector in $\mathbb{R}^k$. That is, a core member is a "payoff" to each player for each of the k agendas. The total share of all the members of N is a feasible vector, namely in $\mathbf{v}(N)$. Finally, it maintains stability in the sense that there is no coalition S that could find a better $y \in \mathbf{v}(S)$. That is, there is no $y \in \mathbf{v}(S)$ that dominates (Pareto) the total share of the S-members, $\sum_{i\in S} x^i$.

4 The Concave and Choquet Integrals w.r.t. SVCs

Definition 3. *Let* $\mathbf{v}$ *be an SVC and*[5] $X \in \mathbb{R}^n_+$. *The* concave integral *of* X *w.r.t.* $\mathbf{v}$ *is*

$$\int^{cav} X d\mathbf{v} = \\ \left\{ \sum_{j=1}^{\ell} \alpha_j y_j;\ \sum_{j=1}^{\ell} \alpha_j \mathbf{1}_{A_j} \leq X, y_j \in \mathbf{v}(A_j), \alpha_j \geq 0, A_j \subseteq N,\ j = 1, ..., \ell \right\}.$$

The concave integral has a natural interpretation in the context of production. Indeed, an SVC, $\mathbf{v}$, reflects the different production possibilities of every coalition (given one unit of time). Then, if each individual $i \in N$ is time constrained by X_i, then

[5] From here on and without explicitly specifying it, X stands for a non-negative vector in $\mathbb{R}^n$.

$\int^{cav} X d\mathbf{v}$ reflects *all* production possibilities given the time constraints profile X. In particular, the concave integral takes into account all the tradeoffs between how much time different coalitions invest in the different projects, when the overall time constraint for individual i (regardless of which coalitions she partakes in) is X_i.

A list $A_1, A_2, ..., A_\ell$ of subsets of N is a *chain* if it is increasing w.r.t. inclusion, that is $A_1 \subseteq A_2 \subseteq ... \subseteq A_\ell$.

Definition 4. *Let $X \in \mathbb{R}^n_+$. The Choquet integral w.r.t. $\mathbf{v}$ is defines as follows:*

$$\int^{Ch} X d\mathbf{v} = \Bigg\{ \sum_{i=1}^{\ell} \alpha_i y_i;\ \sum_{i=1}^{\ell} \alpha_i \boldsymbol{1}_{A_i} \leq X,$$
$$y_i \in \mathbf{v}(A_i), \alpha_i \geq 0, i = 1, ..., \ell \text{ and } A_1, A_2, ..., A_\ell \text{ is a chain} \Bigg\}.$$

5 SVCs and Decision Making

For any two time-constraint profiles $X, Y \in \mathbb{R}^n_+$ define a partial order $\succeq$ as follows:

$$X \succeq Y \quad \text{if and only if} \quad \int X d\mathbf{v} \supseteq \int Y d\mathbf{v}.$$

The interpretation is that a time-constraint profile X is preferred to a time-constraint profile Y if and only if the production possibilities given the SVC $\mathbf{v}$ under the profile X include all the production possibilities under the profile Y. Under comprehensiveness, the condition in Eq. 5 is also equivalent to the Pareto frontier of $\int X d\mathbf{v}$ dominating that of $\int Y d\mathbf{v}$.

The preference relation $\succeq$ given in Eq. (5) is typically partial, that is, there might be two time-constraint profiles X and Y such that neither $X \succeq Y$ nor $Y \succeq X$. This brings us to the issue of selection out of $\int X d\mathbf{v}$ and the completion of $\succeq$.

As discussed in the introduction, unlike the classical theory of capacities, $\mathbf{v}$ and its extension to $\int X d\mathbf{v}$ explicitly model the tradeoffs of investing in different projects conditional on the time investment profile X. Nevertheless, it is quite intuitive to consider how the preferences(over elements in $\mathbb{R}^k_+$) of a third party, say a manager who is responsible to the overall production conditional on market demand, prices, etc., shape the selection out of $\int X d\mathbf{v}$ for every time-constraint profile X. More formally, consider a utility function $U : \mathbb{R}^k_+ \to \mathbb{R}$ aggregating the different values across the k different agendas. Given a time-constraint profile X, the selection out of $\int X d\mathbf{v}$ given the utility function U will be $\text{argmax}_{x \in \int X d\mathbf{v}} U(x)$.

Now, define a preference relation $\succeq^*$ over $\mathbb{R}^n_+$ as follows: for any two time-constraint profiles $X, Y \in \mathbb{R}^n_+$,

$$X \succeq^* Y \quad \text{if and only if} \quad \max_{x \in \int X d\mathbf{v}} U(x) \geq \max_{y \in \int Y d\mathbf{v}} U(y).$$

Notice, $\succeq^*$ is a complete preference relation and is a completion of $\succeq$. That is,

$$X \succeq Y \ \Rightarrow \ X \succeq^* Y.$$

Applications of copulas for a maxmin system to order statistics and to reliability theory

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One of the objectives of this talk will be to extend the work of M. O. and N. Ružić [2], and of F. Durante et al. [1]. In these papers a new line of investigation was started introducing maxmin copulas, closely related to Marshall copulas, but allowing for asymmetric linkages. In joint work in progress with M. Vidmar we go beyond the bivariate case or even the usual multivariate case.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, $\boldsymbol{X} = (X_i)$ a random vector in $\mathbb{R}^n$, for some integer n, and C a copula thereof. For any nonempty $M \subseteq [n]$ we denote

$$\vee_M \boldsymbol{X} := \max\{X_i \,;\, i \in M\} \quad \text{and} \quad \wedge_M \boldsymbol{X} := \min\{X_i \,;\, i \in \mathcal{M}\}.$$

We choose a nonempty set $\mathcal{C}$ of nonempty sets $M \subseteq [n]$ for which we compute $\vee_M \boldsymbol{X}$ and a nonempty set $\mathcal{D}$ of nonempty sets $M \subseteq [n]$ for which we compute $\wedge_M \boldsymbol{X}$. Here we denoted the set of the first n positive integers by $[n]$. We call the $(|\mathcal{C}|+|\mathcal{D}|)$-dimensional random vector

$$X^{\vee\wedge} := ((\vee_M \boldsymbol{X})_{M\in\mathcal{C}}, (\wedge_M \boldsymbol{X})_{M\in\mathcal{D}})$$

a maxmin system. Under certain technical conditions we provide an expression (in terms of C and the marginals $\{F_i\}_{i\in[n]}$) for the copula of this random vector.

Besides the obvious applications to shock models there is an important application of this approach to order statistics. If $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a n-dimensional random vector, then its random vector of order statistics

$$\boldsymbol{X}^{\mathrm{OS}} = (X_{1:n}, \ldots, X_{n:n})$$

is defined (in our notation) by

$$X_{i:n} = \wedge\{\vee_M \boldsymbol{X} \,;\, M \in 2^{[n]}, |M| = i\}$$

for $i \in [n]$. The i-th component of $\boldsymbol{X}^{\mathrm{OS}}$ means the i-th smallest value of components of $\boldsymbol{X}$. So, $\boldsymbol{X}^{\mathrm{OS}}$ corresponds to the permutation of the components of $\boldsymbol{X}$ in the non-decreasing order.

Another important application is through the approach to reliability theory as studied by the group gathered around J.-L. Marichal.

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References

1. F. Durante, M. Omladič, L. Oražem, N. Ružić, *Shock models with dependence and asymmetric linkages*, Fuzzy Sets and Systems, **323** (2017), 152–168.
2. M. Omladič, N. Ružić, *Shock models with recovery option via the maxmin copulas*, Fuzzy Sets and Systems, **284** (2016), 113–128.

On injective constructions of ordered semicategories

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There are quite a lot of papers investigating injective hulls for algebras. Here we only mention some of them which our current work is related to. Injective hulls for posets were studied by Banaschewski and Bruns ([2], 1967) where they got that the injective hull of a poset is its MacNeille completion. After that, Bruns and Lakser, and independently Horn and Kimura constructed injective hulls of semilattices ([3], 1970 and [5], 1971), and their results were soon applied into S-systems over a semilattice by Johnson, Jr., and McMorris ([6], 1972). By the conclusion of Schein ([11], 1974) that there are no non-trivial injectives in the category of semigroups, it took a long time to make further development for the theory of injective hulls on both discrete and ordered (general) semigroups. In 2012, Lambek, Barr, Kennison and Raphael ([9]) studied a kind of category of pomonoids in which the usual category of pomonoids is its subcategory, and found that injective hulls for pomonoids are exactly unital quantales [10, 8]. Later on, Zhang and Laan generalized their results first to the posemigroup case ([14], 2014), and later to S-posets ([15], 2015) and ordered Ω-algebras ([16], 2016). In 2017, Xia, Zhao, and Han ([13]) obtained almost same constructions as in [14], but they described it in a different way.

In what follows we explain how some of preceding results admit far-reaching generalizations in the framework of semicategories. We put the results on S-posets from [15] into that wider perspective. We consider here the "multi-signature" version of modules over posemigroups which turns out to be modules over ordered semicategories.

Our approach sheds a new light on applications of a new kind of fuzzy-like structure. We hope that these results will provide further evidence of what should be considered a good notion of injectivity.

It is well known that the category Pos of posets and order-preserving mappings and the category Sup of sup-lattices and sup-preserving mappings are symmetric monoidal closed categories [7] and we have a so-called *down-set functor* $\mathcal{P}\colon \mathsf{Pos} \to \mathsf{Sup}$ with $X \mapsto \mathcal{P}(X)$ and $f\colon X \to Y \mapsto \mathcal{P}(f)\colon \mathcal{P}(X) \to \mathcal{P}(Y)$, here $\mathcal{P}(f)(Z) = f(Z){\downarrow}$ for all $Z \in \mathcal{P}(X)$ (see [4]). Of course, we have an *inclusion functor* $\mathcal{I}\colon \mathsf{Sup} \to \mathsf{Pos}$ with $X \mapsto X$ and $f\colon X \to Y \mapsto f\colon X \to Y$, for all $X, Y \in ob\,\mathsf{Sup}$, $f\colon X \to Y$ in Sup. An *ordered semicategory* (*category*) is a locally small semicategory (category) such that *hom*-sets are partially ordered and composition on both sides is order-preserving.

A *lax semifunctor* $F\colon \mathcal{C} \to \mathcal{D}$ of ordered categories is given by functions

$$F\colon\ ob\,\mathcal{C} \to ob\,\mathcal{D} \text{ and } F_{X,Y}\colon \mathcal{C}(X,Y) \to \mathcal{D}(FX,FY)$$

for all $X \in ob\,\mathcal{C}$ (with $F_{X,Y}$ usually written only as F), such that

(1) $F_{X,Y}$ is monotone;
(2) $Fg \circ Ff \leqslant F(g \circ f)$,

for all $X, Y, Z \in ob\mathcal{C}$, $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathcal{C}$. A lax semifunctor $F\colon \mathcal{C} \to \mathcal{D}$ is a *semifunctor* if

(2=) $Fg \circ Ff = F(g \circ f)$,

for all $X, Y, Z \in ob\mathcal{C}$, $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathcal{C}$. A semifunctor $F\colon \mathcal{C} \to \mathcal{D}$ is a *2-functor* if $\mathcal{C}$ and $\mathcal{D}$ are categories, and $F\colon \mathcal{C} \to \mathcal{D}$ is a functor.

Notice that the categories Pos of posets and Sup of sup-lattices are ordered categories, with $\mathsf{Pos}(X, Y)$ and $\mathsf{Sup}(X, Y)$ carrying the point-wise order. A *quantaloid* [12] is a locally small category such that *hom*-sets are sup-lattices and composition on both sides is sup-preserving.

Let $\mathcal{S}$ be a (small) ordered semicategory (to avoid set-theoretic problems, we assume $\mathcal{S}$ to be small, that is, to have a set of objects).

An $\mathcal{S}$*-module* is a semifunctor $A\colon \mathcal{S} \to \mathsf{Pos}$ of ordered semicategories.

An $\mathcal{S}$*-morphism* is a lax natural map between $\mathcal{S}$-modules A and B, that is, a family $\alpha = \{\alpha_X \in \mathsf{Pos}(AX, BX) \mid X \in \mathcal{S}\}$ of order-preserving mappings such that for every $f\colon X \to Y$ in $\mathcal{S}$, we have $B(f) \circ \alpha_X \leqslant \alpha_Y \circ A(f)$.

Clearly, $id_A = \{id_{A_X} \in \mathsf{Pos}(AX, AX) \mid X \in \mathcal{S}\} = \{id_{AX} \in \mathsf{Pos}(AX, AX) \mid X \in \mathcal{S}\}$ is an $\mathcal{S}$-morphism.

An $\mathcal{S}$-$\mathcal{Q}$*-module* is an $\mathcal{S}$-module A such that for every two objects $X, Y \in \mathcal{S}$ and for every $f\colon X \to Y$ we have that AX and AY are sup-lattices and $A(f)$ is a sup-preserving mapping, i.e., A yields a semifunctor of ordered semicategories into Sup.

Let us fix the following notation: for every $\mathcal{S}$-module A, every arrow $f\colon X \to Y$ in $\mathcal{S}$, and every element $x \in AX$,

$$f *_A x = A(f)(x).$$

Then the lax naturality property for an $\mathcal{S}$-morphism $\alpha\colon A \to B$ can be expressed as follows: for every arrow $f\colon X \to Y$ in $\mathcal{S}$ and every element $x \in AX$,

$$f *_B \alpha_X(x) \leqslant \alpha_Y(f *_A x).$$

The category $\mathcal{S}-Mod$ of $\mathcal{S}$-modules has $\mathcal{S}$-modules as objects and lax natural maps as morphisms. Clearly, $\mathcal{S} - Mod$ is an ordered category. Hence, for every $\mathcal{S}$-module A and every $X, Y \in \mathcal{S}$, we can consider $*_A$ as a mapping

$$*_A\colon \mathcal{S}(X, Y) \times AX \to AY,$$

which is order-preserving in each variable.

An $\mathcal{S}$-morphism $\gamma\colon A \to A$ of an $\mathcal{S}$-module A is an $\mathcal{S}$*-nucleus* provided that γ is idempotent ($\gamma \circ \gamma = \gamma$) and expanding ($id_A \leqslant \gamma$).

An *order embedding* ε between $\mathcal{S}$-modules A and B is an $\mathcal{S}$-morphism $\varepsilon\colon A \to B$ such that $\varepsilon_X : AX \to BX$ is an order embedding in posets for all $X \in \mathcal{S}$. We denote by $\mathcal{E}_{\mathcal{S}}$ the class of order embeddings between $\mathcal{S}$-modules.

Let $\mathcal{E}_{\leqslant_{\mathcal{S}}}$ be the class of order embeddings $\varepsilon : A \longrightarrow B$ in the category $\mathcal{S} - Mod$ which satisfy the following conditions:

$$f *_B \varepsilon_X(a) \leqslant \varepsilon_Y(b) \Longrightarrow f *_A a \leqslant b,$$

for all $a \in A(X), b \in A(Y)$ and all $f\colon X \to Y$ in $\mathcal{S}$.

Let $\mathcal{C}$ be a category and let $\mathcal{M}$ be a class of morphisms in $\mathcal{C}$. We recall that an object S from $\mathcal{C}$ is $\mathcal{M}$-*injective* in $\mathcal{C}$ provided that for any morphism $h : A \longrightarrow B$ in $\mathcal{M}$ and any morphism $f : A \longrightarrow S$ in $\mathcal{C}$ there exists a morphism $g : B \longrightarrow S$ such that $gh = f$.

A morphism $\eta\colon A \longrightarrow B$ in $\mathcal{M}$ is called $\mathcal{M}$-*essential* (cf. [1]) if every morphism $\psi : B \longrightarrow C$ in $\mathcal{C}$, for which the composite $\psi\eta$ is in $\mathcal{M}$, is itself in $\mathcal{M}$. An object $H \in \mathcal{C}$ is called an $\mathcal{M}$-*injective hull* of an object S if H is $\mathcal{M}$-injective and there exists an $\mathcal{M}$-essential morphism $\eta\colon S \longrightarrow H$. $\mathcal{M}$-injective hulls are unique up to isomorphism (cf. [1, Remark 9.23 and Proposition 9.19]).

One of our main results is the following.

Theorem 1. *Every $\mathcal{S}$-$\mathcal{Q}$-module is $\mathcal{E}_{\leqslant_\mathcal{S}}$-injective and thus $\mathcal{E}_\mathcal{S}$-injective in the category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules.*

Let A be an $\mathcal{S}$-module, and let $\mathcal{P}(A)_\mathcal{S} = \mathcal{I} \circ \mathcal{P} \circ A$ be the composition of lax semifunctor A, and lax functors $\mathcal{P}$ and $\mathcal{I}$. Then $\mathcal{P}(A)_\mathcal{S}$ is clearly an $\mathcal{S}$-$\mathcal{Q}$-module. From Theorem 1 we obtain that $\mathcal{P}(A)_\mathcal{S}$ is $\mathcal{E}_{\leqslant_\mathcal{S}}$-injective in the category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules.

Similarly as in [15, Proposition 5] we obtain the following.

Proposition 1. *Every retract of an $\mathcal{S}$-$\mathcal{Q}$-module in the category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules is an $\mathcal{S}$-$\mathcal{Q}$-module.*

An immediate consequence of Proposition 1 is the following.

Theorem 2. *Let A be an $\mathcal{S}$-module. Then A is $\mathcal{E}_{\leqslant_\mathcal{S}}$-injective in the category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules if and only if A is an $\mathcal{S}$-$\mathcal{Q}$-module.*

Let A be an $\mathcal{S}$-module and $\gamma\colon A \to A$ an $\mathcal{S}$-nucleus on A. Notice that γ_X is a closure operator on AX for all $X \in ob\,\mathcal{S}$. Hence we obtain a function A_γ from $\mathcal{S}$ to Pos such that $A_\gamma X = (AX)_{\gamma_X}$. Moreover, for any $X, Y \in ob\,\mathcal{S}$ and for any morphism $f\colon X \to Y$ in $\mathcal{S}$ we obtain an order-preserving mapping $(Af)_\gamma\colon A_\gamma X \to A_\gamma Y$ given by $(Af)_\gamma(a) = \gamma_Y(Af(a)) \in A_\gamma Y$ for all $a \in A_\gamma Y$ (since $(Af)_\gamma$ is a composition of order-preserving mappings). We put $A_\gamma(f) = (Af)_\gamma$.

We then have the following.

Proposition 2. *Let A be an $\mathcal{S}$-module and $\gamma\colon A \to A$ an $\mathcal{S}$-nucleus on A. Then $A_\gamma\colon \mathcal{S} \to \mathsf{Pos}$ is an $\mathcal{S}$-module.*

Notice that the action on A_γ is defined as follows:

$$f *_{A_\gamma} a = A_\gamma(f)(a) = \gamma_Y(Af(a))$$

for all $a \in A_\gamma X = (AX)_{\gamma_X}$ and all morphisms $f\colon X \to Y$ in $\mathcal{S}$.

In what follows, we will construct an $\mathcal{E}_{\leqslant_\mathcal{S}}$-injective hull for any $\mathcal{S}$-module A in the category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules. The $\mathcal{E}_{\leqslant_\mathcal{S}}$-injective hull of A will be obtained as a quotient of $\mathcal{P}(A)_\mathcal{S}$ with a natural embedding η_A from A into the hull such that $\eta_{A\,X}(a) = a{\downarrow}$ for each $X \in \mathcal{S}$ and $a \in AX$.

Let $X \in ob\,\mathcal{S}$ and let D be a down-set of AX. We then define its closure $\mathsf{cl}_X(D)$ by

$$\mathsf{cl}_X(D) := \{z \in AX \mid (\forall f\colon X \to Y, a \in AX, b \in AY)\; D \subseteq a{\downarrow} \Longrightarrow z \leqslant a,\\ f *_A D \subseteq b{\downarrow} \Longrightarrow f *_A z \leqslant b\}.$$

Notice that $\mathsf{cl}_X(D) \in \mathcal{P}(A)_{\mathcal{S}}(X)$ and $\mathsf{cl} = \{\mathsf{cl}_X \in \mathsf{Pos}(\mathcal{P}(A)_{\mathcal{S}}X, \mathcal{P}(A)_{\mathcal{S}}X) \mid X \in \mathcal{S}\}$ is an $\mathcal{S}$-nucleus on $\mathcal{P}(A)_{\mathcal{S}}$.

Now we are ready to obtain the main result of our paper.

Theorem 3. *For every $\mathcal{S}$-module A, $\mathcal{P}(A)_{\mathcal{S}_{\mathsf{cl}}}$ is the $\mathcal{E}_{\leqslant_{\mathcal{S}}}$-injective hull of A in the category $\mathcal{S} - Mod$ of $\mathcal{S}$-modules.*

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References

1. J. Adámek, H. Herrlich, and G.E. Strecker. Abstract and Concrete Categories: The Joy of Cats. *John Wiley & Sons*, 1990.
2. B. Banaschewski B. and G. Bruns. Categorical construction of the MacNeille completion. *Arch. Math.*, 18:369-377, 1967.
3. G. Bruns and H. Lakser. Injective hulls of semilattices. *Canad. Math. Bull.*, 13:115–118, 1970.
4. D. Hofmann, G.J. Seal and W. Tholen. Monoidal Topology: A Categorical Approach to Order, Metric, and Topology. *Cambridge University Press, Cambridge*, vol. 153, 2014.
5. A. Horn and N. Kimura. The category of semilattices. *Algebra Universalis*, 1:26–38, 1971.
6. C.S. Johnson, Jr. and F.R. McMorris. Injective hulls on certain S-systems over a semilattice. *Proc. Amer. Math. Soc.*, 32:371–375, 1972.
7. G.M. Kelly. Basic concepts of enriched category theory. *Repr. Theory Appl. Categ.*, 10 (2005), vi+137 pp. (electronic). Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].
8. D. Kruml, and J. Paseka. Algebraic and Categorical Aspects of Quantales. *Handbook of Algebra (M. Hazewinkel, ed.), Elsevier* 5 (2008), 323–362.
9. J. Lambek, M. Barr, J.F. Kennison and R. Raphael. Injective hulls of partially ordered monoids. *Theory and Applications of Categories*, 26:338–348, 2012.
10. C. J. Mulvey. §, *Suppl. Rend. Circ. Mat. Palermo ser. II*, 12:99–104, 1986.
11. B.M. Schein. Injectives in certain classes of semigroups. *Semigroup Forum*, 9:159–171, 1974.
12. I. Stubbe. An introduction to quantaloid-enriched categories. *Fuzzy Sets Syst.*, 256:95–116, 2011.
13. C.C. Xia, B. Zhao and S.W. Han. A Note on injective hulls of posemigropus, *Theory Appl. Categ.*, 32:254-257, 2017.
14. X. Zhang and V. Laan. Injective hulls for posemigroups. *Proc. Est. Acad. Sci.*, 63:372–378, 2014.
15. X. Zhang and V. Laan. On injective hulls of S-posets. *Semigroup Forum*, 91:62–70, 2015.
16. X. Zhang and V. Laan. On injective hulls for ordered algebras. *Algebra Universalis*, 76:339–349, 2016.

Probabilistic social choice on graphs

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Finitely many agents have preferences over finitely many alternatives, where these alternatives are the vertices of an undirected connected graph. It is assumed that each preference is single-peaked with respect to some spanning tree of this graph: it has a top-ranked vertex and preference decreases along paths away from this vertex in the spanning tree. Given a profile of individual preferences a probabilistic social choice function (PSCF) assigns a probability distribution over the vertices in the graph. A PSCF is unanimous if it assigns probability one to a vertex if this is every agent's most preferred alternative. It is strategy-proof if each agent, by reporting an insincere preference, can only bring about a probability distribution that is stochastically dominated by the one obtained by reporting sincerely. A PSCF is a random dictatorship if each vertex is assigned the sum of fixed probabilities of agents who have that vertex as their most preferred alternative. We show that each unanimous and strategy-proof PSCF is a random dictatorship if and only if the graph has no leaf. We also characterize all unanimous and strategy-proof PSCF for the case where the graph has a leaf, including the case where the graph is a tree.

A framework for the generation of discrete pseudo-t-norms

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Abstract. By a discrete pseudo-t-norm, we mean an associative binary operation on a finite chain that is monotone in both arguments and such that the top element is an identity. We present a framework within which the in a successive manner all these operations can be represented.

1 Introduction

Fuzzy logic deals with graded truth values. Usually, the set of truth degrees is assumed to be a chain, whose smallest element represents "clearly false" and whose top element represents "clearly true". Furthermore, the conjunction is interpreted by a binary operation on this chain. It is commonly assumed that this operation is associative, commutative, monotone in both arguments, and such that the top element is an identity. If the logic is based on the real unit interval, we are led to what is called a t-norm [KMP]. If the logic is based on a finite chain, we deal with what is sometimes called a discrete t-norm [BaMe]. If, finally, the assumption of commutativity is dropped, we have what we could call in accordance with [FGI] a pseudo-t-norm.

The present contribution deals with operations of the latter type. We will stick in the sequel, however, to the terminology common in algebra: the structures under consideration are negative totally ordered monoids, or negative tomonoids for short [EKMMW].

The following observation gives rise to a method of generating all finite structures of this type. Let T be a finite, negative tomonoid and let 0, α be the smallest two elements. Then the equivalence relation θ identifying 0 and α and nothing else is a congruence. Indeed, by the negativity of T, $\{0, \alpha\}$ is a (semigroup) ideal, hence θ is a Rees congruence [How], and it is easy to see that the induced total order makes the quotient T/θ into a negative tomonoid again. Repeating the same procedure, we eventually end up with the trivial (i.e., the one-element) tomonoid.

In [PeVe1], we have described how to proceed in the opposite direction. Namely, given a tomonoid S, we have determined all those tomonoids that are by one element larger than S and lead by an identification of the smallest two elements back to S. We call these tomonoids the one-element Rees coextensions of S. Starting with the trivial tomonoid, we obviously get in this way all finite, negative tomonoids.

In our present contribution, we reconsider the problem from a different perspective. We do not really care about algorithmic aspects. Instead, we wonder how the procedure can reasonably described on the algebraic side [PeVe2]. Given S as above, we define the free one-element Rees coextension of S, denoted by $\mathcal{R}(S)$, and we show that any one-element Rees coextension is a quotient of $\mathcal{R}(S)$. This is not yet particularly interesting because $\mathcal{R}(S)$ is usually infinite and difficult to describe. However, we show that there is a finite family of finite quotients of $\mathcal{R}(S)$. From each of these quotients, coextensions arise in a straightforward manner, and each of the desired coextensions arises in this way.

2 The free one-element Rees coextension

We recall that a *pomonoid* (where "po" stands for "partially ordered") is a structure $(S; \leqslant, \cdot, 1)$ such that $(S; \cdot, 1)$ is a monoid, $(S; \leqslant)$ is a poset, and, for any $a, b, c, d \in S$, $a \leqslant b$ and $c \leqslant d$ imply $a \cdot c \leqslant b \cdot d$. Moreover, S is called *negative* if 1 is the top element. If S is totally ordered, we call S a *totally ordered monoid*, or simply a *tomonoid*.

Our aim is to describe all finite, negative tomonoids. We shall use the following fact.

Lemma 1. *Let $(S; \cdot, \leqslant, 1)$ be a pomonoid and let I be a (semigroup) ideal as well as a downset. For $a, b \in S$, let $a \;\rho_I\; b$ if $a = b$ or $a, b \in I$. Then ρ_I is a congruence.*

In particular, let T be a finite, negative tomonoid, let 0 be the smallest element and let α be the unique atom of T. Then $\{0, \alpha\}$ is an ideal and a downset. Hence, by Lemma 1, $S = T/\rho_{\{0,\alpha\}}$ is a negative tomonoid, which has one element less than T.

Let us now adopt the opposite viewpoint. Let $(S; \cdot, \leqslant, 1)$ be a finite, negative tomonoid. We assume that S has at least two elements. Let $\bar{S}$ arise from S by removing its smallest element $\dot{0}$ and adding instead two new elements 0 and α. We make $\bar{S}$ into a chain by requiring $0 \leqslant \alpha \leqslant a$ for any $a \in S \setminus \{\dot{0}\}$.

Our intention is to make $\bar{S}$ into a tomonoid such that S is isomorphic to $\bar{S}/\rho_{\{0,\alpha\}}$. We call $\bar{S}$ in this case a *one-element Rees coextension* or simply a *one-element coextension* of S. Obviously, a good part of the structure of $\bar{S}$ is already determined: the total order as well as the product of all pairs $a, b \in S \setminus \{\dot{0}\}$ such that, in S, $ab \neq \dot{0}$.

Example 1. For instance, let us define

$$a \bullet b \;=\; \begin{cases} ab & \text{if } a, b \in S \setminus \{\dot{0}\} \text{ and } ab \neq \dot{0} \text{ in } S, \\ \alpha & \text{if } a = \alpha \text{ and } b = 1, \text{ or } a = 1 \text{ and } b = \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

where $a, b \in \bar{S}$. Note that the products in $\bar{S}$ are either predetermined or else defined to be 0. We may check that $(\bar{S}; \bullet, \leqslant, 1)$ is a one-element coextension of $(S; \cdot, \leqslant, 1)$.

In what follows, we will make use of the fact that pomonoids can be specified by means of "generators and defining relations". The theory behind is slightly more involved than in case of ordinary algebras, because we have to take into account the partial order relation.

Let $(G; \leqslant)$ be a poset. Then there exists a *free pomonoid* $\mathcal{F}(G)$ over G [Sub]. This is the monoid of words $a_1 \ldots a_n$, where $n \geqslant 0$ and $a_1, \ldots, a_n \in G$. The product is defined by concatenation; the identity is given by the empty word; and the partial order on $\mathcal{F}(G)$ is given by

$$a_1 \ldots a_m \leqslant b_1 \ldots b_n \quad \text{if } n = m \text{ and } a_1 \leqslant b_1, \ldots, a_n \leqslant b_n.$$

Identifying G with the words of length 1, we then have that any order-preserving map from G to some pomonoid T extends to a homomorphism from $\mathcal{F}(G)$ to T.

Furthermore, let $\trianglelefteq$ be any binary relation on $\mathcal{F}(G)$. Then there is a smallest congruence $\Theta(\trianglelefteq)$ on $\mathcal{F}(G)$ and a smallest partial order on $\mathcal{F}(G)/\Theta(\trianglelefteq)$ such that $a/\Theta(\trianglelefteq) \leqslant b/\Theta(\trianglelefteq)$ whenever $a \trianglelefteq b$. This means that any homomorphism f from $\mathcal{F}(G)$ to a further pomonoid T such that, for any $a, b \in \mathcal{F}(G)$, $a \trianglelefteq b$ implies $f(a) \leqslant f(b)$, factorises through the quotient $\mathcal{F}(G)/\Theta(\trianglelefteq)$.

We conclude that we may define a pomonoid as generated by a poset G and subject to certain inequalities between elements of the free monoid generated by G.

Definition 1. *Let $\mathcal{R}(S)$ be the free pomonoid over the chain $\bar{S}$, subject to the following conditions:*

(a) $a\,b = c$ *for any* $a, b, c \in \bar{S} \setminus \{0, \alpha\}$ *fulfilling this equation in* S*,*
(b) $\epsilon = 1$*,*
(c) $a\,b \leqslant \alpha$ *if* $(a, b) \in \mathcal{N}$*,*
(d) $0\,a = 0$ *for any* $a \in \bar{S}$*,*

We call $\mathcal{R}(S)$ the free one-element Rees coextension, or simply the free one-element coextension of S.

Here, we have put $\mathcal{N} = \{(a, b) \in \bar{S}^2 \colon a \leqslant \alpha \text{ or } b \leqslant \alpha \text{ or else } ab = \dot{0} \text{ in } S\}$.

Proposition 1. *Let T be a one-element coextension of S. Then there is a congruence θ on $\mathcal{R}(S)$ such that $\mathcal{R}(S)/\theta$ is isomorphic to T.*

We conclude that there is a single pomonoid $\mathcal{R}(S)$ among whose quotients we find all one-element Rees coextensions.

Unfortunately, the pomonoid $\mathcal{R}(S)$ is infinite and not yet the suitable tool to describe systematically all the one-element coextensions of S.

3 One-element coextensions for given borders

Again, let $(S; \cdot, \leqslant, 1)$ be a non-trivial finite, negative tomonoid. Let $(\bar{S}; \bullet, \leqslant, 1)$ be a one-element coextension of S. Then the *left border* of the coextension is the smallest element $\varepsilon_l \in \bar{S} \setminus \{0\}$ such that, in $\bar{S}$, $\varepsilon_l \bullet \alpha = \alpha$. Similarly, the *right border* is the smallest element $\varepsilon_r \in \bar{S} \setminus \{0\}$ such that $\alpha \bullet \varepsilon_r = \alpha$.

It turns out that the left and right border are idempotent elements of $\bar{S} \setminus \{0\}$ and can thus be identified with a pair of idempotent elements of S. We may hence roughly classify the one-element coextensions by means of their borders.

Accordingly, we revise our procedure as follows. We make an assumption from the outset which pair of idempotents of S will play the role of the borders of the coextensions. Then we require further equations to hold in addition to (a)–(d) from Definition 1. Let us fix a pair $\varepsilon_l, \varepsilon_r$ of idempotent elements of S.

Definition 2. *Let $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ be the free pomonoid over the chain $\bar{S}$, subject to the conditions* (a)–(d) *of Definition 1 as well as the following ones, for any $a, b, c \in \bar{S}$:*

(e) $a\,b\,c = b\,c$ *for any* $(b, c) \in \mathcal{N}$ *and* $a \geqslant \varepsilon_l$.
$a\,b\,c = a\,b$ *for any* $(a, b) \in \mathcal{N}$ *and* $c \geqslant \varepsilon_r$.
(f) $a\,b\,c = 0$ *for any* $(b, c) \in \mathcal{N}$ *and* $a < \varepsilon_l$.
$a\,b\,c = 0$ *for any* $(a, b) \in \mathcal{N}$ *and* $c < \varepsilon_r$.

We call $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ the free one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S.

Proposition 2. *Let T be a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S. Then there is a congruence θ on $\mathcal{R}(S)$ such that $\mathcal{R}(S)/\theta$ is isomorphic to T.*

The crucial facts are the following. Most important, $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ is a finite pomonoid. Moreover, the multiplication in $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ is uniquely determined by the products of pairs of elements of $\bar{S}$. This facilitates the representation of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$. Finally, our main result shows that we can determine all coextensions from $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ in a straightforward way.

Theorem 1. *Assume that $0 \neq \alpha$ and let $\varnothing \subset Z \subset [0, \alpha]$ be a downset of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$. For $a, b \in \mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$, let*

$$a\,\theta_Z\,b \quad \text{if and only if} \quad a = b \text{ or } a, b \in Z \text{ or } a, b \in [0, \alpha] \setminus Z.$$

Then θ_Z is a congruence on $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$ and $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)/\theta_Z$ is a one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S.

Up to isomorphism, every one-element $(\varepsilon_l, \varepsilon_r)$-coextension of S arises in this way from a unique downset Z of $\mathcal{R}_{\varepsilon_l,\varepsilon_r}(S)$.

References

[BaMe] B. De Baets, R. Mesiar, Discrete triangular norms, in: S. E. Rodabaugh, E. P. Klement (Eds.), "Topological and algebraic structures in fuzzy sets", Kluwer Academic Publishers, Dordrecht, 2003; pp. 389–400.

[EKMMW] K. Evans, M. Konikoff, J. J. Madden, R. Mathis, G. Whipple, Totally ordered commutative monoids, *Semigroup Forum* **62** (2001), 249–278.

[FGI] P. Flondor, G. Georgescu, A. Iorgulescu, Pseudo-t-norms and pseudo-BL algebras, *Soft Comput.* **5** (2001), 355–371.

[How] J. M. Howie, "An introduction to semigroup theory", Academic Press, London 1976.

[KMP] E. P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Acad. Publ., Dordrecht, 2000.

[PeVe1] M. Petrík, Th. Vetterlein, Rees coextensions of finite, negative tomonoids, *J. Log. Comput.* **27** (2017), 337–356.

[PeVe2] M. Petrík, Th. Vetterlein, Rees coextensions of finite tomonoids and free pomonoids, *Semigroup Forum*, to appear.

[Sub] B. Al Subaiei, "Unitary posets and amalgamations of pomonoids", Ph.D. Thesis, University of Southampton, 2014.

Capacity-valued random choice

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When the characteristics of agents are observed only in coarse manner, a population of observationally identical decision makers might take distinct actions. From the analyst's perspective, choice appears to be random. *Random Utility Models* (RUM)—a set of utility functions and a probability measure thereover—are a powerful and tractable tool in the analysis of such a scenario. The probability of observing x from the decision problem D is the probability of a utility function u such that $x = \arg\max_{z \in D} u(z)$. Properly dealing with indifference has beleaguered the literature on RUMs. Consider the case where with positive probability $u(x) = u(y)$; the probability x is chosen from $D = \{x, y\}$ is undefined by the RUM.

In this paper, we put forth a model of random choice in which precise choice frequencies are identified only up to the frequency they are chosen by strict maximization and discuss how it could be used in different economic environments. We show that such a model (over a linear space) is always representable by a unique random (linear) utility model without *any* restrictions on the measure over utility functions. In other words, the set of random expected utility models and the set of choice rules considered in this paper are in bijection.

The primitive is a *choice capacity* over a linear space (such as von-Neumann-Morgenstern lotteries): Let $\mathcal{D}$ be the set of finite non-empty subsets of this space, and $D \in \mathcal{D}$. A choice capacity is a family of set functions $\{\rho_D : 2^D \to [0,1]\}_{D \in \mathcal{D}}$, that need not be additive. The interpretation is that $\rho_D(A)$ reflects maximal frequency with which elements of $A \subseteq D$ are chosen when the decision problem is D; because of this, ρ need not be additive.

Example 1 *Let $\{a, b\}$ be a set of prizes, with P the set of lotteries thereon. The set of expected utility indices that are realized with positive probability are given by*

$$u_1 = [1, 0], u_2 = [-1, 0], \text{ and } u_3 = [0, 0].$$

Let ξ be the uniform measure over these utility indices. Consider the decision problem $D = \{a, b, c = \frac{1}{2}a + \frac{1}{2}b\} \subset P$ (where we identify outcomes with the degenerate lotteries thereon). Let $\rho^\circ_D(a)$ be the modelers best estimate of the probability that a is chosen from D. Notice, if u_1 is realized then a is definitely chosen, so any reasonable estimate must satisfy $\rho^\circ_D(a) \geq \frac{1}{3}$; if u_2 then a is definitively not chosen, so $\rho^\circ_D(a) \leq \frac{2}{3}$.

When u_3 is realized, the probability a is chosen depends on the tie breaking rule; it is not identified by ξ. If a single tie breaking rule is not consistently employed (if it changes across time or the population, etc), $\rho^\circ_D(a)$ will not be point identified. It will be, however, set identified: we know that $\rho^\circ_D(a) \in [\frac{1}{3}, \frac{2}{3}]$. Similar reasoning shows that $\rho^\circ_D(b) \in [\frac{1}{3}, \frac{2}{3}]$ and $\rho^\circ_D(c) \in [0, \frac{1}{3}]$. Further consider $\rho^\circ_D(\{a, b, c\})$, the probability that a or b or c is chosen. Of course, $\rho^\circ_D(\{a, b, c\})$ can be point identified as 1, as it does not depend on how ties are treated.

As the upper bound of ρ°_D, we have $\rho_D(c) = \frac{1}{3}$, $\rho_D(\{a, b\}) = \rho_D(\{a, b, c\}) = 1$ and all other subsets have a value of $\frac{2}{3}$. Since $\rho_D(\{a, b, c\}) = 1 < \rho_D(a) + \rho_D(b) = \frac{2}{3} + \frac{2}{3}$, ρ_D is not additive. □

Our main representation result provides the conditions on a choice capacity to ensure it maximizes a probability distribution over utilities. That is, to ensure the existence of a probability over utilities, ξ, such that

$$\rho_D(A) = \xi\big(\{u \mid \arg\max_{z \in D} u(z) \cap A \neq \emptyset\}\big).$$

$\rho_D(A)$ reflects the maximal probability that an element of A is chosen when the choice problem is D and preferences are realized according to ξ. When ξ realizes ties with probability 0, ρ is a measure, and all of the [Gul and Pesendorfer(2006)] axioms hold. Thus, our innovation concerns dealing directly with how the non-additivity can enter ρ.

The key axiom, *Convex-Modularity*, limits how non-additive ρ can be. Recall D and ξ from Example 1. We had $\rho_D(a) = \xi(u_1) + \xi(u_3)$ and $\rho_D(b) = \xi(u_2) + \xi(u_3)$. Therefore, $\rho_D(a) + \rho_D(b) \neq \rho_D(\{a, b\}) = 1$ exactly because we have double counted $\xi(u_3)$. This last term—the probability of indifference between a and b—is identified by $\rho_D(c)$. The convex combination of two lotteries is chosen exactly when the two lotteries yield the same utility.

Simple accounting reveals the following modularity relation:

$$\rho_D(\alpha a + (1-\alpha)b) = \rho_D(a) + \rho_D(b) - \rho_D(\{a, b\}).$$

Our Convex-Modularity axiom states that ρ must satisfy a generalized form of the above relation: the probability of the convex combination of two sets is the sum of the probability of the sets, minus the probability of their union.

Theorem 1. *The following are equivalent:*

1. *The CC ρ maximizes a finitely additive RLR ξ.*
2. *The CC ρ satisfies*
   (a) *Let $D \subset D'$, and let $A \subset D$. Then $\rho_D(A) \geq \rho_{D'}(A)$, with equality whenever the extreme points of D and D' coincide.*
   (b) *Let $A, B \subseteq D$ be such that $\alpha A + (1-\alpha)B \subseteq D$ for $\alpha \in (0, 1)$. Then $\rho_D(\alpha A + (1-\alpha)B) = \rho_D(A) + \rho_D(B) - \rho_D(A \cup B)$.*
   (c) *Let $A \subseteq D$. Then $\rho_{\lambda D + z}(\lambda A + z) = \rho_D(A)$. for all $\lambda > 0$ and $z \in \mathbb{R}^n$.*
   (d) *For $D, D' \in \mathcal{D}$, $\rho_{\lambda D + \lambda' D'}$ is continuous in λ, λ' for $\lambda, \lambda' \geq 0$.*

Moreover, every RLR has a unique maximizer and every ρ maximizes at most one RLR.

We now describe several different data generating processes that lead to (the identification of) a choice capacity:

Set Valued Choice. In some environments, a modeler might directly observe the entire set of maximizers associated with a decision problem. In other words, the data available to the modeler is the frequency with which each subset of D is chosen—a measure m_D over 2^D. Taking the observed measures $\{m_D\}_{D\in\mathcal{D}}$ as our primitive, we say that $\{m_D\}_{D\in\mathcal{D}}$ *maximizes* a RLR, ξ, if for all (D, A),

$$m_D(A) = \xi\big(\{u \mid \arg\max_{z\in D} u(z) = A\}\big).$$

Understanding when $\{m_D\}_{D\in\mathcal{D}}$ maximizes a RLR, and when it does, identifying the RLR, seems like an entirely new problem. But worry not, by simply filtering through the world of choice capacities, both questions become simple ones. Construct $\{\rho^m_D\}_{D\in\mathcal{D}}$ as follows:

$$\rho^m_D(A) = \sum_{\substack{B\in 2^D,\\ B\cap A\neq\emptyset}} m_D(B). \tag{1}$$

Theorem 2. *Let $\{\rho_D\}_{D\in\mathcal{D}}$ maximize ξ, then $\{m_D\}_{D\in\mathcal{D}}$ maximizes ξ if and only if*

$$\rho_D = \rho^m_D,$$

for all $D \in \mathcal{D}$, where ρ^m is defined by (1).

Status Quo. Often there is an exogenous default implemented in the case of indifference. For example, if the set of acceptable options includes the status quo, then the status quo is implemented. If our primitive observable data is a choice rule defined over a set *and* an observed status quo alternative, then variation in the default can identify a random choice capacity. Assume that these observable data are being generated by a RUM, such that for each $x \in X$, and each choice problem D we observe an (additive) random choice rule, ρ^x_D, representing choice from D under status quo x, such that $\rho^x_D(y) \leq \rho^y_D(y)$ for all $x, y \in \mathbb{R}^n$, and $D \in \mathcal{D}$. Given a choice problem D, y is chosen more often when it is the status quo then when any other element is.

Say that $\{\rho^x_D\}_{x\in\mathbb{R}^n, D\in\mathcal{D}}$ *maximizes* ξ if

$$\rho^x_D(x) = \xi\big(\{u \mid x \in \arg\max_{z\in D} u(z)\}\big),$$

for all $x \in D$, and $D \in \mathcal{D}$. In conjunction with the assumption that $\rho^x_D(y) \leq \rho^y_D(y)$, it is straightforward to see this characterizes the following class of models: utilities are dawn according to ξ, and in the event of indifference, ties are broken arbitrarily *unless* one of the maximal elements is the status quo, in which case it is chosen.

Then we can recover a choice capacity as follows:

$$\rho^{sq}_D(\{x\}) = \rho^x_D(\{x\}), \tag{2}$$

Although (2) defines ρ^{sq}_D only when the choice is a singleton, it is sufficient to identify a unique choice capacity that satisfies our axioms. This result is formally captured by Lemma 4.

Theorem 3. *Let* $\{\rho_D\}_{D\in\mathcal{D}}$ *maximize* ξ*, then* $\{\rho^x_D\}_{x\in\mathbb{R}^n, D\in\mathcal{D}}$ *maximizes* ξ *for each* x*, if and only if*

$$\rho_D = \rho^{sq}_D$$

where ρ^{sq}_D *is as given by* (2).

Sets of Random Choice Rules. Say a modeler collects data from distinct populations. She may want to know if the data arise from differences in preferences or from (more superficial) differences in tie breaking procedures. In particular, the modeler might observe a collection of (additive) random choice rules $\{\rho^i_D\}_{D\in\mathcal{D},i\in I}$ where I is the set of populations. The modeler wants to ascertain if there is a common ξ such that each ρ^i_D maximizes ξ, up to differences in tie breaking.

We can construct a choice capacity by taking the upper-bound across the measures. When will this choice capacity satisfy our axiomatic restrictions? This is the case exactly when (i) the set of random choice rules arise entirely from differential tie breaking procedures with respect to a common RUM and (ii) every possible tie breaking rule is contained in the convex hull of those employed by some population.

Towards making this definite, for any RLR, ξ, define the set of measures

$$M(\xi, D) = \left\{ \int_{R^n} \tau_u(A)\xi(du) : \tau_u \in \mathcal{D}(\mathbb{R}^n),\ \text{supp}(\tau_u) = \arg\max_{y\in D} u(y) \right\},$$

where supp(τ) is the support of the measure τ. The set $M(\xi, D)$ represents the set of all possible choice rules constructed by first choosing a utility u according to ξ, and subsequently choosing among the maximizers in D according to some tie breaking procedure. An alternative characterization of CCs which maximize RLRs is as follows, using results from work done on belief functions [Wasserman(1990)].

Theorem 4. *The CC* ρ *maximizes* ξ *if and only if* $\rho_D = \max_{m\in M(\xi,D)} m$ *for all* D*.*

Consider again the modeler who observed $\{\rho^i_D\}_{D\in\mathcal{D},i\in I}$. Theorem 4 tells us when the choice capacity given by

$$\rho^I_D(A) = \sup_{i\in I} \rho^i_D(A),$$

will maximize a RLR. But this was not exactly the modelers question. It is possible that some, but not all, tie breaking rules were employed, so that each ρ^i_D maximizes ξ (with respect to some tie breaking rule) but ρ^I_D does not. This state of affairs can be easily captured:

Theorem 5. *For all* $D \in \mathcal{D}$*,* $\{\rho^i_D \mid i \in I\} \subseteq M(\xi, D)$ *if and only if* $\rho^I_D(A) \leq \rho_D(A)$ *for all* $D \in \mathcal{D}$ *and* $A \subseteq D$*, where* ρ *is the unique CC that maximizes* ξ*.*

References

[Gul and Pesendorfer(2006)] Faruk Gul and Wolfgang Pesendorfer. Random expected utility. *Econometrica*, 74(1):121–146, 2006.

[Wasserman(1990)] Larry Alan Wasserman. Prior envelopes based on belief functions. *The Annals of Statistics*, pages 454–464, 1990.

Parsimonious decomposition of 2-additive generalized additive independence models

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Let $N = \{1, \dots, n\}$ be a finite set of n discrete attributes (or criteria). Each attribute $i \in N$ is represented by a set X_i. The alternatives are characterized by a value on each attribute, and are thus represented by an element of the Cartesian product $X = X_1 \times \dots \times X_n$.

In Multiattribute utility theory [6], the additive utility model is one of the best-known representative of preferences in decision making: the overall score can be written as a sum of single-attribute subutility functions:

$$U(x) = \sum_{i \in N} u_i(x_i), x \in X.$$

Note that this model only requires eliciting $u_i(x_i)$. However, such a model cannot be applied in many practical situations (no interaction is possible among attributes). A natural generalization of the additive utility model is to allow some interaction among criteria [3, 1]:

$$U(x) = \sum_{S \in \mathcal{S}} u_S(x_S), x \in X,$$

where $\mathcal{S}$ is a collection of subsets of N, $x_S \in \times_{i \in S} X_i$ is the restriction of x over attributes in coalition S and $u_S : X_S \to \mathbb{R}$. This model is called the Generalized Additive Independence (GAI) model. The additive utility model is a particular case of the GAI model where S is composed of only singletons. There are two major difficulties related to this model. First, its expression is far from being unique. The second difficulty is that the number of monotonicity constraints on the parameters of the model grows exponentially fast in the number of attributes [5]. A GAI model is equivalent to a k-ary capacity [4], i.e., a function $v : \{0, 1, \dots, k\}^N \to \mathbb{R}$ satisfying the monotonicity conditions and the normalization conditions (k-ary capacity is a generalization of the notion of capacity [2]). In this paper, we are interested in the GAI models where the collection $\mathcal{S}$ is made only of singletons and pairs. This particular class is called 2-additive GAI models [5]. The aim of this paper is to provide a fundamental result on decomposition of 2-additive GAI model. We show that for such model, it is always possible to obtain a decomposition into nonnegative monotone nondecreasing terms, and we give an explicit decompsition for 2-additive 2-ary capacities.

References

1. F. Bacchus and A. Grove. Graphical models for preference and utility. In *Proceedings of the Eleventh conference on Uncertainty in artificial intelligence*, pages 3–10. Morgan Kaufmann Publishers Inc., 1995.
2. G. Choquet. Theory of capacities. In *Annales de l'institut Fourier*, volume 5, pages 131–295, 1954.
3. P. Fishburn. Interdependence and additivity in multivariate, unidimensional expected utility theory. *International Economic Review*, 8(3):335–342, 1967.
4. M. Grabisch and C. Labreuche. Capacities on lattices and k-ary capacities. In *EUSFLAT Conf.*, pages 304–307, 2003.
5. M. Grabisch and C. Labreuche. Monotone decomposition of 2-additive generalized additive independence models. *Mathematical Social Sciences*, 92:64–73, 2018.
6. R. L. Keeney and H. Raiffa. *Decision with Multiple Objectives*. Wiley, New York, 1976.

Convexity conditions for the characterization of some copula construction

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Given a binary copula $C\colon [0,1]^2 \to [0,1]$ (i.e., a supermodular binary aggregation function with neutral element 1 [4, 3, 10]), its dual $C^*\colon [0,1]^2 \to [0,1]$ is defined by $C^*(x,y) = x + y - C(x,y)$.

Denoting the product copula by Π, it has been shown by several authors [2, 7] that, for each binary copula C, the function $\Pi(C, C^*)\colon [0,1]^2 \to [0,1]$ given by

$$\Pi(C, C^*)(x,y) = \Pi(C(x,y), C^*(x,y))$$

is always a binary copula.

This result has been generalized into two ways: first of all, in [5] it was shown that, for a binary outer copula D, the function $D(C, C^*)\colon [0,1]^2 \to [0,1]$ given by

$$D(C, C^*)(x,y) = D(C(x,y), C^*(x,y))$$

is a binary copula for each binary inner copula C if and only if the outer copula D is ultramodular and Schur concave on the upper left triangle

$$\Delta = \{(x,y) \in [0,1]^2 \mid x \le y\} \subseteq [0,1]^2.$$

Recall that a binary copula D is ultramodular [9] on Δ if and only if all horizontal and vertical sections of D in Δ are convex, and it is Schur concave [11] on Δ if and only if all sections in Δ with a constant sum of arguments are concave.

Secondly, in [8] all functions $g\colon [0,1] \to [0,1]$ were characterized such that, for each binary copula C, the function $\Pi(C, g(1 - C^*))$ given by

$$\Pi(C, g(1 - C^*))(x,y) = \Pi(C(x,y), g(1 - C^*(x,y)))$$

is a binary copula. Denoting $f(x) = g(1 - x)$ one easily can rewrite their results for functions of the form $\Pi(C, f(C^*))$.

In this contribution we study, for binary copulas D and C and some transformation function $f\colon [0,1] \to [0,1]$, functions of the form $D(C, f(C^*))\colon [0,1]^2 \to [0,1]$ given by

$$D(C, f(C^*))(x,y) = D(C(x,y), f(C^*(x,y))). \tag{1}$$

Denote by $\mathscr{F}\big([0,1]^{[0,1]}\big)$ the set of all functions $f\colon [0,1] \to [0,1]$ which are monotone non-decreasing, 1-Lipschitz, convex and satisfy $f(1) = 1$.

Theorem 1. *Let $f\colon [0,1] \to [0,1]$ be a function and $D\colon [0,1]^2 \to [0,1]$ be a binary copula. The following are equivalent:*

(i) *$f \in \mathscr{F}\big([0,1]^{[0,1]}\big)$ and D is both ultramodular and Schur concave on the upper left triangle Δ of the unit square;*
(ii) *for each binary copula C the function $D(C, f(C^*))\colon [0,1]^2 \to [0,1]$ given by* (1) *is a binary copula.*

Examples of binary copulas which are ultramodular and Schur concave on Δ are the Fréchet-Hoeffding lower bound W, the product Π (which is the greatest ultramodular copula) and the Fréchet-Hoeffding upper bound M (although the latter is not ultramodular on $[0,1]^2$) as well as the Clayton copulas [10] with parameter $\lambda \in [-1,0]$. Also the construction of ordinal sums [6] preserves the ultramodularity and Schur concavity on Δ of the components.

For an affine function $f_a\colon [0,1] \to [0,1]$ given by $f_a(x) = a + (1-a)x$ we have $f_a \in \mathscr{F}\big([0,1]^{[0,1]}\big)$ if and only if $a \in [0,1]$. Similarly, for a quadratic function $f_{a,c}\colon [0,1] \to [0,1]$ given by $f_{a,c}(x) = a + (1-a-c)x + cx^2$ we have $f_{a,c} \in \mathscr{F}\big([0,1]^{[0,1]}\big)$ if and only if $a \in [0,1]$ and $0 \le c \le \min(a, 1-a)$.

Example 1. If we consider the case $D = C = \Pi$ then for each $\lambda \in [-1,0]$ the function $\Pi(\Pi, f_{1+\lambda}(\Pi^*))$ given by

$$\Pi(\Pi, f_{1+\lambda}(\Pi^*))(x,y) = xy + \lambda xy(1-x)(1-y) \tag{2}$$

is a Farlie-Gumbel-Morgenstern copula $C_\lambda^{\mathbf{FGM}}$ (see [10]).

Note that the family of Farlie-Gumbel-Morgenstern copulas is defined by (2) for all $\lambda \in [-1,1]$. For $\lambda > 0$ they can be expressed as $\Pi \cdot f_{1+\lambda}(\Pi^*)$, but in such a case we have $f_{1+\lambda}(0) > 1$, i.e., Theorem 1 does not apply.

But the Farlie-Gumbel-Morgenstern copulas for $\lambda \in \,]0,1]$ can be obtained from those with parameters in $[-1,0[$ using the flipping (see [1]) $C^-\colon [0,1]^2 \to [0,1]$ of a binary copula C given by $C^-(x,y) = x - C(x, 1-y)$, i.e., $\big(C_\lambda^{\mathbf{FGM}}\big)^- = C_{-\lambda}^{\mathbf{FGM}}$. A similar approach can be considered for other parametric families of copulas constructed by means of Theorem 1.

Example 2. If we consider the case $D = W$ and $C = \Pi$ then for each $\lambda \in [-1,0]$ the function $W(\Pi, f_{1+\lambda}(\Pi^*))$ given by

$$W(\Pi, f_{1+\lambda}(\Pi^*))(x,y) = \max((1+\lambda)xy - \lambda(x+y-1), 0)$$

is a Sugeno-Weber copula [6]. Putting $K_\lambda = W(\Pi, f_{1+\lambda}(\Pi^*))$ for $\lambda \in [-1,0]$ and defining, for $\lambda \in \,]0,1]$, $K_\lambda = \big(K_{-\lambda}\big)^-$, the family of copulas $\big(K_\lambda\big)_{\lambda \in [-1,1]}$ is a comprehensive family (i.e., containing the basic copulas W, Π and M) which is continuous and increasing with respect to the parameter λ.

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References

1. B. De Baets, H. De Meyer, J. Kalická, and R. Mesiar. Flipping and cyclic shifting of binary aggregation functions. *Fuzzy Sets and Systems*, 160:752–765, 2009.
2. A. Dolati and M. Úbeda-Flores. Constructing copulas by means of pairs of order statistics. *Kybernetika*, 45:992–1002, 2009.
3. F. Durante and C. Sempi. *Principles of Copula Theory*. CRC Press, Boca Raton, 2015.
4. M. Grabisch, J.-L. Marichal, R. Mesiar, and E. Pap. *Aggregation Functions*. Cambridge University Press, Cambridge, 2009.
5. E. P. Klement, A. Kolesárová, R. Mesiar, and S. Saminger-Platz. On the role of ultramodularity and Schur concavity in the construction of binary copulas. *J. Math. Inequal.*, 11:361–381, 2017.
6. E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms*. Kluwer Academic Publishers, Dordrecht, 2000.
7. A. Kolesárová, R. Mesiar, and J. Kalická. On a new construction of 1-Lipschitz aggregation functions, quasi-copulas and copulas. *Fuzzy Sets and Systems*, 226:19–31, 2013.
8. M. Manstavičius and G. Bagdonas. A class of bivariate copula mappings. *Fuzzy Sets and Systems*, 354:48–62, 2019.
9. M. Marinacci and L. Montrucchio. Ultramodular functions. *Math. Oper. Res.*, 30:311–332, 2005.
10. R. B. Nelsen. *An Introduction to Copulas*. Springer, New York, second edition, 2006.
11. I. Schur. Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie. *Sitzungsber. Berl. Math. Ges.*, 22:9–20, 1923.

Decisions without utilities, belief functions, and the target-based approach under dependence between targets and prospects

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We concentrate attention on single-attribute decision problems where the space $\mathcal{C}$ of possible *consequences* coincides with $\mathbb{R}$, where consequences can be interpreted as monetary amounts, and the *prospects* or *lotteries* X_α $(\alpha \in A)$ are real-valued random variables on a given probability space. In this framework, we will first review briefly the*Target-Based* approach ([1]; see also [4]) and the substantial equivalence of it with the standard setting of decisions under uncertainty, with a bounded and right-continuous utility function u. In practice, a target is a random variable T, stochastically independent of each prospect X_α, and with (normalized) distribution function corresponding to an affine transformation of the function u. Such an approach reveals to be useful in giving direct probabilistic interpretations of some economic properties.

It can be interesting, however, to consider the more general setting which is obtained by allowing some form of stochastic dependence for the pairs (T, X_a). This extension gives rise to a more general framework where one cannot invoke the principle of maximization of expected utility. We will, first of all, point out some relevant implications of conditions of stochastic dependence. In the second part of the talk, some connections will be investigated with different approaches to decisions without utility and with the analysis of belief functions. We will refer in particular to the papers [2, 3].

References

1. R. Bordley, M. LiCalzi (2000). Decision analysis using targets instead of utility functions. *Decisions in Economics and Finance* (74), 23–53.
2. J. Eichberger, D. Kelsey (1999). E-capacities and the Ellsberg paradox. *Theory and Decision* (46), 107–140.
3. T. Kerkvliet, R. Meester (2018). A behavioral interpretation of belief functions. *Journal of Theoretical Probability* (31), 2112–2128.
4. F. L. Spizzichino (2018). On the probabilistic meaning of copula-based extensions of fuzzy measures. Applications to target-based utilities and multi-state reliability systems. *Fuzzy Sets and Systems*, in press.

New types of decomposition integrals for real- and interval-valued functions

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Abstract. In this contribution we will consider a special subclass of decomposition integrals namely the collection integral. This integral is characterized by decomposition systems consisting of only one collections. Some examples of such integrals are presented. Also a condition on the underlying collection is given to ensure that given collection integral extends the Lebesgue integral. We also consider a decomposition integrals for interval-valued functions constructed by two different methods – firstly by Aumann's principle and secondly by interval algebra. Interestingly, both constructions lead to the same integral.

1 Introduction

Integrals play an important role throughout all mathematics and its applications. Decomposition integrals are a common framework for the Lebesgue and nonlinear integrals, including the Choquet [2], PAN [7], Shilkret [6], and concave [4] integrals.

2 Preliminaries

In this contribution we will be interested in non-negative functions $f\colon X \to [0,\infty[$ defined on a fixed finite space $X = \{1,2,\dots,n\} \subsetneq \mathbb{N}$. The class of such functions will be denoted by $\mathcal{F}$.

Also interval-valued functions will be considered. Let $L([0,\infty[)$ denote the class of all closed sub-intervals of $[0,\infty[$, i.e.,

$$e \in L([0,\infty[) \iff e = [a,b], a,b \in \mathbb{R}, 0 \le a \le b.$$

An interval-valued function on 2^X is any function $v\colon X \to L([0,\infty[)$. A class of interval-valued functions will be denoted by $\mathcal{V}$. We say that $f \in \mathcal{F}$ is a selector of $v \in \mathcal{V}$ if and only if $f(x) \in v(x)$ for all $x \in X$. If f is a selector of v we will write $f \in v$.

A set function on X is any function $\mu\colon 2^X \to [0,\infty[$. If a set function μ is grounded, i.e., $\mu(\emptyset) = 0$, and increasing with respect to set inclusion, i.e., $A \subseteq B \subseteq X$ implies $\mu(A) \le \mu(B)$, then μ is called a capacity. A class of all capacities is denoted by $\mathcal{M}$. The class of measures, i.e., additive capacities, is denoted by $\mathcal{M}_+$ and $\mathcal{M}_+ \subsetneq \mathcal{M}$.

A collection $\mathcal{D}$ is any non-empty subset of the power set with excluded empty set, i.e., $\emptyset \ne \mathcal{D} \subseteq 2^X \setminus \{\emptyset\}$. A decomposition system is a non-empty set of collections.

Definition 1. *A decomposition integral [3, 5] with respect to a decomposition system* $\mathcal{H}$ *is an operator* $I_{\mathcal{H}} \colon \mathcal{F} \times \mathcal{M} \to [0, \infty[$ *given by*

$$I_{\mathcal{H}}(f, \mu) = \bigvee_{\mathcal{D} \in \mathcal{H}} \bigvee \left\{ \sum_{A \in \mathcal{D}} a_A \mu(A) \colon \sum_{A \in \mathcal{D}} a_A 1_A \leq f, a_A \geq 0 \right\}.$$

Based on the choice of $\mathcal{H}$ we get different (in general) non-linear integrals. A special choice of $\mathcal{H} = \{\{\{1\}, \{2\}, \dots, \{n\}\}\}$ with restriction to the space $\mathcal{F} \times \mathcal{M}_+$ leads to the Lebesgue integral denoted by Leb.

Definition 2. *An Aumann integral [1] (based on the Lebesgue integral) is an operator* $\mathrm{Aum} \colon \mathcal{V} \times \mathcal{M}_+ \to [0, \infty[$ *given by*

$$\mathrm{Aum}(v, \mu) = \{\mathrm{Leb}(f, \mu) \colon f \in v\}.$$

In other words, the Aumann integral is the envelope of all possible Lebesgue integrals of functions lying in the interval-valued function, i.e., of selectors. Note that the definition can be further extended to any set-valued function.

3 Collection integral

In the first part of this contribution we will introduce a collection integral, i.e., a decomposition integral based on a decomposition system consisting of only one collection.

Definition 3. *A collection integral with respect to a collection* $\mathcal{D}$ *is an operator* $I_{\mathcal{D}}$ *given by*

$$I_{\mathcal{D}} \colon \mathcal{F} \times \mathcal{M} \to [0, \infty[\colon (f, \mu) \mapsto \bigvee \left\{ \sum_{A \in \mathcal{D}} a_A \mu(A) \colon \sum_{A \in \mathcal{D}} a_A 1_A \leq f \right\}.$$

We now present some examples of collection integrals.

Example 1.
- Let $\mathcal{D} = \{A_i\}_{i=1}^k$ be a chain on X. Then

  $$I_{\mathcal{D}}(f, \mu) = \mu(A_k) \min f(A_k) + \sum_{i=1}^{k-1} \mu(A_i) \Big(\min f(A_i) - \min f(A_{i+1}) \Big)$$

  is a collection integral.
- As a special case of the previous example, we can take a chain $\mathcal{D} = \{X\}$ and we obtain

  $$I_{\mathcal{D}}(f, \mu) = \mu(X) \min f(X).$$

It is interesting to know when the collection integral extends the Lebesgue integral. In other words, we would like to know what conditions must a collection $\mathcal{D}$ satisfy to ensure that

$$I_{\mathcal{D}} \restriction_{\mathcal{F} \times \mathcal{M}_+} = \mathrm{Leb}.$$

We show that the necessary and sufficient condition for the above coincidence is that $\{x\} \in \mathcal{D}$ for all $x \in X$.

4 Decomposition integrals for interval-valued functions

There are two natural extensions of decomposition integrals for interval-valued functions. One is based on the Aumann integral and the second one is based on the algebra of intervals.

Definition 4. *A decomposition integral of Aumann type with respect to a decomposition system $\mathcal{H}$ is an operator*

$$A_{\mathcal{H}}\colon \mathcal{V} \times \mathcal{M} \to [0,\infty[\colon (v,\mu) \mapsto \{I_{\mathcal{H}}(f,\mu)\colon f \in v\}\,.$$

Note that we define addition and (non-negative) scalar multiplication of elements of $L([0,\infty[)$ in the following standard way:

$$[a,b] + [c,d] = [a+c, b+d], \qquad \alpha[a,b] = [\alpha a, \alpha b],$$

and the supremum is given by

$$[a,b] \vee [c,d] = [a \vee c, b \vee d].$$

Definition 5. *An interval-valued decomposition integral with respect to a decomposition system $\mathcal{H}$ is an operator $\tilde{A}_{\mathcal{H}}\colon \mathcal{V} \times \mathcal{M} \to [0,\infty[$ such that*

$$\tilde{A}_{\mathcal{H}}(v,\mu) = \bigvee_{\mathcal{D}\in\mathcal{H}} \bigvee \left\{ \sum_{A\in\mathcal{D}} [a_A, b_A]\mu(A)\colon \sum_{A\in\mathcal{D}} [a_A, b_A] 1_A \le f, 0 \le a_A \le b_A \right\}.$$

Interestingly, both of these constructions lead to the same integral.

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References

1. Aumann, R.J.: Integrals of set-valued functions. *Journal of Mathematical Analysis and Applications* **12** (1965), 1–12.
2. Choquet, G.: Theory of capacities. *Annales de l'Institut Fourier* **5** (1954), 131–295.
3. Even, Y., Lehrer, E.: Decomposition-integral: unifying Choquet and concave integrals. *Economic Theory* **56** (2014), 33–58.
4. Lehrer, E.: A new integral for capacities. *Economic Theory* **39** (2009), 157–176.
5. Mesiar, R., Stupňanová, A.: Decomposition integrals. *International Journal of Approximate Reasoning* **54** (2013), 1252–1259.
6. Shilkret, N.: Maxitive measure and integration. *Indagationes Mathematicac* **33** (1971), 109–116.
7. Yang, Q.: The Pan-integral on the fuzzy measure space. *Fuzzy Mathematics* **3** (1985), 107–114.

Not just numbers – logics for representation of preferences and decision making

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In a group our choices are strictly related to our ability to duly express and compare alternatives according to different criteria, e.g. price, utility, feelings, life goals, social conventions, personal values, etc.

Usually preferences are expressed by yes-no answer or by numbers to encode the intensity of preference. These representation are a research field in mathematical logic, in which a yes-no answer is a true-false sentence (classical logic) and intensity is a truth degree of a formula (many-valued logic).

We propose logics and algebraic structures to deal preferences in decision making field.

References

1. Di Nola A., Squillante M., Vitale G. Social Preferences Through Riesz Spaces: A First Approach. In: Collan M., Kacprzyk J. (eds) *Soft Computing Applications for Group Decision-making and Consensus Modeling. Studies in Fuzziness and Soft Computing*, vol 357. Springer, Cham (2018)
2. Cavallo B., D'Apuzzo L., Di Nola A., Squillante M., Vitale G. A General Framework for Individual and Social Choices. In: Flaut C., Hošková-Mayerová Š., Flaut D. (eds) *Models and Theories in Social Systems. Studies in Systems, Decision and Control*, vol 179. Springer, Cham (2019)
3. Vitale G. Risk Analysis via Łukasiewicz Logic (submitted)

New mathematical concepts for nonadditive set functions coming for empirical decision theory under ambiguity

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Nonadditive set functions, also known as fuzzy measures, capacities, or weighting functions, have been used in many domains (Grabisch 2016). This lecture focuses on their applications to decision under uncertainty, where they model ambiguity (unknown probabilities). First a general introduction is given on the history of ambiguity (Keynes 1921, Knight 1921, Savage 1954, Ellsberg 1961), explaining why Gilboa & Schmeidler (1989) was such a breakthrough. Nonclassical measure theory employing Choquet integrals is, unsurprisingly, used in Choquet expected utility theory for ambiguity, but, surprisingly, can also be used in many ambiguity theories that do not directly seem to involve capacities.

Many new questions arise concerning psychological interpretations, economic relevance, and empirical performance. But, also, many theoretical concepts present themselves, calling for the development of new mathematics. For example, under classical expected utility for risk, the well-known Pratt-Arrow (1964) index of utility, $-U''/U'$, comes out as a natural index of risk aversion. Mathematically, it measures the degree of concavity/convexity, and was known before (de Finetti 1952). We need a similar quantitative index of concavity/convexity of capacities. This lecture presents and axiomatizes one.

A second mathematical question is imposed upon us by empirical data–such unforeseen things can happen if mathematical theories are confronted with reality. Although theoretical studies have as yet focused almost exclusively on ambiguity aversion, eagerly applying our familiar and beloved tools of convex analysis and linear functional analysis (in the so-called Anscombe-Aumann model), data put forward another phenomenon: insensitivity. It reflects the degree to which people/capacities do not sufficiently discriminate between different events and take them too much as one blur, moving all their weights in the direction of 0.5 (I assume a $0 - 1$ normalized scale). It is a sort of regression to the mean, reflected in everyday life by the overuse of the uncertainty assessment fifty-fifty. We formalize this property of capacities, introduce an index for it, and axiomatize it. One implication is that Dempster-Shafer belief functions do not fit ambiguity attitudes well.

References

[de Finetti 1952] B. de Finetti, Sulla preferibilità, *Giornale degli Economisti e annali di economia* **11** (1952), 685–709. s

[Ellsberg 1961] D. Ellsberg, Risk, ambiguity and the Savage axioms, *Q. J. Econ.* **75** (1961), 643–669.

[Gilboa& Schmeidler 1989] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, *J. Math. Econ.* **18**(2) (1989), 141–153.

[Grabisch 2016] M. Grabisch, "Set Functions, Games and Capacities in Decision Making", Theory and Decision Library C, vol. 46, Springer, 2016.

[Keynes 1921] J. M. Keynes, "A Treatise on Probability", MacMillan, London, 1921.

[Knight 1921] F. H. Knight, "Risk, Uncertainty, and Profit", Houghton Mifflin, Boston, New York, 1921.

[Pratt-Arrow 1964] J. W. Pratt, Risk aversion in the small and in the large, *Econometrica*, **32** (1964) 122–136.

[Savage 1954] L. J. Savage, "The Foundations of Statistics", John Wiley & Sons, New York, 1954.

Change of base using arrow categories

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A general approach to fuzziness is based on so-called L-fuzzy sets or relations. Such sets or relations assign to each element or pair of elements a certain degree up to which the element or the pair is in the set or relation. These membership values are taken from a complete Heyting algebra L. Formally, an L-fuzzy relation R (or L-relation for short) between a set A and a set B is a function $R : A \times B \to L$. These relations generalize fuzzy relations by replacing the unit interval by an arbitrary complete Heyting algebra and allow, therefore, for incomparable degrees of membership. A suitable abstract theory covering these relations is given by arrow or Goguen categories. These theories have been studied intensively [4–8, 10] including investigations into higher-order fuzziness [11, 12]. In addition to the theoretical studies, these categories have been used to model and specify type-1 and type-2 L-fuzzy controllers [9, 13] as well as L-fuzzy databases [1–3, 15].

First, we define Dedekind categories as suitable categorical framework for relations.

Definition 1. *A Dedekind category $\mathcal{R}$ is a category satisfying the following:*

1. *For all objects A and B the collection $\mathcal{R}[A, B]$ is a complete Heyting algebra. Meet, join, the induced ordering, the least and the greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq$, $\perp\!\!\!\perp_{AB}$, $\top\!\!\!\top_{AB}$, respectively.*
2. *There is a monotone operation $\smile$ (called converse) mapping a relation $Q : A \to B$ to $Q^\smile : B \to A$ such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds: $(Q;R)^\smile = R^\smile ; Q^\smile$ and $(Q^\smile)^\smile = Q$.*
3. *For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law $(Q;R) \sqcap S \sqsubseteq Q;(R \sqcap (Q^\smile;S))$ holds.*
4. *For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $X : A \to B$ the following holds: $X;R \sqsubseteq S \iff X \sqsubseteq S/R$.*

Intuitively, $\sqcap$ and $\sqcup$ are the meet and join operations on relations, $\sqsubseteq$ the induced order on relations,and $\perp\!\!\!\perp_{AB}$ and $\top\!\!\!\top_{AB}$ the empty and universal relation. $Q^\smile$ is the converse of the relation Q and $\mathbb{I}_A$ and ; are the identity relation and composition of relations.

An arrow category adds two operations to a Dedekind category allowing to talk about crispness. A crisp relation is a relation in which each pair is either with full degree 1 or not at all (degree 0). Intuitively, the operations $.^\downarrow$ and $.^\uparrow$ map every relation to greatest crisp relation it contains resp. to the least crisp relation it is included in. The abstract definition is a follows.

Definition 2. *An arrow category $\mathcal{A}$ is a Dedekind category with $\mathbb{T}_{AB} \neq \mathbb{L}_{AB}$ for all A, B and two operations $\uparrow$ and $\downarrow$ satisfying:*

1. $R^{\uparrow}, R^{\downarrow} : A \to B$ *for all* $R : A \to B$
2. $({}^{\uparrow}, {}^{\downarrow})$ *is a Galois correspondence, i.e., we have* $Q^{\uparrow} \sqsubseteq R$ *iff* $Q \sqsubseteq R^{\downarrow}$ *for all* $Q, R : A \to B$.
3. $(R^{\smile} ; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile} ; S^{\downarrow}$ *for all* $R : B \to A$ *and* $S : B \to C$
4. $(Q \sqcap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$ *for all* $Q, R : A \to B$
5. *If* $\alpha_A \neq \mathbb{L}_{AA}$ *is a non-zero scalar then* $\alpha_A^{\uparrow} = \mathbb{I}_A$.

Even though arrow categories are abstract categories it is possible to identify the underlying Heyting algebra of membership values. This algebra is given by the set of scalar relations on object A, i.e., the relations $\alpha : A \to A$ that satisfy $\alpha \sqsubseteq \mathbb{I}_A$ and $\alpha; \mathbb{T}_{AA} = \mathbb{T}_{AA}; \alpha$. In the case of concrete finite L-relations a scalar can be seen as a square matrix with a fixed element from L on the diagonal and 0 everywhere else. A fundamental theorem of arrow categories shows that the Heyting algebras of scalar relation for two different objects A and B of the same arrow category are always isomorphic. We will identity these algebras and denote them by $\mathrm{Sc}(\mathcal{R})$. In addition, we use $\alpha, \beta, \ldots$ to denote elements from $\mathrm{Sc}(\mathcal{R})$ and α_A to denote the corresponding scalar on the object A, i.e., $\alpha_A : A \to A$. All of this shows that arrow categories emphasize the so-called fixed-base approach to L-fuzziness, i.e., all relations of a given arrow category use the same Heyting algebra L as membership values.

It can easily be verified that the substructure of crisp relations, i.e., all relations R with $R^{\downarrow} = R$ (or equivalently $R^{\uparrow} = R$), is a Dedekind category satisfying the so-called Tarski rule, i.e., the equivalence $R \neq \mathbb{L}_{AB}$ iff $\mathbb{T}_{CA}; R; \mathbb{T}_{BD} = \mathbb{T}_{CD}$. We denote this Dedekind category by $\mathcal{A}^{\downarrow}$. Please note that we have $\mathrm{Sc}(\mathcal{A}^{\downarrow}) = \{\mathbb{L}, \mathbb{I}\}$ verifying that $\mathcal{A}^{\downarrow}$ is based on the Boolean values as membership degrees.

In this paper we are interested in the process of changing the base, i.e., an operation that allows to switch from an L_1-relation to an L_2 relation that preserves the relational content but exchanging the membership values from L_1 to L_2. Please note that this is done in setting of abstract arrow categories and not for concrete L-relations where component-wise reasoning can be performed.

Definition 3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrow categories. An functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ is called an arrow functor iff F preserves all operations of an arrow category except $(./.)$, i.e., we have*

1. $F(\mathbb{L}_{AB}) = \mathbb{L}_{AB}$, *and* $F(\mathbb{T}_{AB}) = \mathbb{T}_{AB}$,
2. $F(Q \sqcap R) = F(Q) \sqcap F(R)$, *and* $F(Q \sqcup R) = F(Q) \sqcup F(R)$,
3. $F(Q^{\smile}) = F(Q)^{\smile}$,
4. $F(R^{\uparrow}) = F(R)^{\uparrow}$, *and* $F(R^{\downarrow}) = F(R)^{\downarrow}$.

A change of base is an arrow functor between arrow categories with the same crisp relations that is the identity on those crisp relations, i.e., preserves the purely relational structure.

Definition 4. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrow categories so that $\mathcal{A}_1^{\downarrow} = \mathcal{A}_2^{\downarrow}$. An arrow functor $F : \mathcal{A}_1 \to \mathcal{A}_2$ is called a change of base iff*

1. F is faithful,
2. F restricted to $\mathcal{A}_1^{\downarrow}$ is the identity.

Please note that we could relax the condition $\mathcal{A}_1^{\downarrow} = \mathcal{A}_2^{\downarrow}$ by requiring that the two categories are only isomorphic. For simplicity we will require the stronger version in this paper.

Any change of base induces a Heyting algebra homomorphism between the membership values of the two categories in question.

Theorem 1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be arrow categories and $F : \mathcal{A}_1 \to \mathcal{A}_2$ be a change of base. Then the function $h_F : \mathrm{Sc}(\mathcal{A}_1) \to \mathrm{Sc}(\mathcal{A}_2)$ defined by $h_F(\alpha) = F(\alpha_A)$ for some object A is a homomorphism from $\mathrm{Sc}(\mathcal{A}_1)$ to $\mathrm{Sc}(\mathcal{A}_2)$.*

In the presentation we will show that the opposite implication of the previous theorem is also true. Given a homomorphism $h : \mathrm{Sc}(\mathcal{A}_1) \to \mathrm{Sc}(\mathcal{A}_2)$, then there is a change of base $F_h : \mathcal{A}_1 \to \mathcal{A}_2$ so that $h_{F_h} = h$. Furthermore, we also show that $F_{h_F} = F$ for every change of base F. Consequently, there is a one-one correspondence between the homomorphisms between the membership values and the class of change of bases.

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References

1. Adjei E.: L-Fuzzy Structural Query Language. Brock University (M.Sc. Thesis) (2015).
2. Adjei E., Chowdhury W., Winter M.: L-Fuzzy Databases in Arrow Categories. 15th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS 15), LNCS 9348, 288-304 (2015).
3. Chowdhury W.: An Abstract Algebraic Theory of L-Fuzzy Relations in Relational Databases. Brock University (M.Sc. Thesis) (2015).
4. Furusawa H., Kawahara Y., Winter M.: Dedekind Categories with Cutoff Operators. Fuzzy Sets and Systems 173, 1-24 (2011).
5. Winter M.: A new Algebraic Approach to *L*-Fuzzy Relations Convenient to Study Crispness. Information Science 139, 233-252 (2001).
6. Winter M.: Relational Constructions in Goguen Categories. 6th International Conference on Relational Methods in Computer Science (RelMiCS 6) LNCS 2561, 212-227 (2002).
7. Winter M.: Derived Operations in Goguen Categories. Theory and Applications of Categories 10(11), 220-247 (2002).
8. Winter M.: Representation Theory of Goguen Categories. Fuzzy Sets and Systems 138, 85-126 (2003).
9. Winter M.: Goguen Categories - A Categorical Approach to *L*-fuzzy relations. Trends in Logic 25, Springer (2007).
10. Winter, M.: Arrow Categories. Fuzzy Sets and Systems 160, 2893-2909 (2009).
11. Winter M.: Membership Values in Arrow Categories. Fuzzy Sets and Systems 267, 41-61 (2015).
12. Winter M.: Higher-Order Arrow Categories. 14th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS 14), LNCS 8428, 277-292 (2014).

13. Winter M., Jackson E., Fujiwara Y.: Type-2 Fuzzy Controllers in Arrow Categories. 14th International Conference on Relational and Algebraic Methods in Computer Science (RAMiCS 14), LNCS 8428, 293-308 (2014).
14. Winter M., Jackson E.: Categories of Relations for Variable-Basis Fuzziness. Fuzzy Sets and Systems 298 , 222-237 (2016).
15. Winter M.: Dependencies in Relational Models of Databases. JLAMP 83(5), Special Issue: Festschrift J.N. Oliveira, 1095-1107 (2016).

The decision-making driven approach to specificity of possibilities in the Savage-style setting

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Abstract. In my last paper in "Fuzzy Sets Syst." [6], I proposed a new approach to specificity of comparative possibility distributions based on the idea that the specificity relation definition has to be consistent with the possibilistic decision making (DM): the more specific an underlying possibilistic model is, the narrower a set of the optimal decisions has to be. Let's call such an approach the "DM-driven" one. The DM problem considered in my paper cited above was similar to the statistical problems of point estimation and hypotheses testing. In this work, I study the DM-driven approach to specificity in a more conventional Savage-style setting of the DM problem.

1 Defining the DM-driven specificity relation

In my paper [6] I investigated specificity of comparative possibility distributions [4, 2]. Despite the fact that I considered those distributions as real-valued functions $\pi : S \to [0,1]$ such that $\max_{s\in S} \pi(s) = 1$, where S is a set of states of the world, the obtained results remain suitable in the purely ordinal setting as well. The reason is that the unit interval $[0,1]$ was actually considered in [6] as an ordinal scale with the order-preserving algebraic operations "max" and "min".

Beside the representation of comparative possibility distribution in terms of a real-valued function π, I use a mathematically equivalent representation in terms of a well-ordered partition $\mathrm{WOP}(\pi) = \left(S_1^\pi,\ \ldots,\ S_n^\pi;\ Z^\pi\right)$ of S [1] in this paper:

$$\pi(s) = 0 \Leftrightarrow s \in Z^\pi, \quad \pi(s_1) > \pi(s_2) > 0 \Leftrightarrow s_1 \in S_i^\pi,\ s_2 \in S_j^\pi,\ i < j. \qquad (1)$$

The main result of [6] was a new specificity relation "$\preceq$" of comparative possibility distributions. In terms of well-ordered partitions, $\pi_1 \preceq \pi_2$ can be defined as follows:

SpRel-1. $Z^{\pi_2} \subset Z^{\pi_1}$:
the π_2-impossible states of the world are π_1-impossible as well,
SpRel-2. $\exists r$ such that $S_i^{\pi_2} \cap Z^{\pi_1} = \varnothing,\ \forall i < r$, and $S_j^{\pi_2} \subset Z^{\pi_1},\ \forall j > r$:
according to π_2, the π_1-impossible states are not more likely than all the others,
SpRel-3. $\forall p$, $\exists q$ such that $\bigcup_{i=1}^{p} S_i^{\pi_2} \setminus Z^{\pi_1} = \bigcup_{j=1}^{q} S_j^{\pi_1}$:
on the set $S \setminus Z^{\pi_1}$ of π_1-possible states, the preference ordering induced by π_1 refines the one induced by π_2, that is, $\forall s_1,\ s_2 \in S \setminus Z^{\pi_1}$, if $\pi_1(s_1) \leqslant \pi_1(s_2)$, then $\pi_2(s_1) \leqslant \pi_2(s_2)$.

2 Formulating the DM problem in the Savage-style setting

In [6], the role of the specificity relation "$\preceq$" in decision making was studied. The decision-making problem formulated in [6] was to estimate an unknown value $\xi(s)$ of a latent (unobservable) function ξ in the current state of the world $s \in S$ using a value $\eta(s)$ (known to a decision maker) taken by an observable function η in the same state. The decision maker did not know what was the current state of the world $s \in S$ in a precise way, and that was modeled on the qualitative possibilistic basis. That is, the decision maker knew a joint possibility distribution of ξ and η

$$\pi^{\xi,\eta}(x, y) = \mathit{\Pi}\big(\{\xi(s) = x \text{ and } \eta(s) = y \text{ simultaneously}\}\big),$$

where $\mathit{\Pi} : 2^S \to [0, 1]$ was a maxitive possibility measure expressing *a priori* qualitative imprecise information about the world. Mathematically, the decision-making problem was formulated in [6] as the following optimization problem:

$$\mathit{\Pi}\big(\{s : \ \delta(\eta(s)) \neq \xi(s)\}\big) = \max\big\{\pi^{\xi,\eta}(x, y) \,\big|\, \delta(y) \neq x\big\} \sim \min_{\delta}. \qquad (2)$$

That was a problem of finding estimators δ (the mappings from the range of η to the range of ξ) minimizing possibility of incorrect estimation.

However, a more conventional mathematical model of decision-making is the Savage-style one [5]. According to Savage, the decision maker must to choose optimal act(s) from a set F of all feasible acts. Each act $f \in F$ is a mapping from the set of states S to a set X of potential consequences of decisions that are ranked according to their utility. That is, a real-valued utility function u and the corresponding complete preorder $\leqslant_u$ are defined on X: $x_1 \leqslant_u x_2 \Leftrightarrow u(x_1) \leqslant u(x_2)$.

Therefore, if the decision maker knows for sure that the current state of the world is $s \in S$, then it knows the consequence $f(s)$ of any act $f \in F$ and its utility $u(f(s))$ as well. In this case, a set of optimal acts F_s is as follows:

$$F_s = \big\{f \in F : \ u(f(s)) = u_{\max}(s)\big\}, \quad \text{where} \quad u_{\max}(s) \triangleq \max_{g \in F} u(g(s)). \qquad (3)$$

Under qualitative possibilistic uncertainty, i. e., if the current state of the world is not known precisely, and it is modeled on the qualitative possibilitic basis as described above, a mathematical problem of choosing acts can be formulated in three ways. One of them is to maximize possibility of optimality:

$$P^*(f \mid \pi) \triangleq \max\big\{\pi(s) \,\big|\, f \in F_s\big\} \sim \max_{f \in F}, \qquad (4)$$

another is to minimize possibility of nonoptimality:

$$L_*(f \mid \pi) \triangleq \max\big\{\pi(s) \,\big|\, f \notin F_s\big\} \sim \min_{f \in F}. \qquad (5)$$

Let $F^*(\pi)$ and $F_*(\pi)$ be the sets of all solutions of (4) and (5) respectively. Note that

$$\text{if } \min_{f \in F} L_*(f \mid \pi) \neq 1, \text{ then } F_*(\pi) \subset F^*(\pi). \qquad (6)$$

Indeed, $\max\big\{L_*(f \mid \pi),\, P^*(f \mid \pi)\big\} = 1$. Therefore, if $f \in F_*(\pi)$ and $L_*(f \mid \pi) \neq 1$, then $P^*(f \mid \pi) = 1 = \max_{g\in F} P^*(g \mid \pi)$, i. e., $f \in F^*(\pi)$.

The last formulation of the DM problem is based on the likely dominance rule (LDR) which establishes the preference between acts as follows [3]:

$$f >_\pi g \Leftrightarrow \max\big\{\pi(s) \,\big|\, u(f(s)) > u(g(s))\big\} > \max\big\{\pi(s) \,\big|\, u(g(s)) > u(f(s))\big\},$$
$$f \geqslant_\pi g \Leftrightarrow \text{not } g >_\pi f.$$

Let $F_{\text{LDR}}(\pi)$ be a set of the most preferable acts according to the LDR:

$$F_{\text{LDR}}(\pi) = \{f \in F :\ \forall g \in F,\ f \geqslant_\pi g\}. \tag{7}$$

The theorem below states that those acts are the most preferable according to (5) too:

Theorem 1. $F_{\text{LDR}}(\pi) \subset F_*(\pi)$.

Proof. Assume that $f \in F_{\text{LDR}}(\pi)$ but $f \notin F_*(\pi)$. Then $\exists g \in F$ such that $\max\big\{\pi(s) \,\big|\, u(g(s)) \neq u_{\max}(s)\big\} < \max\big\{\pi(s) \,\big|\, u(f(s)) \neq u_{\max}(s)\big\}$, see (5) and (3). Therefore, there exists integer i s. t. $u(g(s)) = u_{\max}(s),\ \forall s \in S_1^\pi \cup \ldots \cup S_i^\pi$, while $u(f(s_0)) < u_{\max}(s_0) = u(g(s_0))$ for some $s_0 \in S_i^\pi$. Thus, $g >_\pi f$ which contradicts to the initial assumption. □

3 The DM-driven specificity relation in the Savage-style setting

Above, I formulated the problem of decision making in the Savage-style setting in three different ways, see (4), (5), (7). In this section I study the role of the DM-driven specificity relation "$\preceq$" defined by SpRel-1 – SpRel-3 in those DM problems. Firstly,

Theorem 2. *If $\pi_1 \preceq \pi_2$, then $F^*(\pi_1) \subset F^*(\pi_2)$.*

Proof. Obviously, $F^*(\pi) = \{f \in F :\ u(f(s)) = u_{\max}(s) \text{ for some } s \in S_1^\pi\}$, where S_1^π is a set of the most likely states, see (1). According to SpRel-3, $S_1^{\pi_1} \subset S_1^{\pi_2}$. Hence, $F^*(\pi_1) \subset F^*(\pi_2)$. □

The meaning of theorem 2 is exactly as stated in the abstract: the more specific an underlying possibilistic model is, the narrower a set of the optimal decisions has to be. However, criteria (4) is not very decisive (see (6) and theorem 1), so it is not of much interest. Let us study the role of "$\preceq$" with respect to the LDR.

Theorem 3. *If $\pi_1 \preceq \pi_2$, then $f \in F_{\text{LDR}}(\pi_1)$ implies either $f \in F_{\text{LDR}}(\pi_2)$ or $u(f(s)) = u(g(s))$ for any $g >_{\pi_2} f$ and all π_1-possible states of the world $s \in S$.*

Proof. $\forall g >_{\pi_2} f$, $\max\big\{\pi_2(s) \,\big|\, u(f(s)) > u(g(s))\big\} < \max\big\{\pi_2(s) \,\big|\, u(g(s)) > u(f(s))\big\}$. $f \in F_{\text{LDR}}(\pi_2) \Rightarrow \max\big\{\pi_1(s) \,\big|\, u(f(s)) > u(g(s))\big\} \geqslant \max\big\{\pi_1(s) \,\big|\, u(g(s)) > u(f(s))\big\}$. These two inequalities can be met simultaneously if and only if $\max\big\{\pi_1(s) \,\big|\, u(f(s)) > u(g(s))\big\} = \max\big\{\pi_1(s) \,\big|\, u(g(s)) > u(f(s))\big\} = 0$, see SpRel-1 – SpRel-3. That is, $\max\big\{\pi_1(s) \,\big|\, u(f(s)) \neq u(g(s))\big\} = 0$. □

Compared with theorem 2, the meaning of theorem 3 is more complicated. It does not guaranty that $\pi_1 \preceq \pi_2$ implies $F_{\text{LDR}}(\pi_1) \subset F_{\text{LDR}}(\pi_2)$. I. e., if $f \in F_{\text{LDR}}(\pi_1)$, there can exist g satisfying the condition $g >_{\pi_2} f$. However, for any such an act g, there is no difference in utility between the consequences $f(s)$ and $g(s)$ of the acts f and g in any π_1-possible state of the world s, see fig. 1.

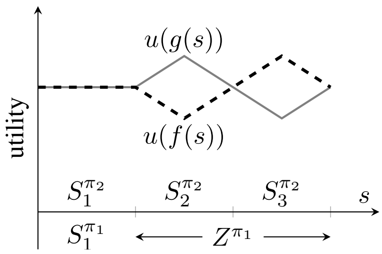

Fig. 1. A graphical representation of possibility distributions $\pi_1 \preceq \pi_2$ and acts f, g such that: (a) according to π_2, g is preferred to f, (b) according to π_1, the consequences of f and g are of equal utility in any possible state of the world $s \in S$.

Let us consider the following situation as an example. *Andrew* and *Barbara* are planning a weekend trip to Vienna by car. *Andrew* is sure that there will be *clear weather* in Vienna on the weekend, so there is no difference between acts $f =$ "*to put an umbrella in the car trunk*" and $g =$ "*to put waterproof jackets*", i. e., f and g are in $F_{\text{LDR}}(\textit{Andrew})$. *Barbara* is less specific and thinks that *clear weather* will be the most likely, *rain with strong wind* will be less likely, and, finally, *rain without wind* will be the least likely. Therefore, the act g is more preferable than f (cause an umbrella is not useful in the case of strong wind): $g >_{\textit{Barbara}} f$. However, this is a fallacious preference caused by *Barbara*'s poor knowledge about weather.

As seen from this example, the fact that conditions $f \in F_{\text{LDR}}(\pi_1)$ and $g >_{\pi_2} f$ are met together does not contradict to the common idea of specificity of incomplete knowledge represented by π_1 and π_2 if the conclusion of theorem 3 is true. Therefore, theorem 3 does allow us to connect the formal definition SpRel-1 – SpRel-2 of the relation "$\preceq$" with the human-understandable idea of specificity.

This is similar to the result obtained in [6] w. r. t. problem (2). This fact emphasizes the universal role of the DM-driven specificity in DM under qualitative possibilistic uncertainty, and it does not matter what setting of the DM problem is in question.

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References

1. S. Benferhat, D. Dubois, H. Prade, Possibilistic and standard probabilistic semantics of conditional knowledge bases, *J. Logic Comput.* **9** (6) (1999), 873–895.
2. D. Dubois, Belief structures, possibility theory and decomposable confidence measures on finite sets, *Comput. Artif. Intell.* **5** (5) (1986), 403–416.
3. D. Dubois, H. Fargier, H. Prade Decision making under ordinal preferences and uncertainty, *Proc. of the 13th Conf. on Uncertainty in AI.* Providence, RI, Morgan & Kaufmann, San Francisco, CA (1997), 157–164.
4. D. Lewis, Counterfactuals and comparative possibility, *J. Philos. Logic.* **2** (4) (1973), 418–446.
5. L. J. Savage, "The Foundations of Statistics", Dover, New York 1972.
6. A. V. Zubyuk, A new approach to specificity in possibility theory: decision-making point of view, *Fuzzy Sets Syst.* (2018), in press.

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