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Abstracts

Many-Valued Logics: Theory and Applications

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Abstracts

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Editors

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MANY-VALUED LOGICS: THEORY AND APPLICATION

ABSTRACTS

Tommaso Flaminio, Lluís Godo, Carles Noguera,
Susanne Saminger-Platz, Thomas Vetterlein
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Since their inception in 1979 the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2022 will be the 39th seminar carrying on this tradition and is devoted to the theme "Many-Valued Logics: Theory and Applications". The goal of the seminar is to present and to discuss recent advances of many-valued and mathematical fuzzy logics and their applications in pure and applied fields.

This volume contains the abstracts of the contributions accepted for presentation at LINZ 2022. The regular contributions are complemented by six invited plenary talks, some of which are intended to give new ideas and impulses from outside the traditional Linz Seminar community.

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New foundations of reasoning via real-valued first-order logics

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Typically the study of inference in many-valued logic answers the following question: given that all premises in a given set Γ are fully true (i.e. take value 1), what other formulas can we see to be fully true as a consequence? The approach followed by [1] goes further and asks: what information can we infer on the assumption that the formulas in are partially true (i.e. take truth values other than 1)? What other formulas could be seen to be partially true or completely false? The goal is to axiomatize inference genuinely involving many truth-values.

In fact, the work in [1] poses the above questions not just for single formulas but for sequences of formulas taking certain combinations of values considered as a single expression called a *multidimensional sentence* (in short, an MD-sentence). More precisely, an MD-sentence is a syntactic object of the form $\langle \sigma_1, \ldots, \sigma_n, S \rangle$ where S is a set of n-tuples of truth-values for the sequence of formulas $\sigma_1, \ldots, \sigma_n$. The semantic intuition is that $\langle \sigma_1, \ldots, \sigma_n, S \rangle$ should be true in an interpretation if the sequence of truth-values that $\sigma_1, \ldots, \sigma_n$ take in that interpretation is one of the n-tuples in S. As it happens, MD-sentences of the form $\langle \sigma, S \rangle$ where S is a union of a finite number of closed (and where the connectives have been given the Łukasiewicz semantics) intervals can be expressed in the extension of Łukasiewicz logic known as Rational Pavelka logic. However, if, for example, S is a left open interval, $(0.5, 1]$, Rational Pavelka logic is unable to deal with this.

In this talk, we will generalize the work from [1] to the first-order and modal contexts. We will study the first-order (as well as modal) logic of multidimensional sentences (generalizing the definition of [1]) when the models considered all have the same fixed domain (which may be of any fixed cardinality, either finite or infinite). The key result is a completeness result that follows the strategy of Fagin et al for the propositional case. We will show how our approach leads to parameterized axiomatizations of the validities of many prominent first-order real-valued logics (since this includes several logics that are not recursively enumerable for validity, our system in general does not yield a recursive enumeration of theorems). Also we will obtain a 0-1 law for finitely-valued versions of these logics. Finally, we will remove the restriction of a fixed domain and provide a completeness theorem for the first-order logic of multidimensional sentences on arbitrary domains.

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Classical and Fuzzy Two-Layered Modal Logics for Uncertainty: Translations and Proof-Theory

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Numerous logical systems have been proposed, and intensively studied in recent years, to cope with reasoning about uncertain events. Among them, two of the most prominent examples are the systems introduced by Fagin, Halpern, and Megiddo [3], which we denote here as $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$.[4]

These systems have since played a crucial role in the logic-based representation of uncertainty in AI; see also the monograph by Halpern [7] and references therein. $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$ employ a rather sophisticated two-layered modal syntax: in their first layer they express classical events (that is, propositions that can only be true or false) by means of the syntax of propositional classical logic; then, they define the atomic statements of the second syntactical layer as linear inequalities (for $\mathrm{Pr}_{\mathrm{lin}}$), or polynomial inequalities (for $\mathrm{Pr}_{\mathrm{pol}}$), of probabilities of these classical events. Each of these inequalities can be seen as the application of a multimodal operator on classical formulas. Finally, such atomic statements may be combined using classical connectives again.

The consequence relation of both logics $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$ is then introduced semantically by means of Kripke frames enriched by a probability measure. Atomic formulas are then interpreted as true when the corresponding inequalities involving the probability of the classcial events, seen as suitable sets of world, hold. This is then extended truth-functionally to complex formulas, by the usual semantics of classical logic.

Despite dealing with the intrinsically graded notion of probability, the semantics of these logics remains essentially bivalent. An alternative approach to reasoning about uncertain events uses the framework of mathematical fuzzy logic and takes sentences like "φ is probable" at face value, that is, identifying its truth degree with the probability of φ. Then, one combines such formulas using connectives of a *suitable fuzzy logic*. Hence, this approach also uses a two-layered modal syntax which is, however, radically simplified. Indeed, it employs only one monadic modality (for "is probable"), instead of

[4] These logics (or more precisely their axiomatic systems) are usually denoted by AX^{prob} and $AX^{prob,\times}$ and, the fuzzy ones introduce below ($\mathrm{Pr}^{Ł}$, $\mathrm{Pr}^{Ł_\triangle}$, and $\mathrm{Pr}^{PŁ_\triangle}$) are traditionally denoted as FP(Ł), FP(Ł$_\triangle$), FP(PŁ), respectively. We have opted here for a uniform but neutral terminology, for ease of reference.

infinitely-many polyadic modalities, as it shifts the syntactical complexity of the atomic statements to the many-valued semantics of the fuzzy logic in question.

The original rendering of this approach in [6, 5] used Łukasiewicz logic Ł to govern modal formulas. The resulting logic, which we denote here as $\mathrm{Pr}^{\text{Ł}}$, was given by using Kripke frames enriched by a probability measures, analogously to $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$. Later, several authors studied numerous similar logical systems by altering not only the upper logic but also the lower one (to speak about probability of fuzzy events) and even their interlinking modalities (to speak about other measures of uncertainty such as necessity, possibility, or belief functions).[5]

In this work, we focus on the logic $\mathrm{Pr}^{\text{Ł}}$ and two of its expansions, $\mathrm{Pr}^{\text{Ł}_\triangle}$ and $\mathrm{Pr}^{\text{PŁ}_\triangle}$, which use stronger fuzzy logics to govern the behavior of modal formulas in the upper syntactical layer, namely the logic $\text{Ł}_\triangle$ expanding Ł with the Baaz–Monteiro projection operator $\triangle$, and its further expansion $\text{PŁ}_\triangle$ with the product conjunction.

Our first contribution is to show that $\mathrm{Pr}_{\mathrm{lin}}$ and $\mathrm{Pr}_{\mathrm{pol}}$ can be translated into the two-layered modal fuzzy logics $\mathrm{Pr}^{\text{Ł}_\triangle}$ and $\mathrm{Pr}^{\text{PŁ}_\triangle}$, and hence, casted into a syntactically simpler framework without losing expressivity. We also provide inverse translations, thus showing that both approaches are indeed much more closely related than it might have seemed at first sight.

Our second contribution is the introduction of Gentzen-style calculus, which we use to provide an alternative proof, arguably simpler, proof of completeness of the logic $\mathrm{Pr}_{\mathrm{lin}}$. Unlike classical Gentzen calculi, which work with sequents, this system is based on more complex syntactical structures, known as hypersequents of relations [1].

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[5] We refer the reader to the survey work [4] and references therein and to the abstract unifying framework for these logics [2]

Some non-mainstream uses of fuzzy logic

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The mainstream applications of fuzzy logic are found in such areas as approximate reasoning, fuzzy control, classification and decision making, expert systems, and natural language modeling. In this talk I aim to give an overview of several less common uses for fuzzy logic, some of which are well known, while others are connected to recent work by my co-authors and myself. These less typical uses of fuzzy logic fall into several broad classes. The first includes cases where fuzzy logic admits constructions that are inconsistent in classical mathematics, such as arithmetic with an unrestricted truth predicate, a naive conception of infinitesimals, and possibly a set theory with unrestricted comprehension axioms. A further kind is the use of fuzzy logic in the semantics of other non-classical logics, aimed at ridding them of certain problematic issues such as the Limit Assumption in the semantics of counterfactuals and the paradox of logical omniscience in epistemic logic. Another is the use of fuzzy logic to eliminate real numbers from a formalism that is thereby simplified, e.g., in probability logic or in the axiomatization of physics. Fuzzy logics can also be used to describe the transmission of qualities other than degrees of truth, such as resources or penalties, insofar as they exhibit the structure of a semilinear residual lattice; in this way, suitable fuzzy logics can be employed to model deontic or resource-aware reasoning. I will briefly survey these non-mainstream use cases for fuzzy logic, and for some of them will describe the state of the art and open problems in more detail.

The general algebraic framework for Mathematical Fuzzy Logic

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Originating as an attempt to provide solid logical foundations for fuzzy set theory [19], and motivated also by philosophical and computational problems of vagueness and imprecision [16] , Mathematical Fuzzy Logic (MFL) has become a significant subfield of mathematical logic [17]. Throughout the years many particular many-valued logics and families of logics have been proposed and investigated by MFL and numerous deep mathematical results have been proven about them (see the three volumes of handbook of MFL [5]). In the early years, there was, however, a great deal of repetition in the papers published on this topic; it was common to encounter articles that studied slightly different logics by repeating the same definitions and essentially obtaining the same results by means of analogous proofs. This unnecessary ballast was delaying the development of MFL while obscuring the reasons behind the main results. Therefore, MFL was an area of science screaming for systematization through the development and application of uniform, general, and abstract methods.

Abstract algebraic logic presented itself as the ideal toolbox to rely on; indeed, this general theory is applicable to all non-classical logics and provides an abstract insight into the fundamental (meta)logical properties at play. However, the existing works in that area (summarized in excellent monographs [2, 14, 15]) did not readily give the desired answers. Despite their many merits, these texts live at a level of abstraction a little too far detached from the intended field of application in MFL. They are indeed great sources of knowledge and inspiration, but there is still a lot of work to be done in order to bring the theory closer to the characteristic particularities of MFL, in particular in first-order logics.

These considerations led us, the authors of this contribution, to writing an extensive series of papers (e.g., [1,3,4,6–8,10–12,18] to name the most important ones) in which we have developed various aspects of the general theory of MFL at different levels of generality and abstraction.

Our first attempt at systematizing this bulk of research was a chapter published in 2011 in the Handbook of Mathematical Fuzzy Logic [9] where we provided rudiments of a well rounded theory constituting solid foundations sufficient (and necessary!) for a rapid development of new particular fuzzy logics demanded by emerging applications. The goal of this talk is to summarize the subsequent 10 years of development and refinements of this theory and present its now matured state of the art as described in our recent monograph [13].

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Weighted Fuzzy Rules for a Relational Data Model

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Abstract. In this contribution, we propose a novel approach to automated fuzzy rule base creation based on underlying observational data. The core of this method lies in adding information to particular fuzzy rule in the form of attached weight given as a value extracted from a relational data model.

1 Sample-based Fuzzy Rules

There is a vast amount of methods for fuzzy rules generation. Usually, they split to those that stems from an expert knowledge or those computed from observed data. A method stemming from observed data formalized in [2] generates one particular fuzzy rule using fuzzy similarity relations for each input data. We summarize it below:

- Consider a (many-sorted) basic many-valued predicate logic (BL$\forall$) with the connectives $\&, \to, \wedge, \vee$ and quantifiers $\forall, \exists$.
- Let T be a theory with a binary predicate F of a type (t_1, t_2), let $\approx_j$ be a similarity predicate in T for the sort t_j, $j = 1, 2$. Moreover, let c_i, d_i be constants of the sorts t_1, t_2, respectively, for $i = 1, 2, \ldots, n$.
- The following formulas are the axioms of T:
  - Similarity axioms for $\approx_1, \approx_2$.
  - Extensionality of F: $((x \approx_1 x') \,\&\, (y \approx_2 y') \,\&\, F(x, y)) \to F(x', y')$.
  - Functionality of F: $((x \approx_1 x') \,\&\, F(x, y) \,\&\, F(x', y')) \to (y \approx_2 y')$.
  - Examples of F: $\bigwedge_i F(c_i, d_i)$.
- Define:

$$\mathrm{Mamd}_F(x, y) \equiv_{\mathrm{df}} \bigvee_i \big((x \approx_1 c_i) \,\&\, (c \approx_2 d_i)\big),$$

$$\mathrm{Rules}_F(x, y) \equiv_{\mathrm{df}} \bigwedge_i \big((x \approx_1 c_i) \to (c \approx_2 d_i)\big).$$

  We call Mamd_F and Rules_F *relational data models.*

In the theory T, we prove many interesting properties, e.g.,

$$\mathrm{Mamd}_F(x, y) \to F(x, y), \quad F(x, y) \to \mathrm{Rules}_F(x, y),$$

$$\bigvee_i (x \approx_1 c_i)^2 \to (\mathrm{Mamd}_F(x, y) \leftrightarrow \mathrm{Rules}_F(x, y)).$$

Unfortunately, this approach without any further fuzzy rules reduction method is suitable only for a small number of data samples. One of the possible solutions is to fix the number of rules and use quantifiers of Fuzzy General Unary Hypotheses Automaton (FGUHA) methods [4, 6] (a generalization of GUHA methods [3]) to confirm/reject each particular designed fuzzy rule. Examples of successful solutions to real-world problems from various fields having a vague nature can be found in [7].

2 Weighted Fuzzy Rules from the Mamd Model

Another solution to the above-described problem can be given by adding to each fuzzy rule a special weight based on the given sample data. We do it using the so-called normal forms for fuzzy logic formulas introduced in [5] and the relational data model Mamd as follows:

- Let S be a theory over BL$\forall$ with binary predicates F, G of a type (t_1, t_2), let $\approx_j$ be a similarity predicate in S for the sort t_j, $j = 1, 2$. moreover, let c_i, d_i be constants of the sorts t_1, t_2, respectively, for $i = 1, 2, \ldots, n$, and p_j, q_j be constants of the sorts t_1, t_2, respectively, for $j = 1, 2. \ldots, k$.
- Define:

$$\mathrm{DNF}_G(x,y) \equiv_{\mathrm{df}} \bigvee_j \big((x \approx_1 p_j) \mathbin{\&} (y \approx_2 q_j) \mathbin{\&} G(p_j, q_j)\big),$$

$$\mathrm{CNF}_G(x,y) \equiv_{\mathrm{df}} \bigwedge_j \big(((x \approx_1 p_j) \mathbin{\&} (y \approx_2 q_j)) \to G(p_j, q_j)\big).$$

- The following formulas are the axioms of S:
  - Similarity axioms for $\approx_1, \approx_2$.
  - $G(x,y) \leftrightarrow \mathrm{Mamd}_F(x,y)$.

In the theory S, we consider a relational data model in the form of Mamd; then we can prove the extensionality of Mamd, i.e.,

$$S \vdash ((x \approx_1 x') \mathbin{\&} (y \approx_2 y') \mathbin{\&} \mathrm{Mamd}_F(x,y)) \to \mathrm{Mamd}_F(x', y').$$

Consequently, S proves:

$$\mathrm{DNF}_G(x,y) \to G(x,y), \quad G(x,y) \to \mathrm{CNF}_G(x,y),$$

$$\bigvee_j (x \approx_1 p_j)^2 \to (\mathrm{DNF}_G(x,y) \leftrightarrow \mathrm{CNF}_G(x,y)).$$

Note that functionality of $\mathrm{DNF(CNF)}_{\mathrm{Mamd}_F}$ is not provable. Many more interesting results on approximate inference with $\mathrm{DNF}_{\mathrm{Mamd}_F}$ and $\mathrm{CNF}_{\mathrm{Mamd}_F}$ are directly obtained from the results on normal forms published in [1].

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Are finite affine topological spaces worthy of study?

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Abstract. Motivated by the theorem of S. A. Morris stating that every topological space is homeomorphic to a subspace of a product of a finite (3-element) topological space, we show that this result is no longer valid for affine topological spaces (inspired by affine sets of Y. Diers), including, e.g., many-valued topology.

1 Introduction

A well-known result of S. A. Morris [8] states that every topological space is homeomorphic to a subspace of a product of copies of the *Davey topological space* (or just *Davey space*), i.e., the space $\mathcal{D} = (D, \tau_D)$ with a 3-element underlying set $D = \{0, 1, 2\}$ equipped with a topology $\tau_D = \{\varnothing, \{1\}, \{0, 1, 2\}\}$. Stating differently, $\mathcal{D}$ is an extremal coseparator in the category **Top** of topological spaces and continuous maps [1]. In view of this result and also to answer the criticism of some of the researchers claiming that "finite topological spaces are not in the slightest bit interesting", it is stated in [8] that "perhaps there is something of interest in finite spaces after all".

There is another point supporting a general interest in finite topological spaces, i.e., the well-known concept of *Sierpinski space* $\mathcal{S} = (\{0, 1\}, \{\varnothing, \{1\}, \{0, 1\}\})$. This space plays an important role in general topology, e.g., a topological space is T_0 iff it can be embedded into some power of $\mathcal{S}$. Stating differently, $\mathcal{S}$ is an $\mathcal{M}$-coseparator in the category $\mathbf{Top}_0$ of T_0 topological spaces, where $\mathcal{M}$ stands for the class of all topological embeddings (initial injective maps) in $\mathbf{Top}_0$. Motivated by the importance of the notion of Sierpinski space, E. G. Manes [7] introduced its analogue for concrete categories under the name of *Sierpinski object*. An object S of a concrete category **C** is a *Sierpinski object* provided that for every **C**-object C, the hom-set $\mathbf{C}(C, S)$ is an initial source.

To find a way of interaction between different approaches to lattice-valued topology, S. Solovjovs [10] introduced an affine approach to lattice-valued topology, originating in the notion of *affine set* of Y. Diers [4]. Observe that a classical topological space (X, τ) consists of a set X and a topology τ, where τ is a subset of the powerset $\mathcal{P}X$ and has the algebraic structure of *frame* [6]. The affine approach replaces the standard contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{\mathrm{op}}$ (where **Set** is the category of sets, and

CBAlg is the category of complete Boolean algebras) with a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{\mathrm{op}}$ (where $\mathbf{X}$ is a category, and $\mathbf{A}$ is a variety of algebras), and requires τ to be a subalgebra of TX. Suitable variety $\mathbf{A}$ and functor T provide not only the classical topological spaces, but also the closure spaces of, e.g., [2] and numerous lattice-valued topological frameworks.

This talk tries to investigate the role of *finite* spaces in affine topology. There exists an affine analogue of the Sierpinski space in terms of the Sierpinski object of E. G. Manes [3, 5], which (in general) is no longer finite. This talk provides an affine analogue of the Davey space and shows its simple relation to the affine Sierpinski space. Since the affine Davey space is (in general) no longer finite as well, a simple message we want to convey is that finite spaces play a (probably) less important role in affine topology (for example, in lattice-valued topology) than they do in the classical topology.

2 Affine topology

This section recalls from, e.g., [3] the main preliminaries on affine topological spaces.

Definition 1. *Let $\Omega = (n_\lambda)_{\lambda \in \Lambda}$ be a family of cardinal numbers indexed by a (possibly, proper or empty) class Λ. An Ω-*algebra *is a pair $(A, (\omega_\lambda^A)_{\lambda \in \Lambda})$ made of a set A and a family of maps $A^{n_\lambda} \xrightarrow{\omega_\lambda^A} A$. An Ω-*homomorphism *$(A, (\omega_\lambda^A)_{\lambda \in \Lambda}) \xrightarrow{\varphi} (B, (\omega_\lambda^B)_{\lambda \in \Lambda})$ is a map $A \xrightarrow{\varphi} B$ such that $\varphi \circ \omega_\lambda^A = \omega_\lambda^B \circ \varphi^{n_\lambda}$ for every $\lambda \in \Lambda$. $\mathbf{Alg}(\Omega)$ is the category of Ω-algebras, concrete over the category* **Set** *(with the forgetful functor $|-|$).*

Definition 2. *Let $\mathcal{M}$ (resp. $\mathcal{E}$) be the class of Ω-homomorphisms with injective (resp. surjective) underlying maps. A* variety of Ω-algebras *is a full subcategory of $\mathbf{Alg}(\Omega)$ closed under the formation of products, $\mathcal{M}$-subobjects, and $\mathcal{E}$-quotients. The objects (resp. morphisms) of a variety are called* algebras *(resp.* homomorphisms*).*

We fix a variety $\mathbf{A}$ (one can think of the variety **Frm** of frames, which provides an example for all the results in the talk) and let $\mathbf{A}^{\mathrm{op}}$ denote the dual of the category $\mathbf{A}$.

Definition 3. *Given a functor $\mathbf{X} \xrightarrow{T} \mathbf{A}^{\mathrm{op}}$, $\mathbf{Af\,Spc}(T)$ is the concrete category over $\mathbf{X}$, whose objects (T-*affine spaces*) are pairs (X, τ), where X is an $\mathbf{X}$-object and τ is an $\mathbf{A}$-subalgebra of TX; and whose morphisms (T-*affine morphisms*) $(X_1, \tau_1) \xrightarrow{f} (X_2, \tau_2)$ are $\mathbf{X}$-morphisms $X_1 \xrightarrow{f} X_2$ such that $(Tf)^{\mathrm{op}}(\alpha) \in \tau_1$ for every $\alpha \in \tau_2$.*

In this talk we restrict ourselves to a particular form of the functor T shown below.

Proposition 1. *Every $\mathbf{A}$-algebra A gives rise to a functor* **Set** $\xrightarrow{\mathcal{P}_A}$ $\mathbf{A}^{\mathrm{op}}$ *defined by $\mathcal{P}_A(X_1 \xrightarrow{f} X_2) = A^{X_1} \xrightarrow{\mathcal{P}_A f} A^{X_2}$ where $(\mathcal{P}_A f)^{\mathrm{op}}(\alpha) = \alpha \circ f$.*

The case $\mathbf{A} = \mathbf{CBAlg}$, $A = 2$ gives the contravariant powerset functor $\mathbf{Set} \xrightarrow{\mathcal{P}} \mathbf{CBAlg}^{\mathrm{op}}$ defined on a map $X \xrightarrow{f} Y$ by $\mathcal{P}Y \xrightarrow{(\mathcal{P}f)^{\mathrm{op}}} \mathcal{P}X$, $(\mathcal{P}f)^{\mathrm{op}}(S) = \{x \in X \mid f(x) \in S\}$.

Observe that $\mathbf{Af\,Spc}(\mathcal{P}_A)$ is the category $\mathbf{ASet}(\Omega)$ of affine sets of [5] based in the notion of affine set of Y. Diers. For $\mathbf{A} = \mathbf{Frm}$, $\mathbf{Af\,Spc}(\mathcal{P}_2)$ is the category **Top** of topological spaces. For $\mathbf{A} = \mathbf{Frm}$ or $\mathbf{A} = \mathbf{UQuant}$ (unital quantales), $\mathbf{Af\,Spc}(\mathcal{P}_A)$ is the category A-**Top** of fixed-basis lattice-valued topological spaces of S. E. Rodabaugh [9].

From now on, we fix an **A**-algebra L and consider the category $\mathbf{Af\,Spc}(\mathcal{P}_L)$, denoted $\mathbf{Af\,Spc}(L)$, whose objects will be called *affine spaces*. We also assume that L has at least two elements, excluding thus possible trivial cases of the empty and a singleton algebra. The category $\mathbf{Af\,Spc}(L)$ is topological over **Set** and therefore is (co)complete.

Definition 4. *An affine space (X, τ) is said to be*

1. indiscrete *provided that τ is the algebra generated by the empty set;*
2. T_0 *provided that for every distinct $x, y \in X$, there is $\alpha \in \tau$ such that $\alpha(x) \neq \alpha(y)$.*

The case $\mathbf{A} = \mathbf{Frm}$ and $L = 2$ gives the classical properties of topological spaces.

3 Affine Sierpinski and Davey spaces

This section outlines briefly the main results on affine Sierpinski and Davey spaces.

Definition 5. Affine Sierpinski space S *is the pair $(|L|, \langle 1_L \rangle)$, where $\langle 1_L \rangle$ is the subalgebra of $L^{|L|}$ generated by the identity map 1_L.*

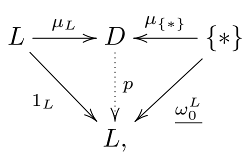
$\mathbf{A} = \mathbf{Frm}$ and $L = 2$ give the classical Sierpinski space $\mathcal{S} = (\{0,1\}, \{\varnothing, \{1\}, \{0,1\}\})$.

Theorem 1. *An affine space (X, τ) is T_0 iff it can be embedded into some power of* S.

Corollary 1. S *is an $\mathcal{M}$-coseparator in the category $\mathbf{Af\,Spc}_0(L)$ of T_0 affine spaces, where $\mathcal{M}$ stands for the class of all embeddings in $\mathbf{Af\,Spc}_0(L)$.*

The classical Sierpinski space can be embedded into the Davey space. Moreover, a topological space is an extremal coseparator in **Top** iff it contains an indiscrete subspace with two elements and a Sierpinski subspace [1]. To preserve these classical relationships between topological spaces, we introduce a particular assumption on the variety **A** underlying the category $\mathbf{Af\,Spc}(L)$. The variety **A** should have at least one nullary operation ω_0 (a constant, which is then an element of every algebra of the variety **A**).

Definition 6. Affine Davey space D *is the pair (D, τ_D), where $D = |L| \coprod \{*\}$ and $\tau_D = \langle p \rangle$, in which the map $D \xrightarrow{p} L$ is given by the following commutative diagram:*



where μ_L, $\mu_{\{*\}}$ are coproduct injections, and $\underline{\omega_0^L}$ is the constant map with value ω_0^L.

Theorem 2. *Every affine space can be embedded into some power of* D.

Corollary 2. *The space* D *is an extremal coseparator in the category* $\mathbf{Af\,Spc}(L)$.

The classical case $\mathbf{A} = \mathbf{Frm}$ and $L = \{\bot, \top\}$ gives the next topological spaces D.

1. Taking $\omega_0^L = \bot$, one obtains the 3-element set $D = \{0, 1, 2\}$ (we use notation "2" instead of "$*$") and the topology $\tau_D = \{\varnothing, \{1\}, \{0, 1, 2\}\}$, namely, D is precisely the classical Davey space $\mathcal{D}$ of S. A. Morris [8].
2. Taking $\omega_0^L = \top$, one obtains the same 3-element set $D = \{0, 1, 2\}$, but the topology τ_D has now the form of $\{\varnothing, \{1, 2\}, \{0, 1, 2\}\}$, namely, D is the second possible form of the Davey space $\mathcal{D}$ of S. A. Morris [8].

The cardinality of the underlying set of the affine spaces S and D depends on the cardinality of the algebra L and can be arbitrarily large. For example, in case of the variety of frames, as soon as we substitute the two-element frame 2 underlying the classical topology with an infinite frame, we obtain infinite affine Sierpinski and Davey spaces. The setting of affine topology shows that it is not the underlying set of a topological space, which plays the main role, but the algebra underlying the respective powersets.

Theorem 3. *The affine Davey space* D *contains an indiscrete* 2*-element space and the affine Sierpinski space* S *as a subspace.*

Despite the fact that the cardinality of the affine Davey space D can be arbitrarily large, its contained non-trivial indiscrete space has exactly two elements.

We observe that all the obtained results can be restated in terms of an affine analogue of topological systems of S. Vickers [11] considered in, e.g., [3].

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New Results on Fuzzy Quantifiers over Fuzzy Domains

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Abstract. We introduce a novel approach to fuzzy quantification based on fuzzy domains. We present its motivation and show new results on operations of restriction and freezing.

Fuzzy quantification is an interesting research field that connects ideas from logic, linguistics, fuzzy set theory, and other areas. Our approach is based on notions from the theory of generalized quantifiers (see [6, 5]). Fuzzy quantifiers over crisp domains [4] generalizing the bivalent generalized quantifiers in the sense of [6] allow one to better model natural language quantifiers such as *many, a few*, etc.

In [3], we proposed a novel approach to fuzzy quantification based on the notion of *fuzzy domain*. We were mainly motivated by the impossibility to define the important operation of *relativization* for fuzzy quantifiers over crisp domains in a natural and satisfactory way. Similar considerations hold also for further important operations of *restriction* and *freezing*, and also for the so-called *living-on property* [2].

Relativization. Basically, this operation transforms a quantifier with one argument into a closely related quantifier with two arguments. Let us denote a truth value of a quantifier Q applied to arguments $A_1, \dots, A_n$ over a domain D by $Q_D(A_1, \dots, A_n)$, where $A_1, \dots, A_n$ are either subsets of D (bivalent quantification) or fuzzy subsets of D (fuzzy quantification). In case of relativization of bivalent quantifiers, a new quantifier Q^{rel} with two arguments arises from a quantifier Q with one argument as

$$(Q^{\mathrm{rel}})_M(A, B) := Q_A(A \cap B). \tag{1}$$

However, in the case of fuzzy quantification, there is a problem: a type mismatch in the expression on the right side of (1): in a position for the domain of Q, a classical set is expected, not a fuzzy set A. In [3, Theorem 5.1], we proved that for fuzzy quantifiers over crisp domains, relativization cannot be defined in a satisfactory way. But relativization plays a prominent rôle in characterizing the important class of quantifiers with two arguments that are models of natural language quantifiers [6, Chapter 4.5]. This motivated us to seek a way how to successfully define this operation for fuzzy quantifiers.

Fuzzy domains. As a solution to the problem explained above, we came up with the idea of defining a *fuzzy domain* as a pair (M, C), where M is a set and C is a fuzzy subset of M. It allows us (leaving technical details aside) to define relativization of fuzzy quantifiers over fuzzy domains in a way that resembles the definition (1) for bivalent quantifiers:

$$(Q^{\mathrm{rel}})_{(M,C)}(A, B) := Q_{(M,C\cap A)}(A \cap B). \tag{2}$$

Notice that, using fuzzy domains, we are able to "move" the fuzzy set A to the position for a domain of Q. Relativization defined by (2) permits us to show properties analogous to those valid for bivalent quantifiers. We also defined operations with fuzzy domains (union, intersection, and difference) and binary relations of equality and the so-called *equality up to negligible elements* that does not take into account elements with zero membership degree. It can be shown that this relation is a congruence with respect to union, intersection, and difference of fuzzy domains.

Freezing. This operation can be seen as an opposite to relativization, transforming quantifiers with two arguments into related one-argument quantifiers. For bivalent quantifiers, the quantifier Q with its first argument frozen to A (denoted Q^A) is defined as

$$(Q^A)_M(B) := Q_{(M\cup A)}(A, B).$$

Again, for the definition of freezing, it is necessary to "move" A to the domain position. We will discuss various possibilities on how to define freezing for fuzzy quantifiers over fuzzy domains and show their properties. We will also define a related operation of *restriction* (transforming a one-argument quantifier to another one-argument quantifier restricted to a fixed set) and show the result that demonstrates a way how these three notions (relativization, restriction, and freezing) are interrelated. Finally, we will define and discuss an important property of a quantifier *living on a set* [1] in the framework of fuzzy quantifiers over fuzzy domains.

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Finite Quantales and Applications

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Abstract

Let Sup be the symmetric and monoidal closed catgeory of complete lattices and join-preserving maps. A quantale is a *semigroup* in Sup and a unital quantale (i.e. a quantale $\mathfrak{Q} = (\mathfrak{Q}, *)$ with unit e) is a *monoid* in Sup.

Let L be a complete lattice. Then the set $[L, L]$ of all join-preserving self-maps $L \xrightarrow{f} L$ of L is a complete lattice w.r.t. the pointwise order and a unital quantale w.r.t. the composition of maps.

Theorem 1. *Every unital quantale $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ is a unital subquantale of of the unital quantale $[\mathfrak{Q}, \mathfrak{Q}]$ of all join-preserving self-maps of $\mathfrak{Q}$.*

Proof. Since $\mathfrak{Q}$ is unital, every element $\alpha \in \mathfrak{Q}$ can be identified with a join-preserving map $\mathfrak{Q} \xrightarrow{g_\alpha} \mathfrak{Q}$ defined by:

$$g_\alpha(\beta) = \alpha * \beta, \qquad \beta \in \mathfrak{Q}.$$

Then we obtain:

$$g_e = \mathrm{id}_{\mathfrak{Q}} \quad \text{and} \quad g_{\alpha_1 * \alpha_2} = g_{\alpha_1} \circ g_{\alpha_2}, \quad \alpha_1, \alpha_2 \in \mathfrak{Q}.$$

We conclude from Theorem 1 that every quantale multiplication $*$ can be understood as a *composition of maps*. Hence from a logical point of view we interpret the product $\alpha * \beta$ as β *and then* α.

Further, we recall the following terminology. An element δ of a quantale $\mathfrak{Q}$ is *cyclic*, if for all $\alpha, \beta \in \mathfrak{Q}$ the equivalence $\alpha * \beta \le \delta \Leftrightarrow \beta * \alpha \le \delta$ holds. The right (resp. left) implication of $\mathfrak{Q}$ is given by:

$$\alpha \searrow \beta = \bigvee \{\gamma \in \mathfrak{Q} \mid \alpha * \gamma \le \beta\} \quad \text{and} \quad \alpha \swarrow \beta = \{\gamma \in \mathfrak{Q} \mid \gamma * \beta \le \alpha\}.$$

An element δ of a quantale $\mathfrak{Q}$ is dualizing, if for all $\alpha \in \mathfrak{Q}$ the following condition holds:

$$\delta \swarrow (\alpha \searrow \delta) = \alpha = (\delta \swarrow \alpha) \searrow \delta.$$

Every quantale with a dualizing element is unital. A quantale $\mathfrak{Q}$ is a *Girard quantale* if $\mathfrak{Q}$ has a *cyclic and dualizing* element.

Theorem 2. *If L is a completely distributive lattice, then the quantale $[L, L]$ of all join-preserving self-maps of L is a Girard quantale and the join-preserving map $L \xrightarrow{d} L$ defined by*

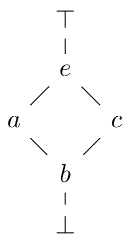
$$d(z) = \bigvee\{x \in L \mid z \not\leq x\}, \qquad z \in L$$

is a cyclic and dualizing element of $[L, L]$.

For the proof of Theorem 2 we refer to [1, Example 2.6.16].

With regard to finite quantales we now consider a simple *finite set* consisting of three elements only. This is the *first step to many-valuedness*.

Example 1. Let C be the set consisting of *three* elements. Then on C there exists a unique lattice structure given by the 3-chain $C_3 = \{\bot, a, \top\}$ with $\bot < a < \top$. Then the quantale $[C_3, C_3]$ of all join-preserving self-maps (i.e. of all isotone self-maps) of C_3 has the following Hasse diagram, multiplication table and table of right implication (cf. [3, Subsec. 5.1]):



$*$	$\bot$	b	a	c	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
b	$\bot$	$\bot$	b	$\bot$	b	c
a	$\bot$	$\bot$	a	$\bot$	a	$\top$
c	$\bot$	b	b	c	c	c
e	$\bot$	b	a	c	e	$\top$
$\top$	$\bot$	a	a	$\top$	$\top$	$\top$

$\searrow$	$\bot$	b	a	c	e	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
b	c	e	e	$\top$	$\top$	$\top$
a	c	c	e	c	e	$\top$
c	$\bot$	a	a	e	e	$\top$
e	$\bot$	b	a	c	e	$\top$
$\top$	$\bot$	$\bot$	a	$\bot$	a	$\top$

Obviously b is the (unique) cyclic and dualizing element of $[C_3, C_3]$. Further, the *three unital quantales* on C_3 are the following unital subquantales of $[C_3, C_3]$:

$$\{\bot, a, e\}, \quad \{\bot, b, e\} \quad \text{and} \quad \{\bot, e, \top\}.$$

Since the previous subquantales are commutative, it is interesting to see that $[C_3, C_3]$ contains also the following *four non-commutative and unital subquantales*:

— Two of them are idempotent and non-integral: $\mathfrak{Q}_4^\ell = \{\bot, a, e, \top\}$ and $\mathfrak{Q}_4^r\{\bot, c, e, \top\}$.
— Two of them are non-idempotent, but integral: $\{\bot, b, a, e\}$ and $\{\bot, b, c, e\}$.

Finally, we show how we can apply finite quantales to health care problems. Our first example is related to WHO-FIC's ICD (International Statistical Classification of Diseases and Related Health Problems). Even though diseases are ICD coded as two-valued in the sense that either a disease is diagnosed, and a disease code is registered in the patient record, or the disease is not seen to be present, we may view the diagnosing process using three stages, including the final stage either confirming a diagnosis or rejecting it, together with a 'diagnosis in progress' stage as the «intermediate state». A similar approach appears in [2]. Let us make the situation more precise.

The rejection of a diagnose means that there does not exist sufficient support for a confirmation of the diagnose. The 'diagnosis in progress' means that there seems to exist sufficient support for a confirmation, but the diagnosing process has not been completed. Finally we have the state of a confirmation of the diagnose. In this sense their exist an *order* between these stages, and all three stages form a 3-chain $C_3 = \{\bot, a, \top\}$ with $\bot < a < \top$, where the rejection is the bottom element $\bot$, the confirmation of the diagnose is the top element $\top$, and the 'diagnosis in progress' is in between $\bot$ and $\top$, and is denoted by a.

In a simplified scenario, the care situation can be viewed as involving four persons $P_1,\ P_2,\ P_3,\ P_4$, where

— P_1 is an optimistic medical expert.
— P_2 is a medical expert hesitating to make drastic decisions.
— P_3 is a pessimistic medical expert.
— P_4 is not a medical expert.

All four persons leave the rejection unchanged. The optimistic expert changes the state of 'diagnosis in progress' already into a confirmation of the diagnose and consequently leaves the confirmation of the diagnose unchanged. In this sense her/his decision is *coherent*, and so P_1 can be identified with the following join-preserving self-map f_1 of C_3:

$$f_1(\bot) = \bot, \quad f_1(a) = \top, \quad f_1(\top) = \top.$$

The hesitatintg expert does not change anything and can consequently be identified with the identity map 1_{C_3} of C_3. The pessimistic expert keeps the confirmation of the diagnose, but changes the state 'diagnosis in progress' into rejection. Hence her/his decision is also coherent, and so P_3 can be identified with the following join-preserving self-map of C_3:

$$f_3(\bot) = \bot, \quad f_3(a) = \bot, \quad f_3(\top) = \top.$$

Finally the non-medical expert is changing anything into a rejection and is identified with the constant map f_4 taking always the value $\bot$.

We now assume that the diagnosing process is controlled by the optimistic and pessimistic expert only. Then a diagnose of the disease by P_1 *and* P_2 can be expressed by the *composition* of the respective control maps f_1 and f_3. It is remarkable to see that the result of the final diagnose is always 2-valued, but depends on the *order* of the observations, because $f_1 \circ f_3 = f_3$ and $f_3 \circ f_1 = f_1$. Moreover, since f_1 and f_3 are elements of the unital quantale $[C_3, C_3]$ of all join-preserving self-maps of the 3-chain, we can consider the subquantale of $[C_3, C_3]$ generated by $\{f_1, f_3\}$. It is easily seen that this quantale corresponding to the optimistic and pessimistic expert has the form $\{f_1, f_3, f_4\}$ and coincides with the subquantale $\mathbb{L}([C_3, C_3])$ of all left-sided elements of $[C_3, C_3]$, which is isomorphic to the unique non-commutative, left-sided and idempotent quantale $\mathfrak{Q}_3^\ell$ on $C_3 = \{\bot, a, \top\}$.

If we now consider the right action $\boxdot$ on C_3 w.r.t. the unital subquantale $\mathfrak{Q}_4^r$ (cf. Example 1) determined by

$\boxdot$	$\bot$	c	e	$\top$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$
a	$\bot$	$\bot$	a	$\top$
$\top$	$\bot$	$\top$	$\top$	$\top$

then we can rediscover the control maps of the optimistic and pessimistic expert as follows. The right action by $\top$ corresponds to f_1 — i.e. $a \boxdot \top = \top = f_1(a)$ and

$\top \boxdot \top = \top = f_1(\top)$. The right action by c coincides with f_3 — i.e. $a \boxdot c = \bot = f_3(a)$ and $\top \boxdot c = \top = f_3(\top)$. Further, the right action by the unit e is exactly the control map f_2 of the hesitating expert. Hence the *algebraic model* of the diagnosing process is given by right module $(C_3, \boxdot)$ over the finite, non-commutative, unital quantale $\mathfrak{Q}_4^r$ in the sense of Sup. In particular, the *interaction* between the optimistic and the pessimistic medical expert is expressed by the quantale $\mathfrak{Q}_4^r$.

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Fuzzy Judgment Aggregation
— Background, Observations, Ideas

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We revisit some essential examples and features of judgment aggregation and explore various routes for generalizing the classical setting to fuzzy logic. While classical impossibility results carry over to the many-valued setting, also possibilities for circumventing well-known forms of impossibilities in social choice theory arise. Fuzzy logic may actually appear on different levels and in various forms in judgment aggregation. In the simplest scenario, judgments remain bivalent, but the aggregation may deliver fuzzy outputs. On another level, also the individual judgments that are to be aggregated may be fuzzy. Finally, we will consider intervals of truth values (or rather: attitude values), both as inputs and outputs of the aggregation function. We will also clarify the relation between fuzzy and probabilistic opinion pooling. A further observation is the connection between fuzzy judgment aggregation and fuzzy quantifiers that remains to be exploited in future work.

Rather than reporting on technical results, the presentation is intended as an invitation to join the ongoing discovery of the varied landscape of many-valued judgment aggregation that largely remains unexplored so far.

On the role of Dunn and Fisher Servi axioms in relational frames for Gödel modal logics

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Extending modal logics to a non-classical propositional ground has been, and still is, a fruitful research line that encompasses several approaches, ideas and methods. In the last years, this topic has significantly impacted on the community of many-valued and mathematical fuzzy logic that have proposed ways to expand fuzzy logics (t-norm based fuzzy logics, in the terminology of Hájek [8]) by modal operators so as to capture modes of truth that can be faithfully described as "graded".

In this line, one of the fuzzy logics that has been an object of major interest without any doubt is the so called *Gödel logic*, i.e., the axiomatic extension of intuitionistic propositional calculus given by the *prelinearity axiom*: $(\varphi \to \psi) \vee (\psi \to \varphi)$. As first observed by Horn in [9], prelinearity implies completeness of Gödel logic with respect to totally ordered Heyting algebras, i.e., *Gödel chains*. Indeed, prelinear Heyting algebras form a proper subvariety of that of Heyting algebras, usually called the variety of Gödel algebras and denoted $\mathbb{G}$ whose subdirectly irreducible elements are totally ordered. Furthermore, in contrast with the intuitionistic case, $\mathbb{G}$ is locally finite, whence the finitely generated free Gödel algebras are finite.

Modal extensions of Gödel logic have been intensively discussed in the literature [2, 3, 10]. Following the usual methodological and philosophical approach to fuzzy logic, they have been mainly approached semantically by generalizing the classical definition of Kripke model $\langle W, R, e\rangle$ by allowing both the evaluation of (modal) formulas and the accessibility relation R to range over a Gödel algebra, rather than the classical two-valued set $\{0, 1\}$ (see [1] for a general approach). More precisely, a model of this kind, besides evaluating formulas in a more general structure than the classical two-element boolean algebra, regards the accessibility relation R as a function from the cartesian product $W \times W$ to a Gödel algebra $\mathbf{A}$ so that, for all $w, w' \in W$, $R(w, w') = a \in A$ means that a is the *degree of accessibility* of w' from w.

Here, we put forward a novel approach to Gödel modal logic that leverages on the duality between finite Gödel algebras and finite forests. This line, that was previously presented in [7], is deepened and extended by the present approach.

In particular, we ground our investigation on finite Gödel modal algebras and their dual structures, that is, the prime spectra of finite Gödel algebras ordered by reverse-inclusion. These ordered structures can be regarded as the prelinear version of posets and they are known in the literature as *finite forests*: finite posets whose principal downsets are totally ordered. In general, Gödel algebras with modal operators form a variety denoted by $\mathbb{GAO}$ for *Gödel algebras with operators*. Hence, the algebras we are concerned with are those belonging to the finite slice of $\mathbb{GAO}$. The associated relational structures based on forests, as we briefly recalled above, might hence be regarded as the prelinear version of the usual relational semantics of intuitionistic modal logic. Accessibility relations $R_\Box$ and $R_\Diamond$ on finite forests are defined, in our frames, by ad hoc properties that we express in terms of (anti)monotonicity on the first argument of the relations themselves. These relational frames will be called *forest frames*.

Furthermore, we put forward a comparison between our approach to the ones that have been proposed for intuitionistic modal logic and, in particular, those developed by Palmigiano in [12] and Orłowska and Rewitzky in [11]. By analyzing the role that these different relational frames (namely, those presented by Palmigiano, Orłowska and Rewitzky, and ours) have in proving a Jónsson-Tarski like representation theorem for Gödel algebras with modal operators, we realized that forest frames situate in a middle level of generality between those of Palmigiano and those of Orłowska and Rewitzky. The former being the less and the latter being the more general ones.

More in detail, we observe that, if we start from any Gödel algebra with operators $(\mathbf{A}, \Box, \Diamond)$, its associated forest frame $(\mathbf{F_A}, R_\Box, R_\Diamond)$ allows to construct another algebraic structure $(\mathbf{S_{F_A}}, \beta_\Box, \delta_\Diamond)$ isomorphic to the starting one. Interestingly, the forest frame $(\mathbf{F_A}, R_\Box, R_\Diamond)$ is not the unique one that reconstructs $(\mathbf{A}, \Box, \Diamond)$ up to isomorphisms. Indeed, for every Gödel algebra with operators $(\mathbf{A}, \Box, \Diamond)$, there are non-isomorphic forest frames, Palmigiano-like, and Orłowska and Rewitzky-like frames, that determine the same original modal algebra $(\mathbf{A}, \Box, \Diamond)$ up to isomorphism.

We start by considering the most general way to define the operators $\Box$ and $\Diamond$ on Gödel algebras, and by investigating the relational structures corresponding to the resulting algebraic structures. Later on, we focus on particular and well-known extensions. Precisely, we consider two main extensions of Gödel algebras with operators: (1) the one obtained by adding the Dunn axioms, typically studied in the fragment of positive classical (and intuitionistic) logic [5, 4]; (2) the one determined by adding the Fischer-Servi axioms [6]. From the algebraic perspective, adding these two sets of identities to Gödel algebras with operators identifies two proper subvarieties of $\mathbb{GAO}$, that we respectively denoted by $\mathbb{DGAO}$ and $\mathbb{FSGAO}$.

In contrast with the case of general Gödel algebras with operators, whose relational structures need two independent relations to treat the modal operators, the structures belonging to $\mathbb{DGAO}$ and $\mathbb{FSGAO}$ only need, for their Jónnson-Tarski like representation, frames with only one accessibility relation. In addition, we study in detail the relational structures corresponding to two further subvarieties of $\mathbb{GAO}$. The first one is the variety obtained as the intersection $\mathbb{DGAO} \cap \mathbb{FSGAO}$. The algebras belonging to such variety have been called *bi-modal Gödel algebras* in [3] and a modal algebra $(\mathbf{A}, \Box, \Diamond) \in \mathbb{DGAO} \cap \mathbb{FSGAO}$ is characterized by the property stating that, for every boolean element $b \in A$, both $\Box b$ and $\Diamond b$ are boolean as well. The second subvariety that we consider refines $\mathbb{DGAO}$. Indeed, any algebra $(\mathbf{A}, \Box, \Diamond)$ belongs to this class iff it satisfies Dunn axioms plus the requirement that $\Box a$ and $\Diamond a$ are boolean for all $a \in A$.

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A case for constants

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This talk will offer a bird's eye view of research in the area of fuzzy logic with rational constants, Rational Pavelka logic RPL (i.e., an expansion of Łukasiewicz logic Ł with rational constants) being a prime example, and a few other systems—particularly product logic P expanded with rational constants—providing a context. Fuzzy logic with rational constants responds to a preference for greater expressivity of the propositional language, allowing for indicating degrees of truth in formulas and for estimating the validity of derivations of such formulas. This was a desideratum already in the papers [7, 19], now considered classics in the area and (along with the seminal papers of Zadeh, esp. [22]) the origin of an important route to mathematical fuzzy logic in its current advanced state of development.

In Goguen's papers—[6] and particularly [7]—the introduction of semantics into the syntax (Goguen's blueprint for propositional constants) is just one important idea among many. Another one is his axiomatic approach to algebras of truth values (leading to subvarieties of residuated lattices), as well as the discussion he provides on capturing axiomatically the properties of particular vague predicates (essentially yielding theories over specific fuzzy logics). Goguen in fact introduced the product connectives, but his approach was in accordance with the line of research represented by Łukasiewicz, Rose and Rosser, or Chang. Much later, these ideas echo in the debates that philosophers promoting fuzzy plurivaluationism held with logicians attempting to read it as axiomatic theories over fuzzy logic and classes of models thereof [20, 2, 21].

I will discuss some relatively well known topics in the area of RPL: implicit definability of rational elements of the MV-algebra on the real unit interval [9, 13] and Beth property [16], that contribute to the impression that Ł and RPL are very closely related systems (cf. also [11]) or Pavelka completeness [19, 8, 9, 3], a result that singles out Ł for expansion with constants.

Complexity results for logics of continuous t-norms with rational constants [10] include, i.a., a direct proof that expansion of Ł with rational constants does not affect complexity classification (while for P this problem seems still to be open); [13] remarks that the result for RPL follows from that for Ł, using implicit definability of the rationals. In a many-valued logic, arguably a natural generalization of classical satisfiability, tautologousness, or finite consequence problems are provided by optimization problems: finding the maximal value for a given term or determining the validity/provability degree of a given term under a given (finite, possibly empty) theory [14]. In the standard MV-algebra, these problems turn out to be hard to solve even approximately [15], with or without rational constants in the language.

The metalogical property of structural completeness is an example of an attribute where Ł and RPL differ profoundly. Łukasiewicz logic is known to be structurally incomplete [4] and the lattice of its extensions is dually isomorphic to the lattice of quasivarieties of MV-algebras, known to be $\mathcal{Q}$-universal [1]. RPL is hereditarily structurally complete, there being no consistent extensions [5]. For the sake of a comparison, product logic P is hereditarily structurally complete and the lattice of extensions is a three-element chain; but the lattice of extensions of product logic with rational constants (RP) is dually isomorphic to the lattice of quasivarieties of rational product algebras, which turns out to be $\mathcal{Q}$-universal [5], and the only structurally complete extensions are the logic of the RP-algebra on the rationals in $[0, 1]$ and the three proper axiomatic extensions of RP term-equivalent to the extensions of P.

The talk will conclude by summing up the viewpoint where RPL is a viable system from both a philosophical and an application-oriented perspective; thus nodding to the tenets of Goguen's essay [7]. The framework of RPL may also subsume logics with graded syntax over Łukasiewicz logic (with rational constants), following esp. the work of Hájek [8] and Novák [17, 18], given that the latter can be viewed as a syntactic fragment of RPL [12].

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On the algebraic structure of the $\mathfrak{Q}$-valued power set

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Abstract.

Let Sup be the catgeory of complete lattices and join-preserving maps. The tensor product of complete lattices has been constructed by Mowat in his Ph.D thesis 1968. Since the mid 1970s it is known that Sup is a symmetric, monoidal closed category w.r.t. Mowat's tensor product (cf. [2, 10], [3, Sect. 2.1.2]). Hence algebra is available in Sup — e.g. *semigroups* in Sup are *quantales*. Due to the universal property of the tensor product (cf. [3, Def. 2.1.7]) quantales can be characterized as complete lattices provided with a semigroup operation $X \times X \xrightarrow{*} X$, which is join-preserving in each variable separately — i.e.

$$\alpha * \big(\bigvee A\big) = \bigvee_{\beta \in A} (\alpha * \beta), \quad \big(\bigvee A\big) * \beta = \bigvee_{\alpha \in A} (\alpha * \beta), \quad \alpha, \beta \in \mathfrak{Q},\, A \subseteq \mathfrak{Q}.$$

A *monoid* in Sup is a *unital* quantale.

Semigroups/monoids in Sup play a significant role for many-valued logics — e.g. on the 3-chain $C_3 = \{\bot, a, \top\}$ with $\bot < a < \top$ there exist 12 quantales, 4 of them are non-commutative and 3 of them are unital. All these 3 unital quantales are commutative:

- The integral and idempotent quantale is the restriction of the Gödel quantale to C_3 (cf. [1]).
- The integral quantale with a dualizing element is the set of truth values of Łukasiewicz three-valued logic.
- The idempotent quantale with a dualizing element has Peirce's Ψ-operator as multiplication and is the set of truth values for the triadic logic (cf.[4, 6]).

On the set consisting with 4 elements there exist 4 non-commutative and unital quantales having necessarily the 4-chain as underlying lattice:

- two of them are idempotent and non-integral,
- two of them are integral, but not idempotent.

The second important algebraic structure in Sup are modules. Let $\mathfrak{Q} = (\mathfrak{Q}, *, e)$ be a unital quantale (i.e. a monoid in Sup). A *right $\mathfrak{Q}$-module* in Sup is a complete lattice M provided with a right action $M \otimes \mathfrak{Q} \xrightarrow{\boxdot} M$ (cf. [7, p. 174]). Again due to the universal property of the tensor product a right action $\boxdot$ can be characterized as a

binary map $M \times \mathfrak{Q} \xrightarrow{\boxdot} M$, which is join-preserving in each variable separately and satisfies the following conditions:

$$t \boxdot e = t \quad \text{and} \quad (t \boxdot \alpha) \boxdot \beta = t \boxdot (\alpha * \beta), \qquad t \in M,\, \alpha, \beta \in \mathfrak{Q}.$$

The simplest mainstream example of a right $\mathfrak{Q}$-module is the unital quantale $\mathfrak{Q}$ itself provided with the *right quantale multiplication* as right action. Further, the *right implication* of $\mathfrak{Q}$ determined by $\alpha \searrow \beta = \bigvee\{\gamma \in \mathfrak{Q} \mid \alpha * \gamma \leq \beta\}$ for all $\alpha, \beta \in \mathfrak{Q}$ induces a right action $\boxdot$ on the dual lattice $\mathfrak{Q}^{\dagger}$ of $\mathfrak{Q}$ as follows:

$$\beta \boxdot \alpha = \alpha \searrow \beta, \qquad \beta \in \mathfrak{Q}^{\dagger},\, \alpha \in \mathfrak{Q}.$$

In this context the *associativity law* of $\boxdot$ is equivalent to the *exportation and importation law* of monoidal logic.

Further, the subquantale $\mathbb{L}(\mathfrak{Q})$ of all left-sided elements of $\mathfrak{Q}$ is always a right $\mathfrak{Q}$-module. Hence there exists a large variety of right modules in Sup.

A *right $\mathfrak{Q}$-module homomorphism* (i.e. morphism of right actions ([7, p. 174]) $(M, \boxdot) \xrightarrow{h} (N, \boxdot)$ is a join-preserving map $M \xrightarrow{h} N$, which also preserves the respective right actions — i.e. $h(t \boxdot \alpha) = h(t) \boxdot \alpha$ for all $t \in M$ and $\alpha \in \mathfrak{Q}$. Right $\mathfrak{Q}$-modules with right $\mathfrak{Q}$-module homomorphisms form a category $\mathsf{Mod}_r(\mathfrak{Q})$.

It is not difficult to see that the forgetful functor $\mathsf{Mod}_r(\mathfrak{Q}) \xrightarrow{\mathcal{G}} \mathsf{Set}$ has a left adjoint functor $\mathsf{Sup} \xrightarrow{\mathcal{F}} \mathsf{Mod}_r(\mathfrak{Q})$ sending a set X to the right $\mathfrak{Q}$-module $\mathfrak{Q}^X$ with the pointwise right quantale multiplication as right action — i.e.

$$(f * \alpha)(x) = f(x) * \alpha, \qquad x \in X, \alpha \in \mathfrak{Q},\, f \in \mathfrak{Q}^X.$$

On maps $X \xrightarrow{\varphi} Y$ the functor $\mathcal{F}$ acts as follows:

$$\mathfrak{Q}^X \xrightarrow{\varphi^{\rightarrow}} \mathfrak{Q}^X, \quad \varphi^{\rightarrow}(f)(y) = \bigvee\{f(x) \mid \varphi(x) = y\}, \quad f \in \mathfrak{Q}^X,\, y \in Y.$$

It is important to note that here $\varphi^{\rightarrow}$ is not simply *Zadeh's forward operator* (cf. [9, (2.9.2) on p. 103]), but has the fundamental algebraic property of a *right $\mathfrak{Q}$-module homomorphism*.

Fact (1) The $\mathfrak{Q}$-valued power set of X is the free right $\mathfrak{Q}$-module $\big(\mathfrak{Q}^X, *\big)$ on X, and is consequently uniquely determined by X up a right $\mathfrak{Q}$-module isomorphism.

(2) The $\mathfrak{Q}$-valued power set $\big(\mathfrak{Q}^X, *\big)$ is isomorphic to the tensor product $\mathcal{P}(X) \otimes \mathfrak{Q}$ (cf. [5, p. 10]), where the complete lattice $\mathcal{P}(X)$ is the ordinary power set of X.

(3) Every right $\mathfrak{Q}$-module $(M, \boxdot)$ is a quotient of $(\mathfrak{Q}^M, *)$ and the corresponding quotient morphism $\mathfrak{Q}^M \xrightarrow{\pi} M$ has the form:

$$\pi(f) = \bigvee_{s \in M} s \boxdot f(s), \qquad f \in \mathfrak{Q}^M.$$

(4) Since the right action $\mathfrak{Q}^X \otimes \mathfrak{Q} \xrightarrow{*} \mathfrak{Q}^X$ is join-preserving, the right adjoint map $\mathfrak{Q}^X \xrightarrow{*^{\vdash}} \mathfrak{Q}^X \otimes \mathfrak{Q}$ of $*$ is determined by

$$\big(*^{\vdash}(g)\big)(f) = \bigvee\{\alpha \in \mathfrak{Q} \mid f * \alpha \leq g\} = \bigwedge_{x \in X} (f(x) \searrow g(x)), \;\; f, g \in \mathfrak{Q}^X. \qquad (1)$$

Obviously, the evaluation of $*^{\vdash}$ at $f, g \in \mathfrak{Q}^X$ — i.e. $d(f,g) = \big(*^{\vdash}(g)\big)(f)$ (see (1)), induces a $\mathfrak{Q}$-preorder d on $\mathfrak{Q}^X$ — the so-called *inclusion* $\mathfrak{Q}$*-order*. Then the $\mathfrak{Q}$-valued power set of X viewed as $\mathfrak{Q}$-preordered set $(\mathfrak{Q}^X, d)$ is always $\mathfrak{Q}$*-join complete*. In fact, for every $\mathfrak{Q}$-fuzzy set F in $\mathfrak{Q}^X$ (i.e. $\mathfrak{Q}^X \xrightarrow{F} \mathfrak{Q}$) the $\mathfrak{Q}$*-join* of F is given by

$$f_0(x) = \bigvee_{f \in \mathfrak{Q}^X} f(x) * F(f), \qquad x \in X,$$

since the following relation holds for all $g \in \mathfrak{Q}^X$ (cf. [3, Def. 3.3.7, Thm 3.3.8]):

$$d(f_0, g) = \bigwedge_{f \in \mathfrak{Q}^X} F(f) \searrow d(f,g).$$

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Quantizing partially ordered structures

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Abstract. In this submission, we consider a compact dagger quantaloid $\mathbf{qRel}$ whose objects are called *quantum sets*, and which can be regarded as a non-commutative (or quantum) generalization of the category $\mathbf{Rel}$ of ordinary sets and relations. We show that in this category one can find quantum generalizations of partial orders in a similar way as one can do in fuzzy set theory by means of Q-relations for some quantale Q. In this way, we obtain quantum generalizations of several ordered structures, for instance the quantum generalization of the power set. Moreover, we present a quantum generalization of complete partial orders. The latter are used in computer science for the denotational semantics of programming languages; we use their quantum generalizations for the denotational semantics of quantum programming languages.

1 Introduction

Quantales were originally introduced as non-commutative generalizations of locales, but are nowadays central in fuzzy set theory as well [1]. The idea is that given two sets X and Y, any relation $X \to Y$ can be respresented by a function $X \times Y \to 2$. Here, the two-point set 2 represents the truth values, and turns out to be an example of a quantale. If we now replace 2 by an arbitrary quantale Q, we obtain the 'fuzzification' of relations. Hence, we define a Q-relation $X \to Y$ to be a function $X \times Y \to Q$. We can now form the category Q-$\mathbf{Rel}$ of ordinary sets and Q-relations, which generalizes the category of ordinary sets and ordinary relations $\mathbf{Rel}$. The category Q-$\mathbf{Rel}$ turns out to be a *quantaloid*, i.e., a category enriched over suplattices. Moreover, Q-$\mathbf{Rel}$ is a *dagger* category: for any morphism $r : X \to Y$, there exists an *opposite* or *dual* morphism $r^\dagger : Y \to X$, which is defined by $r^\dagger(x, y) = r(y, x)$ via the isomorphism $X \times Y \cong Y \times X$. It turns out that this provides sufficient structure to generalize many concepts of ordinary set theory. For instance, a Q-function $f : X \to Y$ is a Q-relation such that $f^\dagger \circ f \geq 1_X$ and $f \circ f^\dagger \leq 1_Y$, whereas a Q-partial order on a set X is a Q-relation $r : X \to X$ such that $1_X \leq r$ (reflexivity), $r \circ r \leq r$ (transitivity), and $r \wedge r^\dagger = 1_X$ (antisymmetry).

We are not exactly doing fuzzy set theory, but we are interested in *quantizing* set theory, i.e., finding a non-commutative generalization of the category of sets. In order to do so, we consider a dagger quantaloid $\mathbf{qRel}$, whose objects are called *quantum sets*.

We will give precise definitions below. The category $\mathbf{qRel}$ was originally introduced in [2], and proven in [3] to be equivalent to a full subcategory of the category of von Neumann algebras and Weaver's quantum relations [6]. The latter category is also an example of a dagger quantaloid. However, in contrast to this ambient category, the category $\mathbf{qRel}$ is *compact*, i.e., for each object $\mathcal{Y}$ (we call *quantum sets*), there is a dual object $\mathcal{Y}^*$ such that the homsets $\mathbf{qRel}(\mathcal{X} \times \mathcal{Y}, \mathcal{Z})$ and $\mathbf{qRel}(\mathcal{X}, \mathcal{Y}^* \times \mathcal{Z})$ are bijective. Here, $\times$ refers to the monoidal product on $\mathbf{qRel}$, which is not cartesian, but since it generalizes the usual product of sets, we use the same notation.

Since $\mathbf{Rel}$ can be embedded into $\mathbf{qRel}$, we call the morphisms in the latter simply *relations*, although they are not defined as subsets of the product of two quantum sets. We can now generalize classical structures in the same way as one can do in Q-$\mathbf{Rel}$: a function $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets is a relation between quantum seets such that $F^\dagger \circ F \geq I_\mathcal{X}$ and $F \circ F^\dagger \leq I_\mathcal{Y}$, where $I_\mathcal{X}$ is the identity on $\mathcal{X}$. Also in this case, a function between quantum sets is not a function in the classical sense, i.e., it does not assign a unique element of $\mathcal{Y}$ regarded as an ordinary set to any element of $\mathcal{X}$. In fact, any function between quantum sets corresponds to a normal $*$-homomorphism between the corresponding von Neumann algebras. However, just as in the case of relations, there is a fully faithful functor of $\mathbf{Set}$ into the category $\mathbf{qSet}$ of quantum sets and functions, which shows that the definition of a function between quantum sets extends the notion of a function between ordinary sets. Moreover, $\mathbf{qSet}$ has almost the same categorical properties as the category $\mathbf{Set}$ of ordinary sets and functions, namely it is symmetric monoidal closed, complete and cocomplete. The only difference is that the monoidal product on $\mathbf{Set}$ is cartesian, whereas the monoidal product on $\mathbf{qSet}$ is only semicartesian, reflecting the quantum nature of the category.

This suggests a way of quantizing mathematical structures as follows: first we consider the (typically cartesian) category $\mathbf{C}$ of objects and morphisms corresponding to the mathematical structure we want to quantize. Usually, the objects of $\mathbf{C}$ are tuples consisting of a set and relations satisfying some identities in terms of the categorical properties of $\mathbf{Rel}$ that captures the mathematical structure. Then we form a (typically semicartesian monoidal) category $\mathbf{qC}$ whose objects are now quantum sets equipped with relations satisfying the same identities as those of $\mathbf{C}$. The quantization is 'sound' if there is a fully faithful functor $\mathbf{C} \to \mathbf{qC}$ such that the underlying quantum sets of the objects in the image of the functor correspond to commutative von Neumann algebras. We quantize posets in this way, and use the thus-obtained quantum posets to construct denotational models for quantum programming languages.

2 Quantum sets

We give a brief overview of the category $\mathbf{qRel}$. Its objects, called *quantum sets*, are collections $\mathcal{X}$ of finite-dimensional Hilbert spaces, and are in a bijective correspondence with von Neumann algebras of the form $\bigoplus_{X \in \mathcal{X}} L(X)$, where $L(X)$ denotes the space of all linear maps on the Hilbert space X. A *relation* $R : \mathcal{X} \to \mathcal{Y}$ between quantum sets is an assignment that to each Hilbert space X in $\mathcal{X}$ and each Hilbert space Y in $\mathcal{Y}$ assigns a subspace $R(X, Y)$ of $L(X, Y)$, the space of all linear maps $X \to Y$.

Composition is defined like the composition of matrices: given $R : \mathcal{X} \to \mathcal{Y}$ and $S : \mathcal{Y} \to \mathcal{Z}$, we define $S \circ R : \mathcal{X} \to \mathcal{Z}$ by $(S \circ R)(X, Z) = \bigvee_{Y \in \mathcal{Y}} S(Y, Z) \cdot R(X, Y)$, where $S(Y, Z) \cdot R(X, Y)$ is the subspace of $L(X, Z)$ spanned by linear maps of the form sr, where $r \in R(X, Y)$ and $s \in S(Y, Z)$. The supremum $\bigvee$ is the supremum in the set of all subspaces of $L(X, Z)$. The identity $I_{\mathcal{X}}$ on $\mathcal{X}$ is the relation given by $I_{\mathcal{X}}(X, X') = \delta_{X,X'}\mathbb{C}1_X$ for each two Hilbert spaces X and X' in $\mathcal{X}$. The dagger $R^\dagger : \mathcal{Y} \to \mathcal{X}$ of a relation $R : \mathcal{X} \to \mathcal{Y}$ is given by $R^\dagger(Y, X) = \{r^* : r \in R(X, Y)\}$, where r^* is the adjoint of the linear map $r : X \to Y$. The set $\mathbf{qRel}(\mathcal{X}, \mathcal{Y})$ of relations $\mathcal{X} \to \mathcal{Y}$ becomes a complete modular ortholattice when ordered by $R \leq S$ if and only if $R(X, Y) \subseteq S(X, Y)$ for each Hilbert space X in $\mathcal{X}$ and each Hilbert space Y in $\mathcal{Y}$. Here, the orthocomplement $\neg R$ of a relation $R : \mathcal{X} \to \mathcal{Y}$ is the largest relation $S : \mathcal{X} \to \mathcal{Y}$ such that $\mathrm{Tr}(R^\dagger \circ S) = 0$, where Tr is the trace on $\mathbf{qRel}$, whose existence follows from the compactness of $\mathbf{qRel}$. It follows that $\mathbf{qRel}$ is a quantaloid. The monoidal product $\mathcal{X} \times \mathcal{Y}$ of quantum sets $\mathcal{X}$ and $\mathcal{Y}$ is the quantum set consisting of the Hilbert spaces $X \otimes Y$ for X in $\mathcal{X}$ and Y in $\mathcal{Y}$, where $\otimes$ is the usual tensor product of Hilbert spaces. Given relations $R : \mathcal{X} \to \mathcal{V}$ and $S : \mathcal{Y} \to \mathcal{Z}$, we define the monoidal product of relations $R \times S : \mathcal{X} \times \mathcal{Y} \to \mathcal{V} \times \mathcal{W}$ by $(R \times S)(X \otimes Y, V \otimes W) = R(X, V) \otimes S(Y, W)$. Finally, the dual $\mathcal{X}^*$ of $\mathcal{X}$ is the quantum set consisting of the duals X^* of the Hilbert spaces X in $\mathcal{X}$. We have a fully faithful functor $`(-) : \mathbf{Rel} \to \mathbf{qRel}$ that to each ordinary set $S$ assigns the quantum set $`S$ consisting of the Hilbert spaces $\mathbb{C}_s$ for $s \in S$, i.e., one-dimensional Hilbert spaces labeled by elements of S. Note that $`S$ corresponds to the von Neumann algebra $\bigoplus_{s \in S} L(\mathbb{C}_s)$, which is indeed commutative. Furthermore, given a relation $r : S \to T$ between ordinary sets, we define the relation $`r : `S \to `T$ between the corresponding quantum sets as the relation $`r(\mathbb{C}_s, \mathbb{C}_t) = L(\mathbb{C}_s, \mathbb{C}_t)$ if $(s, t) \in r$, and $`r(\mathbb{C}_s, \mathbb{C}_t) = 0$ otherwise.

One then can define a function as in $\mathbf{Rel}$, namely a morphism $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets such that $F^\dagger \circ F \geq I_{\mathcal{X}}$ and $F \circ F^\dagger \leq I_{\mathcal{Y}}$. We define $\mathbf{qSet}$ to be the category of quantum sets and functions in this sense. The functor $`(-) : \mathbf{Rel} \to \mathbf{qRel}$ now restricts to a fully faithful functor $`(-) : \mathbf{Set} \to \mathbf{qSet}$.

3 Quantum posets

Following Weaver [6, Definition 2.6], we define a *quantum poset* to be a pair $(\mathcal{X}, R)$ consisting of a quantum set $\mathcal{X}$ and a relation $R \in \mathbf{qRel}(\mathcal{X}, \mathcal{X})$ such that $I_{\mathcal{X}} \leq R$, $R \circ R \leq R$, and $R \wedge R^\dagger \leq I_{\mathcal{X}}$. A *monotone* map $F : (\mathcal{X}, R) \to (\mathcal{Y}, S)$ is simply a function $F : \mathcal{X} \to \mathcal{Y}$ satisfying $F \circ R \leq S \circ F$. We denote the category of quantum posets with monotone maps by $\mathbf{qPOS}$, which generalizes the category $\mathbf{POS}$ of posets and monotone maps as follows from the following theorem:

Theorem 1. *[4] The category $\mathbf{qPOS}$ is complete, has all coproducts, and is symmetric monoidal closed. Moreover, there is a fully faithful functor $`(-) : \mathbf{POS} \to \mathbf{qPOS}$ defined by $(S, \sqsubseteq) \mapsto (`S, `\sqsubseteq)$.*

The power set construction can be extended to a functor $\mathbf{Rel} \to \mathbf{Set}$ that is right adjoint to the inclusion of $\mathbf{Set}$ into $\mathbf{Rel}$. Its counit is the opposite $\ni$ of the membership relation $\in$. The composition of these adjoint functors yields the power set monad Pow on $\mathbf{Set}$.

Moreover, any poset $(S, \sqsubseteq)$ can be embedded into $\mathrm{Pow}(S)$ ordered by inclusion via the function $d : s \mapsto \downarrow s$. This function is determined by the condition $\ni \circ d = (\sqsupseteq)$, where $\ni$, d, and $\sqsupseteq$ are regarded as morphisms in $\mathbf{Rel}$. Compactness of $\mathbf{qRel}$ allows us to prove a similar theorem in the quantum case:

Theorem 2. *[4, Theorems 9.3 & 9.5] The inclusion* $\mathbf{qSet} \to \mathbf{qRel}$ *has a right adjoint* $\mathrm{qPow} : \mathbf{qRel} \to \mathbf{qSet}$*, whose counit we denote by* $\ni$*. Given any quantum set* $\mathcal{X}$*, there is a canonical order* Q *on* $\mathrm{qPow}(\mathcal{X})$ *(the quantum analog of the inclusion order) such that for any order* R *on* $\mathcal{X}$*, there is an order embedding* $D : (\mathcal{X}, R) \to (\mathrm{qPow}(\mathcal{X}), Q)$*, which is the unique function such that* $\ni_{\mathcal{X}} \circ D = R^{\dagger}$ *(where* $R^{\dagger}$ *is the inverse order of* R*).*

The most important application of quantum posets lies in the denotational semantics of quantum programming languages, i.e., the translation of any phrase in the given programming language to a mathematical function in such a way that the function is the composition of the functions corresponding to the phrase's subphrases. Since it is virtually impossible to debug quantum programs, it is pertinent to find different tools for the verification of quantum programs such as denotational semantics. For ordinary programming languages, denotational semantics is often done in terms of complete partial orders (cpos), i.e., posets in which any monotonically increasing sequence has a supremum. In [5], we constructed denotational models for quantum programming languages based on the quantization of the category $\mathbf{CPO}$ of cpos:

Theorem 3. *[5, Theorem 5.6] There is a symmetric monoidal closed, complete and cocomplete subcategory* $\mathbf{qCPO}$ *of* $\mathbf{qPOS}$ *such that the functor* $`(-) : \mathbf{POS} \to \mathbf{qPOS}$ *restricts to a fully faithful functor* $\mathbf{CPO} \to \mathbf{qCPO}$.

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Injective hulls in a category of V-semigroups

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Abstract. This talk describes injective objects and hulls in a category of V-semigroups considered as semigroup objects in the monoidal category V-**Cat** of V-categories, namely, categories enriched in a unital commutative quantale V.

1 Introduction

The concept of lattice-valued (pre)order plays an important role in lattice-valued mathematics [3]. In particular, I. Stubbe [9] gave a brief survey of quantaloid-enriched categories as a convenient setting for "fuzzy logicians and fuzzy set theorists", which incorporated the above notion of lattice-valued (pre)order. Moreover, P. Eklund et al. [5] considered quantale-enriched categories as a tool to develop "many-valued order theory" as part of their aim to offer "a new way to approach many-valuedness in mathematics". This talk considers quantale-enriched categories as a setting for doing lattice-valued categorical algebra. We show an explicit description of injective objects and hulls in a category of generalized semigroups viewed as semigroup objects in the monoidal category V-**Cat** of V-categories (categories enriched in a unital commutative quantale V).

We observe that injective objects and hulls themselves have a significant place in the study of categories. Recall from, e.g., [1] that given a category **C** and a class $\mathcal{M}$ of **C**-morphisms, a **C**-object C is called $\mathcal{M}$-*injective* (or just *injective* if the class $\mathcal{M}$ is clear from the context) provided that for every morphism $A \xrightarrow{m} B$ in $\mathcal{M}$ and every **C**-morphism $A \xrightarrow{f} C$, there exists a **C**-morphism $B \xrightarrow{g} C$ such that $g \cdot m = f$. A morphism $A \xrightarrow{m} B$ in $\mathcal{M}$ is called $\mathcal{M}$-*essential* provided that a **C**-morphism $B \xrightarrow{h} C$ belongs to $\mathcal{M}$ whenever the composite $h \cdot m$ does. An $\mathcal{M}$-*injective hull* of a **C**-object A is a pair (m, B) consisting of an $\mathcal{M}$-injective object B and an $\mathcal{M}$-essential morphism $A \xrightarrow{m} B$. Many familiar mathematical constructions can be regarded as an injective hull, e.g., the Mac Neille completion of a poset or the algebraic closure of a field.

In view of the importance of injective objects, a number of researchers studied their concrete realizations in various categories of interest. In particular, G. Bruns and H. Lakser [2] showed that the injective objects in the category of meet-semilattices are exactly the frames (complete lattices with finite meets distributing over arbitrary joins). J. Lambek et al. [6] extended this result by showing that the injective objects in the category of partially ordered monoids (po-monoids) and submultiplicative order-preserving maps with respect to a special class of monomorphisms ("embeddings") are precisely the unital quantales. Moreover, X. Zhang and V. Laan [10] extended the above result even further through showing that the injective objects with respect to a certain class of order embeddings in the category of partially ordered semigroups (po-semigroups) with submultiplicative morphisms (employing the setting of [6] but discarding the requirements related to the existence of the unit element) are exactly the quantales. Finally, the result of [10] was generalized by X. Zhang and J. Paseka [12] for the setting of S-semigroups (po-semigroups equipped with an action of a po-semigroup S), where the injective objects appeared to be the S-semigroup quantales (S-semigroups, which are quantales, and where the action of S satisfies an additional distributivity condition).

Inspired by the above results, this talk characterizes the injective objects in a specific category of V-semigroups equipped with a left V-action, where V stands for a unital commutative quantale. Following the setting of M. M. Clementino and A. Montoli [4] employed for the study of categorical behaviour of V-groups, we consider V-semigroups as semigroup objects in the monoidal category V-**Cat** of V-categories (categories enriched in V) and V-functors. Since the case $V = 2$ provides preordered semigroups, our result extends the above one of [10], i.e., in case of V-semigroups, we get that the injective objects are exactly the quantale algebras of, e.g., [8] (an analogue of algebras over a commutative ring with identity) satisfying additional requirements related to the enrichment in the quantale V. Moreover, we also show that the machinery of injective hulls of po-semigroups proposed in [10] could be transferred to our extended setting. In particular, we give an explicit description of injective hulls of V-semigroups.

2 Injective objects

This section provides a brief outline of the obtained results on injective objects of V-semigroups. From now on, $V = (V, \otimes, k)$ stands for a unital commutative quantale. Observe that for every element $u \in V$, the map $V \xrightarrow{-\otimes u} V$ has a right adjoint map $V \xrightarrow{\hom(u,-)} V$ (it follows that $v \otimes u \leqslant w$ iff $v \leqslant \hom(u, w)$ for every $u, v, w \in V$).

Definition 1. *A* V-category *is a pair* (X, a) *with a set* X *and a* V*-relation* $X \overset{a}{\longrightarrow\!\!\!\!\!\!\!+\;\;} X$ *(a map* $X \times X \xrightarrow{a} V$*) such that* $k \leqslant a(x, x)$ *for every* $x \in X$ *(*reflexivity*) and* $a(x, y) \otimes a(y, z) \leqslant a(x, z)$ *for every* $x, y, z \in X$ *(*transitivity*). Given* V*-categories* (X, a) *and* (Y, b)*, a map* $X \xrightarrow{f} Y$ *is a* V-functor $(X, a) \xrightarrow{f} (Y, b)$ *provided that* $a(x, x') \leqslant b(f(x), f(x'))$ *for every* $x, x' \in X$*.* V-**Cat** *is the category of* V*-categories.*

Observe that 2-**Cat** is isomorphic to the category **Prost** of preordered sets and monotone maps (where $2 = (\{\bot_2, \top_2\}, \wedge, \top_2)$ is a two-element unital quantale), and P_+-**Cat** is isomorphic to the category **Met** of generalized metric spaces [7] and non-expansive maps (where $\mathsf{P}_+ = ([0, \infty]^{\mathrm{op}}, +, 0)$ is the extended real half-line with the dual partial order). We also notice that the quantale V provides a V-category $(V, \hom)$.

Every V-category (X, a) induces a preorder on its underlying set X (*induced preorder*) by $x \leqslant x'$ iff $k \leqslant a(x, x')$. A V-category is said to be *separated* if its induced preorder is a partial order. Given V-categories (X, a), (Y, b), one defines a V-category $(X, a) \otimes (Y, b)$ by $(X \times Y, a \otimes b)$ with $(a \otimes b)((x, y), (x', y')) = a(x, x') \otimes b(y, y')$.

Definition 2. *A V-*semigroup *is a triple $(X, +, a)$ such that (1) $(X, +)$ is a semigroup; (2) (X, a) is a V-category; (3) $(X, a) \otimes (X, a) \xrightarrow{+} (X, a)$ is a V-functor ($a(x, x') \otimes a(y, y') \leqslant a(x + y, x' + y')$ for every $(x, y), (x', y') \in X \times X$). Given V-semigroups (X, a) and (Y, b), a V-functor $(X, a) \xrightarrow{f} (Y, b)$ is V-*submultiplicative *provided that $f(x) + f(x') \leqslant f(x + x')$ for every $x, x' \in X$. V-**Sgr** is the category of V-semigroups.*

We notice that the category 2-**Sgr** is isomorphic to the category of preordered semigroups and submultiplicative preorder-preserving maps in the sense of [10].

Definition 3. *A V-*act-semigroup *is a V-semigroup (X, a) with a left action $V \times X \xrightarrow{*} X$ of V where (1) $u * (v * x) = (u \otimes v) * x$ for every $u, v \in V$, $x \in X$; (2) $k * x = x$ for every $x \in X$; (3) $(V, \hom) \otimes (X, a) \xrightarrow{*} (X, a)$ is a V-functor ($\hom(v, v') \otimes a(x, x') \leqslant a(v * x, v' * x')$ for every $(v, x), (v', x') \in V \times X$). Given V-act-semigroups (X, a) and (Y, b), a V-submultiplicative V-functor $(X, a) \xrightarrow{f} (Y, b)$ is V-*act-submultiplicative *provided that $v * f(x) \leqslant f(v * x)$ for every $v \in V$, $x \in X$. V-**Act-Sgr** is the category of V-act-semigroups. V-**Act-Sgr**$_s$ is its full subcategory of separated V-act-semigroups.*

There exists a full embedding 2-**Sgr** $\xhookrightarrow{E}$ 2-**Act-Sgr**, $E(X, a) = (X, a, *)$ with $\bot_2 * x = x = \top_2 * x$ for every $x \in X$. The category V-**Act-Sgr** provides a quantale-valued analogue of the category of S-posets and S-submultiplicative order-preserving maps for a po-monoid S of [11] (which considers partial orders instead of preorders).

Definition 4. *Given V-act-semigroups (X, a) and (Y, b), a V-act-submultiplicative V-functor $(X, a) \xrightarrow{f} (Y, b)$ is a V-*embedding *provided that the map $X \xrightarrow{f} Y$ is injective and $b(f(x_1) + \ldots + f(x_n), f(x)) \leqslant a(x_1 + \ldots + x_n, x)$ for every $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, $x_1, \ldots, x_n, x \in X$. $\mathcal{M}$ stands for the class of all V-embeddings.*

Theorem 1. *A separated V-act-semigroup (Z, c) is $\mathcal{M}$-injective iff $(Z, \leqslant)$ is a V-algebra ($\leqslant$ is the induced partial order) such that $c(\bigvee S, z) = \bigwedge_{s \in S} c(s, z)$ for every $z \in Z$, $S \subseteq Z$, and $c(u_1 * z_1 + \ldots + u_n * z_n, z) = \hom(u_1 \otimes \ldots \otimes u_n, c(z_1 + \ldots + z_n, z))$ for every $n \in \mathbb{N}$, $u_1, \ldots, u_n \in V$, $z_1, \ldots, z_n, z \in Z$.*

The case $V = 2$ shows that the result of [10] on injective objects for po-semigroups holds in a bigger category 2-**Act-Sgr**$_s$ of po-semigroups, which are also 2-acts.

3 Injective hulls

This section shows an outline of the obtained results on injective hulls of V-semigroups.

Definition 5. *Given V-act-semigroups (X, a) and (Y, b), a V-act-submultiplicative V-functor $(X, a) \xrightarrow{f} (Y, b)$ is a V-*ih-embedding *provided that the map $X \xrightarrow{f} Y$ is injective and $b(v_1 * f(x_1) + \ldots + v_n * f(x_n), f(x)) \leqslant a(v_1 * x_1 + \ldots + v_n * x_n, x)$ for every $n \in \mathbb{N}$, $x_1, \ldots, x_n, x \in X$, $v_1, \ldots, v_n \in V$. $\mathcal{M}_{ih}$ is the class of all V-ih-embeddings.*

Definition 6. *A V-functor $(X, a) \xrightarrow{j} (X, a)$ is a V-*nucleus *on a V-category (X, a) provided that $k \leqslant a(x, j(x))$ for every $x \in X$ and $j(j(x)) = j(x)$ for every $x \in X$.*

Definition 7. *Every V-act-submultiplicative V-functor $(X,a) \xrightarrow{j} (X,a)$, which is additionally a V-nucleus on a separated V-category (X,a), provides a separated* quotient V-act-semigroup $(X_j, +_j, *_j, a_j)$ *such that* $X_j = \{x \in X \mid j(x) = x\}$, $a_j(x,x') = a(x,x')$, $x +_j x' = j(x+x')$, *and* $v *_j x = j(v * x)$ *for every* $x, x' \in X$, $v \in V$.

V-act-semigroups (X,a) give separated V-act-semigroups $\mathcal{P}(X,a) = (X_V, +_V, *_V, a_V) := (V\text{-}\mathbf{Cat}((X,a^\circ),(V,\hom)), +_V, *_V, [-,-])$, $a^\circ(x,x') = a(x',x)$, $(f +_V g)(x) = \bigvee_{x_1,x_2 \in X} f(x_1) \otimes g(x_2) \otimes a(x, x_1 + x_2)$, and $[f,g] = \bigwedge_{x \in X} \hom(f(x), g(x))$.

Proposition 1. *Given a separated V-act-semigroup (X,a), a map $X_V \xrightarrow{\mathsf{j}} X_V$ defined by* $\mathsf{j}(f) = \bigvee\{g \in X_V \mid a_V(\mu(x_1) +_V f +_V \mu(x_2), \mu(x_3)) \leqslant a_V(\mu(x_1) +_V g +_V \mu(x_2), \mu(x_3))$ *and* $a_V(\mu(x_1) +_V f, \mu(x_3)) \leqslant a_V(\mu(x_1) +_V g, \mu(x_3))$ *and* $a_V(f +_V \mu(x_2), \mu(x_3)) \leqslant a_V(g +_V \mu(x_2), \mu(x_3))$ *and* $a_V(f, \mu(x_3)) \leqslant a_V(g, \mu(x_3))$ *for every* $x_1, x_2, x_3 \in X\}$ *is a V-act-submultiplicative V-functor and a V-nucleus on $\mathcal{P}(X,a)$.*

Theorem 2. *Given a separated V-act-semigroup (X,a), $(\eta, (X_{V\mathsf{j}}, a_{V\mathsf{j}}))$ is an $\mathcal{M}_{ih}$-injective hull of (X,a), where $(X,a) \xrightarrow{\eta} (X_{V\mathsf{j}}, a_{V\mathsf{j}})$ is defined by $\eta(x) = \mathsf{j}(a(-,x))$.*

The case $V = 2$ shows that the result of [10] on injective hulls of po-semigroups is valid in a bigger category 2-$\mathbf{Act\text{-}Sgr}_s$ of po-semigroups, which are also 2-acts.

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Judgment aggregation. An overview on a logical aggregation problem, from impossibility results to feasible rules

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Judgment aggregation extends the problems of preference aggregation and voting theory to more general decision problems, where diverse individual beliefs, judgments or viewpoints have to be aggregated into a consistent collective opinion. Despite the simplicity of the problem, seemingly natural aggregation procedures fail to return consistent collective outcomes, leading to the so-called doctrinal paradox.

In this tutorial I will give an overview of judgment aggregation. After mentioning some of the impossibility theorems that characterised the early phase of this field, I will turn to the definition and the investigation of concrete aggregation rules, and outline some future lines of research.

Logic Tensor Networks

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The talk will present Logic Tensor Networks: a logical framework and and a platform that integrates learning based on neural networks with constraints expressed in first-order many-valued/fuzzy logic. LTN supports a wide range of reasoning and learning tasks with logical knowledge and numeric data using rich symbolic knowledge representation in first-order logic which semantic is defined in terms of embeddings, and real functions implemented by neural networks. LTN has been successfully used to solve tasks in which background knowledge plays an important role such as semantic image interpretation.

A Catalogue of Every Quantale of Order up to 9

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Quantales are algebraic structures that are preordered sets on the one hand, and semigroups on the other. They can be used as a basis for many-valued logics, where some truth values can be interpreted as stronger than others, or incomparable depending on what the order structure in question looks like. In the finite case, the preorder induces a complete lattice, and in addition, the semigroup and order relation uniquely determine two adjoint operators, referred to as implications, that behave like multi-valued analogues of the traditional Boolean implication. There is a growing body of literature surrounding these structures, including recent efforts to apply quantales in practical applications such as healthcare or circuit design. To this end there is a need to understand quantales as a design space, with a more detailed outlook of the numbers involved as well as concrete examples than previous work has emphasized.

In this work, every quantale on up to 9 elements has been enumerated up to isomorphism, catalogued and classified with respect to various properties using computer software. This improves on previous work by the author that enumerated all quantales on up to 6 elements. A number of concrete examples of quantales with peculiar properties are presented, along with observations and patterns of theoretical and practical interest. For example, just by looking at the numbers we can now claim that there are over 19 million quantales on 8 elements, hundreds of millions on 9 elements, and well over 29 million on the nine-element chain lattice alone. Among other things, we study the subquantales of the enumerated quantales, find all Frobenius quantales on order 9 or less that are not Girard, and reflect on how the choice of underlying lattice dictates what properties a quantale can have. We also define never before considered properties of quantales, a few problems and conjectures based on what we have seen in the data.

To enumerate the quantales, the program Mace4 was used. In order to do this in reasonable time as well as to keep the workload on the program within its capabilities, a branching scheme based on isomorphically invariant properties was utilized. To classify the quantales with respect to different properties, custom Python scripts and C++ code were used to post-process the enumerated quantales. A companion iOS / Mac app has also been developed to enable easier browsing of quantales of order up to 6.

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Duality, unification, and admissibility in the positive fragment of Łukasiewicz logic

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MV-algebras, the equivalent algebraic semantics in the sense of Blok Pigozzi of Łukasiewicz logic, have a deep connection with interesting geometrical objects. Indeed, the category of finitely presented MV-algebras with homomorphisms is dually equivalent to a category whose objects are rational polyhedra and the morphisms are so-called $\mathbb{Z}$-maps [12]. This connection allows the study of relevant algebraic and logical properties from the geometrical point of view, such as, for instance, the study of projective algebras and amalgamation, or correspondingly, the investigation of interpolation and unification problems [6, 8, 9, 12, 14].

Looking at Łukasiewicz logic as a substructural logic, thus as an axiomatic extension of the Full Lambek Calculus with exchange and weakening [10], we consider its positive (i.e., 0-free) fragment. The latter is also algebraizable, and its corresponding equivalent algebraic semantics is the variety of Wajsberg hoops. Wajsberg hoops are interesting structures also from a purely algebraic point of view. They play an important role in the theory of hoops [4], which are naturally ordered commutative monoids, and they have a particular connection with lattice-ordered abelian groups (abelian ℓ-groups for short). In fact, the variety WH of Wajsberg hoops is generated by its totally ordered members, that are, in loose terms, either negative cones of abelian ℓ-groups, or *intervals* of abelian ℓ-groups [2] (equivalently, MV-algebras, via Mundici's Γ functor [13]). In the context of the algebraic semantics of many-valued logics, the relevance of Wajsberg hoops is also related to the study of the equivalent algebraic semantics of Hájek Basic Logic and its positive subreducts (BL-algebras and basic hoops). Given the well-known decomposition result in terms of Wajsberg hoops for totally ordered BL-algebras given by Aglianò and Montagna [1], the understanding of Wajsberg hoops is key to obtain interesting results in this framework.

We show that finitely presented Wajsberg hoops have an interesting geometrical dual as well, in particular, with what we will call *pointed* rational polyhedra. More precisely, we show how finitely presented Wajsberg hoops are dually equivalent to a (non-full) subcategory of rational polyhedra with $\mathbb{Z}$-maps, given by rational polyhedra in unit cubes $[0, 1]^n$ that contain the lattice point $\mathbf{1} = (1, \ldots, 1)$, and *pointed* $\mathbb{Z}$-maps, that are $\mathbb{Z}$-maps that respect the lattice point $\mathbf{1}$. In particular, we use a key result in [2] to first show that Wajsberg hoops are equivalent to a (non full) subcategory of finitely presented MV-algebras, and then we suitably restrict the Marra-Spada duality [12].

The connection with the MV-algebraic duality with rational polyhedra allows the use of the deep results obtained by Cabrer and Mundici about finitely generated projective MV-algebras [8, 9, 6] to describe finitely generated projective Wajsberg hoops. In particular, we show that no MV-algebra (or more precisely, its 0-free reduct) is projective in the variety of Wajsberg hoops, and actually that finitely generated nontrivial projective Wajsberg hoops are necessarily unbounded. Interestingly enough, this implies that, in particular, the (0-free reduct of the) two-element Boolean algebra **2** is not projective in the variety of residuated lattices, while **2** is projective in every variety of bounded commutative integral residuated lattices, and in the variety of all bounded commutative integral residuated lattices it is the only finite projective algebra [3].

The fact that Wajsberg hoops are the equivalent algebraic semantics of the positive fragment of Łukasiewicz logic allows us to use our algebraic and geometric investigation to derive some analogies and differences between Łukasiewicz logic and its positive fragment. In particular, we show that while deducibility in the fragment coincides with deducibility of positive terms in Łukasiewicz logic, the same is not true for admissibility of rules.

Moreover, via the algebraic approach to unification problems developed by Ghilardi [11], we will show that the unification type of the variety of Wajsberg hoops, and thus of the positive fragment of Łukasiewicz logic, is nullary. This is in close analogy with the case of MV-algebras, and indeed our proof adapts to pointed rational polyhedra the pathological example given in [12] for the case of Łukasiewicz logic.

Moreover, via the algebraic approach to admissibility developed in [7], we show that while the exact unification type of Łukasiewicz logic is finitary, the one of its positive fragment is unitary. This in particular implies decidability of the admissibility of rules in Wajsberg hoops and the positive fragment of Łukasiewicz logic.

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