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Abstracts

Copulas
Theory and Applications

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Abstracts

Fabrizio Durante
Susanne Saminger-Platz
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Since their inception in 1979 the Linz Seminars on Fuzzy Set Theory have emphasized the development of mathematical aspects in the context of fuzzy sets by bringing together researchers in fuzzy sets and established mathematicians whose work outside the fuzzy setting can provide direction for further research. The philosophy of the seminar has always been to keep it deliberately small and intimate so that informal critical discussions remain central.

LINZ 2023 will be the 40th seminar carrying on this tradition and is devoted to the theme "Copulas – Theory and Applications". The goal of the seminar is to present and to discuss recent advances of copulas and their applications in pure and applied fields.

This volume contains the abstracts of the contributions accepted for presentation at LINZ 2023. The regular contributions are complemented by four invited plenary talks, providing insights and new impulses on the various aspects of the topics at LINZ 2023.

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Contents

A simple extension of Azadkia & Chatterjee's rank correlation to a vector of endogenous variables

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Abstract. We propose a direct and natural extension of Azadkia & Chatterjee's rank correlation T introduced in [2] to a set of $q \geq 1$ endogenous variables. The approach builds upon converting the original vector-valued problem into a univariate problem and then applying the rank correlation T to it. The novel measure T^q then quantifies the scale-invariant extent of functional dependence of an endogenous vector $\mathbf{Y} = (Y_1, \ldots, Y_q)$ on a number of exogenous variables $\mathbf{X} = (X_1, \ldots, X_p)$, $p \geq 1$, characterizes independence of $\mathbf{X}$ and $\mathbf{Y}$ as well as perfect dependence of $\mathbf{Y}$ on $\mathbf{X}$ and hence fulfills all the desired characteristics of a measure of predictability. Aiming at maximum interpretability, we provide various general invariance and continuity conditions for T^q as well as novel ordering results for conditional distributions, revealing new insights into the nature of T.

Measure of predictability

In regression analysis the main objective is to estimate the functional relationship $\mathbf{Y} = f(\mathbf{X}, \boldsymbol{\varepsilon})$ between a set of $q \geq 1$ response variables $\mathbf{Y} = (Y_1, \ldots, Y_q)$ and a set of $p \geq 1$ exogenous variables $\mathbf{X} = (X_1, \ldots, X_p)$ where $\boldsymbol{\varepsilon}$ is a model-dependent error. In view of constructing a good model, the question naturally arises to what extent $\mathbf{Y}$ can be predicted from the information provided by the multivariate exogenous variable $\mathbf{X}$, and which of the exogenous variables are relevant for the model at all.

We refer to κ as a *measure of predictability* for the q-dimensional random vector $\mathbf{Y}$ given the p-dimensional random vector $\mathbf{X}$ if it satisfies the following axioms (cf, e.g., [3],[7] for the case $q = 1$):

(A1) $0 \leq \kappa(\mathbf{Y}|\mathbf{X}) \leq 1$.
(A2) $\kappa(\mathbf{Y}|\mathbf{X}) = 0$ if and only if $\mathbf{Y}$ and $\mathbf{X}$ are independent.
(A3) $\kappa(\mathbf{Y}|\mathbf{X}) = 1$ if and only if $\mathbf{Y}$ is perfectly dependent on $\mathbf{X}$, i.e., there exists some measurable function $\mathbf{f} : \mathbb{R}^p \to \mathbb{R}^q$ such that $\mathbf{Y} = \mathbf{f}(\mathbf{X})$ almost surely.

In addition to the above-mentioned three axioms, it is desirable that additional information improves the predictability of $\mathbf{Y}$. This yields the following two closely related properties of a measure of predictability κ which prove to be of utmost importance for this paper (cf, e.g, [5, 6]):

(P1) *Information gain inequality*: $\kappa(\mathbf{Y}|\mathbf{X}) \leq \kappa(\mathbf{Y}|(\mathbf{X}, \mathbf{Z}))$ for all random vectors $\mathbf{X}$, $\mathbf{Z}$ and $\mathbf{Y}$.

(P2) *Characterization of conditional independence*: $\kappa(\mathbf{Y}|\mathbf{X}) = \kappa(\mathbf{Y}|(\mathbf{X},\mathbf{Z}))$ if and only if $\mathbf{Y}$ and $\mathbf{Z}$ are conditionally independent given $\mathbf{X}$.

Due to axioms (A2) and (A3), the values 0 and 1 of a measure of predictability κ have a clear interpretation. However, the meaning of κ , taking values in the interval $(0,1)$, is not specified. This justifies the investigation of dependence orderings $\prec$ that are compatible with κ in the following sense:

(P3) *Monotonicity*: If $(\mathbf{X},\mathbf{Y}) \prec (\mathbf{X}',\mathbf{Y}')$, then $\kappa(\mathbf{Y}|\mathbf{X}) \leq \kappa(\mathbf{Y}'|\mathbf{X}')$.

In view of interpreting the values of a measure of predictability κ, it is further desirable to have an understanding of the measures' performance in terms of convergence, such as

(P4) *Continuity*: If $(\mathbf{X}_n,\mathbf{Y}_n)_{n\in\mathbb{N}}$ converges (in some sense) to $(\mathbf{X},\mathbf{Y})$, then $\lim_{n\to\infty}\kappa(\mathbf{Y}_n|\mathbf{X}_n) = \kappa(\mathbf{Y}|\mathbf{X})$.

Recently, for $q = 1$, a particularly suitable candidate for such a measure has been studied by [2]: Their so-called 'simple measure of conditional dependence' T is given (in its unconditional form) by

$$T(Y|\mathbf{X}) := \frac{\int_{\mathbb{R}} \mathrm{var}(P(Y \geq y \,|\, \mathbf{X}))\,\mathrm{d}P^Y(y)}{\int_{\mathbb{R}} \mathrm{Var}(\mathbb{1}_{\{Y\geq y\}})\,\mathrm{d}P^Y(y)} \tag{1}$$

and is based on [4]. Although the literature is rich concerning measures of predictability for a single endogenous variable (see, e.g., [6]), the measure T is special in that it not only satisfies the information gain inequality (P1) but also characterizes conditional independence (P2). Using these properties, we show that the functional T^q defined by

$$\begin{aligned} T^q(\mathbf{Y}|\mathbf{X}) &:= \frac{\sum_{i=1}^q \left[T(Y_i|(\mathbf{X},Y_{i-1},\dots,Y_1)) - T(Y_i|(Y_{i-1},\dots,Y_1))\right]}{\sum_{i=1}^q \left[1 - T(Y_i|(Y_{i-1},\dots,Y_1))\right]} \\ &= 1 - \frac{q - \sum_{i=1}^q T(Y_i|(\mathbf{X},Y_{i-1},\dots,Y_1))}{q - \sum_{i=1}^q T(Y_i|(Y_{i-1},\dots,Y_1))}\,, \\ \text{with} \quad & T(Y_1|\emptyset) := 0\,, \end{aligned} \tag{2}$$

is a natural extension of Azadkia & Chatterjee's rank correlation coefficient to a measure of predictability for a vector $(Y_1,\dots,Y_q)$ of $q \geq 1$ endogenous variables. To illustrate that T^q is a correct choice for an extension of T, we first observe that for $q = 1$ the above defined functional T^q reduces to T . Further, due to the information gain inequality for T , each summand of the nominator in (2) is non-negative. Since T characterizes conditional independence, the ith summand of the nominator vanishes if and only if Y_i and $\mathbf{X}$ are conditionally independent given $(Y_{i-1},\dots,Y_1)$. Hence, as a consequence of the chain rule for conditional independence, $\mathbf{X}$ and $\mathbf{Y}$ are independent if and only if $T^q(\mathbf{Y}|\mathbf{X}) = 0$. The denominator, which only takes values in the interval $[1,q]$, guarantees that T^q is normalized and thus, similarly, $T(\mathbf{Y}|\mathbf{X}) = 1$ if and only if Y_i is a function of $\mathbf{X}$ for all $i \in \{1,\dots,q\}$.

The measure of predictability T^q

To the best of the authors' knowledge, so far the only measure of predictability applicable to a vector $\mathbf{Y} = (Y_1, \ldots, Y_q)$ of $q \geq 1$ endogenous variables has been introduced by [3] and employs the vector-valued structure of $\mathbf{Y}$ for its evaluation. In [1], we show that T^q defined by (2) is a measure of predictability for $\mathbf{Y}$ given $\mathbf{X}$, where T^q exhibits a simple expression, is fully non-parametric, has no tuning parameters and is well-defined without any distributional assumptions. Further, T^q fulfils the information gain inequality (P1), characterizes conditional independence (P2), satisfies the so-called data processing inequality, is self-equitable, and exhibits numerous invariance properties. In particular, for continuous marginal distributions, T^q depends only on the underlying copulas.

To tackle the monotonicity property (P3), we introduce the Schur order for conditional distributions $\leq_S$ which turns out to be a natural ordering of predictability hidden behind the properties of T. All the fundamental properties of $\leq_S$, i.e., the characterization of (conditional) independence and complete directed dependence as well as the information gain inequality, are inherited by T. Since the Schur order satisfies a dimension reduction principle to bivariate copulas that are conditionally increasing in sequence (CIS), it follows that also T^q only depends on transformations to bivariate CIS copula.

To address the continuity property (P4), we establish general continuity results for T^q using a characterization of conditional weak convergence in [8]. Applying these results, we obtain continuity of T^q in classes of elliptical and l_1-norm symmetric distributions. Further, for elliptical distributions, a characterization is given for the case where T^q attains the values 0 and 1, respectively.

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References

1. J. Ansari and S. Fuchs. A simple extension of Azadkia & Chatterjee's rank correlation to a vector of endogenous variables *Available at https://arxiv.org/abs/2212.01621*, 2023.
2. M. Azadkia and S. Chatterjee. A simple measure of conditional dependence. *Ann. Stat.*, 49 (6):3070–3102, 2021.
3. N. Deb, P. Ghosal, and B. Sen. Measuring association on topological spaces using kernels and geometric graphs. *Available at http://128.84.4.18/abs/2010.01768*, 2020.
4. H. Dette, K. F. Siburg, and P. A. Stoimenov. A copula-based non-parametric measure of regression dependence. *Scand. J. Statist.*, 40(1):21–41, 2013.
5. S. Fuchs. Quantifying directed dependence via dimension reduction. *J. Multivariate Anal., to appear*, 2023.

6. F. Griessenberger, R. Junker, and W. Trutschnig. On a multivariate copula-based dependence measure and its estimation. *Electron. J. Statist.*, 16:2206–2251, 2022.
7. F. Han and Z. Huang. Azadkia-Chatterjee's correlation coefficient adapts to manifold data. *Available at https://arxiv.org/abs/2209.11156v1*, 2022.
8. T. J. Sweeting. On conditional weak convergence. *J. Theor. Probab.*, 2(4):461–474, 1989.

On infinite-order s-vine copula processes

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Abstract. We construct stationary and ergodic time series with serial dependence behaviour described by stationary d-vine (or s-vine) copulas. We pay particular attention to the case where the s-vine is generated by an infinite sequence of copulas. Such models are shown to yield a rich class of processes generalizing classical Gaussian ARMA processes and allowing both non-Gaussian marginal behaviour and non-Gaussian and non-linear serial dependence. We explain how these models can be estimated and give examples showing their superiority to classical models for certain datasets.

1 Introduction

Let $(X_t)_{t\in\mathbb{Z}}$ be a strictly stationary time series with continuous marginal distribution F_X and let $(U_t)_{t\in\mathbb{Z}}$ be the process of uniformly-distributed random variables given by $U_t = F_X(X_t)$ for all t. In this paper we are interested in processes where the dynamics of $(U_t)_{t\in\mathbb{Z}}$ are described explicitly by a set of copula functions.

The main examples of such models in the literature are first-order Markov copula processes [7, 10] and their higher-order d-vine generalizations [16, 3, 6, 14]. These are based on pair copula decompositions [11, 4, 1], i.e. models constructed from bivariate copulas. While this literature concerns processes constructed from a finite set of bivariate copulas, we consider the generalization to processes defined by an infinite sequence of bivariate copula functions, as suggested in [5]. We derive some of properties of these processes, show how they may be estimated and highlight some open theoretical issues.

Our interest in these processes stems from the fact that they can be used to define non-Gaussian analogues of stationary Gaussian processes. We can obtain processes with both non-Gaussian marginal behaviour and non-Gaussian and non-linear serial-dependence behaviour. Moreover, we can propose natural non-Gaussian analogues to classical ARMA, seasonal ARMA and ARFIMA models that share a degree of structure with their classical counterparts [5].

2 Notation

Let $(C_k)_{k\in\mathbb{N}}$ be a sequence of bivariate copulas with continuous partial derivatives of all orders and densities c_k that are strictly positive on $(0,1)^2$. For $k \in \mathbb{N}$ let the functions

$R_k : (0,1) \times (0,1)^k \to (0,1)$ and $R_k^* : (0,1) \times (0,1)^k \to (0,1)$ be defined in a recursive, interlacing fashion by $R_1(x;u) = h_1^{(1)}(u,x)$, $R_1^*(x;u) = h_1^{(2)}(x,u)$ and, for $k \geq 2$,

$$R_k(x;\boldsymbol{u}) = h_k^{(1)}\left(R_{k-1}^*(u_k;\boldsymbol{u}_{[k-1:1]}), R_{k-1}(x;\boldsymbol{u}_{[1:k-1]})\right)$$
$$R_k^*(x;\boldsymbol{u}) = h_k^{(2)}\left(R_{k-1}^*(x;\boldsymbol{u}_{[1,k-1]}), R_{k-1}(u_k;\boldsymbol{u}_{[k-1:1]})\right)$$

where $h_k^{(i)}(u_1,u_2) = \frac{\partial}{\partial u_i} C_k(u_1,u_2)$ and where the notation $\boldsymbol{u}_{[i,j]} = (u_i,\ldots,u_j)'$ denotes sub-vectors of the vector $\boldsymbol{u}$ consisting of contiguous components, which may be in ascending or descending order of the indices according to whether $j > i$ or $j < i$. For reasons that become clearer in Section 3 below, we refer to the functions R and R^* as forward and backward Rosenblatt functions respectively. We denote the inverse of the Rosenblatt forward function by $Q_k(z;\boldsymbol{u})$ so that $R_k(Q_k(z;\boldsymbol{u});\boldsymbol{u}) = z$ for all $(z,\boldsymbol{u}) \in (0,1) \times (0,1)^k$ and $k \in \mathbb{N}$.

3 Construction of s-vine copulas

Let $(Z_t)_{t\in\mathbb{N}}$ by a sequence of iid uniform innovations. Suppose we set $U_1 = Z_1$ and

$$U_k = Q_{k-1}(Z_k;\boldsymbol{U}_{[k-1:1]}), \tag{1}$$

for $k = 2,\ldots,n$ and some fixed $n \geq 2$. The random vector $\boldsymbol{U} = (U_1,\ldots,U_n)^\top$ will have a joint distribution given by a copula $C^{(n)}$ with density

$$c_{(n)}(\boldsymbol{u}) = \prod_{k=1}^{n-1}\prod_{j=k+1}^{n} c_k\Big(R_{k-1}^*(u_{j-k};\boldsymbol{u}_{[j-k+1:j-1]}), R_{k-1}(u_j;\boldsymbol{u}_{[j-1:j-k+1]})\Big). \tag{2}$$

This is the density of a a d-vine copula subject to translation-invariance conditions, which we will refer to as a stationary d-vine or s-vine copula, since it belongs to the larger class of copulas for stationary multivariate time series introduced in [14].

For $k = 1,\ldots,n-1$, $t \geq 1$ and $t+k \leq n$, the copula C_k is the copula of the conditional distribution of (U_t, U_{t+k}) given the variables in between; we refer to this as the kth partial copula. The Rosenblatt functions are the conditional distribution functions

$$R_k(x;\boldsymbol{u}) = \mathbb{P}(U_{t+k} \leq x \mid U_{t+k-1} = u_1,\ldots,U_t = u_k)$$
$$R_k^*(x;\boldsymbol{u}) = \mathbb{P}(U_t \leq x \mid U_{t+1} = u_1,\ldots,U_{t+k} = u_k)$$

and hence the construction (1) is simply the familiar method of generating a realization from a copula using the inverse of Rosenblatt's transformation [15].

4 Construction of s-vine processes

We are interested in constructing strictly stationary processes whose higher-dimensional marginal distributions have copulas with densities of the form (2), that is processes which conform to the following definition.

Definition 1 (S-vine process). *A strictly stationary time series $(X_t)_{t\in\mathbb{Z}}$ is an s-vine process if for every $t \in \mathbb{Z}$ and $n \geq 2$ the distribution of the vector $(X_t, \ldots, X_{t+n-1})$ is absolutely continuous and admits a unique copula $C_{(n)}$ with a joint density of form $c_{(n)}$ in (2). An s-vine process $(U_t)_{t\in\mathbb{Z}}$ is an s-vine copula process if its univariate marginal distribution is standard uniform.*

We consider constructing a process $(U_t)_{t\in\mathbb{N}}$ starting from a single uniform random variable U_1 and iterating construction (1) ad infinitum for the sequence of copulas $(C_k)_{k\in\mathbb{N}}$.

If the copula sequence satisfies $C_k = C^{\perp}$ (independence copula) for $k > p$ we refer to this as the finite-order case. In this case, for $k > p$, we have that

$$U_k = Q_p(Z_k; \boldsymbol{U}_{[k-1:k-p]}),$$

so U_k only depends on the previous p values. It can be shown that this gives a stationary and ergodic Markov process of order p [5] and there is literature on rates of mixing [7, 2, 8, 12]. Thus, in this case, construction (1) is an exact simulation algorithm for a realization $\boldsymbol{U} = (U_1, \ldots, U_n)^\top$ of any length n from an s-vine copula process. The random vector $\boldsymbol{X} = (F_X^{-1}(U_1), \ldots, F_x^{-1}(U_n)))^\top$ is a realization from an s-vine process $(X_t)_{t\in\mathbb{Z}}$ with marginal distribution F_X.

We are particularly interested in the infinite-order case where the copula sequence satisfies $C_k \to C^{\perp}$ as $k \to \infty$ but where, for every $n \in \mathbb{N}$, there exists $k \geq n$ such that $C_k \neq C^{\perp}$. When each C_k is a Gaussian copula with parameter α_k then a result of [9] can be used to show that absolute summability of the sequence $(\alpha_k)_{k\in\mathbb{N}}$ is a sufficient condition for (1) to describe the construction of a stationary and ergodic process. On the other hand, if $\alpha_k = (k+1)^{-1}$ for all k, then it is shown in [5] that this yields a non-ergodic process. In the case of non-Gaussian copula sequences there remain a number of open questions about the conditions on the copula sequence for ergodic and mixing behaviour. We will present some partial results and formulate some conjectures.

We develop an estimation methodology for finite and infinite-order s-vine processes based on the Kendall partial rank autocorrelation function (kpacf), which is the sequence of Kendall tau values $(\tau_k)_{k\in\mathbb{N}}$ corresponding to the copula sequence $(C_k)_{k\in\mathbb{N}}$. This allows us to propose a new interpretation of the concept of a non-Gaussian ARMA(p,q) process of any order (p, q); the idea can also be applied to develop non-Gaussian extensions of other Gaussian processes such as seasonal ARMA models and ARFIMA models. We also develop a model checking methodology based on a natural definition for model residuals.

5 Applications

We will give some specific examples of s-vine processes for modelling macroeconomic time series. In particular, we will show some results for a non-Gaussian seasonal ARIMA model for rate of inflation data. The estimation of models has been implemented in the R library `tscopula` [13].

References

1. AAS, K., CZADO, C., FRIGESSI, A., AND BAKKEN, H. Pair-copula constructions of multiple dependence. *Insurance: Mathematics and Economics 44*, 2 (2009), 182–198.
2. BEARE, B. Copulas and temporal dependence. *Econometrica 78*, 395–410 (2010).
3. BEARE, B., AND SEO, J. Vine copula specifications for stationary multivariate Markov chains. *Journal of Time Series Analysis 36* (2015), 228–246.
4. BEDFORD, T., AND COOKE, R. M. Vines–a new graphical model for dependent random variables. *Annals of Statistics 30*, 4 (08 2002), 1031–1068.
5. BLADT, M., AND MCNEIL, A. Time series with infinite-order partial copula dependence. *Dependence Modeling 10* (2022), 87–107.
6. BRECHMANN, E. C., AND CZADO, C. Copar - multivariate time series modeling using the copula autoregressive model. *Applied Stochastic Models in Business and Industry 31*, 4 (2015), 495–514.
7. CHEN, X., AND FAN, Y. Estimation of copula-based semi-parametric time series models. *Journal of Econometrics 130*, 2 (2006), 307–335.
8. CHEN, X., WU, W. B., AND YI, Y. Efficient estimation of copula-based semiparametric Markov models. *Annals of Statistics 37*, 6B (2009), 4214–4253.
9. DEBOWSKI, L. On processes with summable partial autocorrelations. *Statistics and Probability Letters 77* (2007), 752–759.
10. IBRAGIMOV, R. Copula-based characterizations for higher-order Markov processes. *Econometric Theory 25*, 819–846 (2009).
11. JOE, H. *Multivariate Models and Dependence Concepts*. Chapman & Hall, London, 1997.
12. LONGLA, M., AND PELIGRAD, M. Some aspects of modeling dependence in copula-based Markov chains. *Journal of Multivariate Analysis 111* (2012), 234–240.
13. MCNEIL, A., AND BLADT, M. *tscopula: Time Series Copula Models*, 2022. R package version 0.3.1 available on CRAN.
14. NAGLER, T., KRÜGER, D., AND MIN, A. Stationary vine copula models for multivariate time series. *Journal of Econometrics 227*, 2 (2022), 305–324.
15. ROSENBLATT, M. Remarks on a multivariate transformation. *Annals of Mathematical Statistics 23* (1952), 470–472.
16. SMITH, M., MIN, A., ALMEIDA, C., AND CZADO, C. Modeling Longitudinal Data Using a Pair-Copula Decomposition of Serial Dependence. *Journal of the American Statistical Association 105*, 492 (2010), 1467–1479.

Dependence orders and copula issues arising from public rescue of bank defaults

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Abstract. In this talk we will analyze the problem of how the budget constraints for the rescue of bank defaults influences the joint distribution of times to default of a banking system. In particular, we will propose and discuss new models with different features and we will analyze the induced dependence.

1 Introduction

A common feature of the financial crises is the debate on which players to rescue to prevent possible systemic developments. Rescue and support are obviously limited by a budget constraint and can take different forms. Direct injection of funds in the banks in the form of equity is the most usual way. Other popular forms of help consist of freeing the financial system from some common source of risk: providing deposit insurance prevents contagion from the banking system to the real economy and buying troubled assets (as in the TARP program during the big financial crisis) can stop contagion across financial institutions. The same systemic threat can originate from operational risk events, such as the breach of the security of a digital system, exposed to failure of its components because of external cyber attacks or physical disasters.

A question that arises is how the presence of this partial safety net can affect the joint default distribution of the banks in the system. More particularly, we are interested in assessing the effect of this safety net on the marginal times to default of the financial institutions and their dependence structure, represented by their copula function and the corresponding non-parametric dependence measures. In this talk, we address this problem by combining tools and models that are typical of copula functions theory and those coming from reliability theory.

In a technical setting, consider a system whose individual lifetimes have an assigned dependence structure and marginal distributions. Assume you can repair, or rescue, a limited number of components, subject to a bounded cost function. The question that is addressed in this talk is how this repair and rescue may affect both the marginal lifetimes, their dependence and the lifetime of the system. Reliability theory suggests that the structural links among the banks may make a difference. The role of each bank is given by the structure of the whole banking system: in case of system made of all

SIFI banks then the default of just one of them implies the default of the whole system, recalling a system in which the components are assembled in series, while in the case of a system made of "small" banks, the end of the entire system coincides with the default of the last bank. In the financial jargon, a banking system in series calls for a first-to-default rescue, while in a parallel system, preserving the system requires a last-to-default rescue.

The connection with reliability theory is clear if we refer to repairable systems (see Spizzichino, 2021, for a review on multivariate survival models for reliability systems). As the simplest case, consider a system with d independent components, and assume we have resources to fix the individual components n times. Repair and rescue is assumed to occur on a first come first served basis (that corresponds to the case of the rescue of a series system). The question is: are the components still independent? The qualitative answer is no, because the lifetime of each components depends on the end of the others, that take out repair resources. The quantitative answer, that is how dependence may change, is instead addressed in this talk. In fact, even in this particular situation, the problem is not trivial.

In order to include some underlying fundamental dependence we will consider the case in which the d components lifetimes are conditionally i.i.d. with a particular attention to the case in which they are linked by a Gumbel copula being this choice particularly suited when dealing with exponentially distributed lifetimes (see Cherubini and Mulinacci, 2017).

2 Aims and Provisional results

Let, for simplicity, assume $d = 2$. We denote with $(Y_1^{(n)}, Y_2^{(n)})$ the random vector of the residual lifetimes when n $(n \geq 0)$ repairments are allowed. If $(Z_1^{(n+1)}, Z_2^{(n+1)})$ do represent the additional corresponding lifetimes in case of failure and repair, then the residual lifetimes in the case of $n+1$ repairs are given by

$$\begin{aligned} Y_1^{(n+1)} &= Y_1^{(n)} + Z_1^{(n+1)} \mathbf{1}_{\{Y_1^{(n)} < Y_2^{(n)}\}} \\ Y_2^{(n+1)} &= Y_2^{(n)} + Z_2^{(n+1)} \mathbf{1}_{\{Y_1^{(n)} < Y_2^{(n)}\}} \end{aligned} \tag{1}$$

and the distribution of $\left(Y_1^{(n)}, Y_2^{(n)}\right)$ for all $n \geq 1$ can be obtained recursively, starting from the initial distribution of $\left(Y_1^{(0)}, Y_2^{(0)}\right)$ whose dependence structure represents the *fundamental* dependence of the system. Even in this simple case, the solution is quite involved.

In fact, if we consider the particular case in which no fundamental dependence is assumed (that is that $Y_1^{(0)}$ and $Y_2^{(0)}$ are independent), $Y_1^{(0)}$ and $Z_1^{(n)}$ are exponentially distributed with parameter λ_1 and $Y_2^{(0)}$ and $Z_2^{(n)}$ are exponentially distributed with parameters λ_2, and if, additionally, for $i = 1, 2$ and $n \geq 0$, $Z_i^{(n+1)}$ is independent of $(Y_1^{(n)}, Y_2^{(n)})$, then, if $\lambda_s = \lambda_1 + \lambda_2$, the joint survival distribution of $(Y_1^{(n)}, Y_2^{(n)})$ is

$$\bar{F}(t_1, t_2) = \begin{cases} \left(\frac{\lambda_s}{\lambda_1}\right)^n e^{-\lambda_2 t_2} \left[C_{\lambda_1 t_1}(n) - C_{\lambda_1 t_2}(n-1)\right] + C_{\lambda_s t_2}(n-1), & \text{if } t_1 \leq t_2 \\ \left(\frac{\lambda_s}{\lambda_2}\right)^n e^{-\lambda_1 t_1} \left[C_{\lambda_2 t_2}(n) - C_{\lambda_2 t_1}(n-1)\right] + C_{\lambda_s t_1}(n-1), & \text{if } t_1 > t_2 \end{cases},$$

where $C_x(n) = e^{-x} \sum_{j=0}^{n} \frac{x^j}{j!}$, and the analysis of its probabilistic properties will be the object of part of the talk. As a second step, we will add some fundamental dependence assuming that the components are conditionally i.i.d. and we will analyze its contribution to the resulting distribution. A similar analysis will be conducted in the case of more than two components.

However, even though there is clearly some parallelism with reliability theory, we will discuss differences and analogies and if and how much reliability theory can actually help.

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References

1. F. Spizzichino, Aging and Positive Dependence, In Wiley StatsRef: Statistics Reference Online, (2021).
2. U. Cherubini, S. Mulinacci, The Gumbel-Marshall-Olkin distribution, In: Ubeda Flores M., de Amo Artero E., Durante F., Fernandez Sanchez J. (eds) Copulas and dependence models with applications. Contributions in honor of Roger B. Nelsen. Springer. (2017), 21-31 (2017)

Multivariate Archimedean copulas and the Williamson transform

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1 Introduction

Archimedean copulas are popular and well-known mainly due to their simple algebraic structure. We revisit the close interrelation between Archimedean copulas C_γ and probability measures γ on $(0, \infty)$ established via the so-called Williamson transform as studied in [7] (also see [8, Theorem 1.11]) and derive novel and simple expressions for the level set masses and the Kendall distribution function in terms of γ. Moreover, we show that within the class of multivariate Archimedean copulas uniform convergence is equivalent to weak convergence of the corresponding probability measures on $(0, \infty)$ and that singularity and regularity properties of the Archimedian copulas may directly be derived from properties of the probability measures on $(0, \infty)$. Using the afore-mentioned results we conclude that the subfamily of all absolutely continuous Archimedean copulas as well as the family of all singular Archimedean copulas is dense in the set of all Archimedean copulas.

2 Notation and preliminaries

Throughout this contribution $\mathcal{C}^d$ denotes the family of all d-dimensional copulas for some fixed integer $d \geq 3$. To simplify notation, we write vectors in bold symbols. The d-stochastic measure associated with a copula $C \in \mathcal{C}^d$ will be denoted by μ_C, i.e., $\mu_C([\mathbf{0}, \mathbf{x}]) = C(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{I}^d$, whereby $[\mathbf{0}, \mathbf{x}] := [0, x_1] \times [0, x_2] \times \ldots \times [0, x_d]$ and $\mathbb{I} := [0, 1]$. For the $(1, \ldots, d-1)$-marginal of $C \in \mathcal{C}^d$ we will write $C^{1:d-1}$, i.e.,

$$C^{1:d-1}(x_1, x_2, \ldots, x_{d-1}) := C(x_1, x_2, \ldots, x_{d-1}, 1),$$

for all $(x_1, ..., x_{d-1}) \in \mathbb{I}^{d-1}$. If a d-dimensional random vector $\mathbf{X}$ has distribution function C in the sequel we will simply write $\mathbf{X} \sim C$.
Given an arbitrary metric space (S, d), the Borel σ-field on S will be denoted by $\mathcal{B}(S)$ and we write λ_d for the Lebesgue measure on $\mathcal{B}(\mathbb{I}^d)$. In what follows *Markov kernels* (regular conditional distributions) will play a prominent role: A mapping $K : \mathbb{I}^{d-1} \times \mathcal{B}(\mathbb{I}) \to \mathbb{I}$ from $\mathbb{I}^{d-1}$ to $\mathbb{I}$ is a $(d-1)$-*Markov kernel* if and only if $\mathbf{x} \mapsto K(\mathbf{x}, E)$

is $\mathcal{B}(\mathbb{I}^{d-1})$-$\mathcal{B}(\mathbb{I})$-measurable for every fixed $E \in \mathcal{B}(\mathbb{I})$ and $E \mapsto K(\mathbf{x}, E)$ is a probability measure on $\mathcal{B}(\mathbb{I}^{d-1})$ for arbitrary $\mathbf{x} \in \mathbb{I}^{d-1}$. It is well-known (see, e.g., [1]) that every copula $C \in \mathcal{C}^d$ induces a Markov kernel $K_C : \mathbb{I}^{d-1} \times \mathcal{B}(\mathbb{I}) \to \mathbb{I}$ and that K_C is unique only $\mu_{C^{1:d-1}}$-almost everywhere. For more background on conditional expectation and Markov kernels we refer to [2, Section 5] and [5, Section 8].

A continuous, non-increasing, function $\psi : [0, \infty) \to [0, 1]$ fulfilling $\psi(0) = 1$, $\lim_{z\to\infty} \psi(z) = 0 =: \psi(\infty)$ and being strictly decreasing on $[0, \inf\{z \in [0, \infty] : \psi(z) = 0\}]$ (with the convention $\inf \emptyset := \infty$) is called an Archimedean generator. Moreover, the map $\varphi : [0, 1] \to [0, \infty]$ defined by $\varphi(y) := \inf\{z \in [0, \infty] : \psi(z) = y\}$ for every $y \in [0, 1]$ denotes the so-called pseudo-inverse of an Archimedean generator ψ. A copula $C \in \mathcal{C}^d$ is called *Archimedean* (in which case we write $C \in \mathcal{C}^d_{\text{ar}}$) if there exists some Archimedean generator ψ with

$$C(\mathbf{x}) = \psi(\varphi(x_1) + \cdots + \varphi(x_d))$$

for every $\mathbf{x} \in \mathbb{I}^d$. It can be shown that the latter is the case if ψ is a d-monotone Archimedean generator on $[0, \infty)$, i.e., ψ is an Archimedean generator fulfilling that $(-1)^{d-2}\psi^{(d-2)}$ exists on $(0, \infty)$, is non-negative, non-increasing and convex on $(0, \infty)$ (whereby $g^{(m)}$ denotes the m-th derivative of a function g). Following [4], in order to have a one-to-one correspondence between copulas and their generator we implicitly assume that all generators are normalized in the sense that $\varphi(\frac{1}{2}) = 1$, or equivalently, $\psi(1) = \frac{1}{2}$ holds.

3 Main results

According to [7], we may characterize generators of d-dimensional Archimedean copulas in terms of the Williamson transform $\mathcal{W}_d\gamma$ of probability measures γ on $(0, \infty)$. We recall the following result (see [7] and [8, Theorem 1.11]) where we write f_+^m for the m-th power of the positive part f_+ of a function f, i.e., $f_+^m := (f_+)^m$:

Theorem 1. *Let $\psi\colon [0, \infty) \to \mathbb{I}$ be a function and $d \geq 2$. Then the following two conditions are equivalent:*

(1) ψ is the generator of a d-dimensional Archimedean copula C_ψ.
(2) There exists a unique probability measure γ on $\mathcal{B}([0, \infty))$ with $\gamma(\{0\}) = 0$ such that

$$\psi(z) = \int_{[0,\infty)} (1 - tz)_+^{d-1} \mathrm{d}\gamma(t) =: (\mathcal{W}_d\gamma)(z), \tag{1}$$

holds for every $z > 0$. In other words, ψ is the Williamson transform of (the Williamson measure) γ.

It is worth noting that the normalization property $\psi(1) = \frac{1}{2}$ translates to the probability measure γ as $\int_{\mathbb{I}} (1 - t)^{d-1} \mathrm{d}\gamma(t) = \frac{1}{2}$.

According to [7], for every $t \in (0, 1]$ the t-level set of $C \in \mathcal{C}^d$ can be written as

$$\begin{aligned} L_t &:= \left\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : C(\mathbf{x}, y) = t\right\} \\ &= \left\{(\mathbf{x}, y) \in \mathbb{I}^{d-1} \times \mathbb{I} : \sum_{i=1}^{d-1} \varphi(x_i) + \varphi(y) = \varphi(t)\right\}. \end{aligned} \tag{2}$$

Denoting the Kendall distribution function of $C \in \mathcal{C}^d$ by $F_K^d(t) := \mathbb{P}(C(\mathbf{X}) \leq t)$ for $t \in [0, 1]$ and assuming $\mathbf{X} \sim C$, the next theorem yields a simple representation of the level-set masses and the Kendall-distribution function of an Archimedean copula in terms of its corresponding Williamson measure γ:

Theorem 2 ([3]). *Let C be a d-dimensional Archimedean copula with generator ψ and Williamson measure γ. Then*

$$\mu_C(L_t) = \gamma(\{\tfrac{1}{\varphi(t)}\}),\ t \in (0, 1] \tag{3}$$

holds for every $t \in (0, 1]$. Moreover, the Kendall distribution function F_K^d of C fulfills

$$F_K^d(t) = \gamma([0, \tfrac{1}{\varphi(t)}]) \tag{4}$$

for every $t \in (0, 1]$.

As a consequence of Theorem 1 we obtain that a sequence of Archimedean copulas converges uniformly if and only if their corresponding Williamson-measures converge weakly on $(0, \infty)$ (again see [3]):

Theorem 3. *Suppose that $C, C_1, C_2, \ldots$ are d-dimensional Archimedean copulas with generators $\psi, \psi_1, \psi_2, \ldots$ and let $\gamma, \gamma_1, \gamma_2, \ldots$ denote the corresponding Williamson measures. Then the following assertions are equivalent:*

(1) $(C_n)_{n\in\mathbb{N}}$ converges uniformly to C.
(2) $(\gamma_n)_{n\in\mathbb{N}}$ converges weakly on $(0, \infty)$ to γ.

Following [6], every Markov kernel K_C can be decomposed into the sum of three sub-Markov kernels from $\mathbb{I}^{d-1}$ to $\mathcal{B}(\mathbb{I})$ as

$$K_C(\mathbf{x}, \cdot) = K_C^{dis}(\mathbf{x}, \cdot) + K_C^{sing}(\mathbf{x}, \cdot) + K_C^{abs}(\mathbf{x}, \cdot), \tag{5}$$

whereby each measure $K_C^{dis}(\mathbf{x}, \cdot)$ is discrete, each $K_C^{sing}(\mathbf{x}, \cdot)$ is singular and has no point masses and $K_C^{abs}(\mathbf{x}, \cdot)$ is absolutely continuous on $\mathcal{B}(\mathbb{I})$. Using the fact that the marginal $C^{1:d-1}$ of $C \in \mathcal{C}_{ar}^d$ is absolutely continuous (see [7]) and letting $c^{1:d-1}$ denote the corresponding density, in what follows we will refer to the three measures $\mu_C^{dis}, \mu_C^{sing}, \mu_C^{abs}$, defined by

$$\begin{aligned} \mu_C^{dis}(G) &= \int_{\mathbb{I}^{d-1}} K_C^{dis}(\mathbf{x}, G_\mathbf{x}) c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) \\ \mu_C^{sing}(G) &= \int_{\mathbb{I}^{d-1}} K_C^{sing}(\mathbf{x}, G_\mathbf{x}) c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) \\ \mu_C^{abs}(G) &= \int_{\mathbb{I}^{d-1}} K_C^{abs}(\mathbf{x}, G_\mathbf{x}) c^{1:d-1}(\mathbf{x}) \mathrm{d}\lambda_{d-1}(\mathbf{x}) \end{aligned} \tag{6}$$

for every $G \in \mathcal{B}(\mathbb{I})$, as the discrete, the singular, and the absolutely continuous component of μ_C. The next theorem (see [3]) states, how the singularity/regularity of γ carries over to the corresponding Archimedean copula.

Theorem 4. *Suppose that $C \in \mathcal{C}^d_{ar}$ has generator ψ and Williamson measure γ. Then the following assertions hold:*

(1) If γ is absolutely continuous then $\mu_C^{abs}(\mathbb{I}^d) = 1$, i.e., C is absolutely continuous.
(2) If γ is discrete then $\mu_C^{dis}(\mathbb{I}^d) = 1$.
(3) If γ is singular without point masses then $\mu_C^{sing}(\mathbb{I}^d) = 1$.

Referring to $\mathcal{C}^d_{ar,abs}$ as the family of all absolutely continuous d-dimensional Archimedean copulas, $\mathcal{C}^d_{ar,dis}$ as the family of all $C \in \mathcal{C}^d_{ar}$ with $\mu_C^{dis}(\mathbb{I}^d) = 1$, and $\mathcal{C}^d_{ar,sing}$ as the family of all $C \in \mathcal{C}^d_{ar}$ with $\mu_C^{sing}(\mathbb{I}^d) = 1$, combining Theorem 4 and Theorem 3, yields that all the afore-mentioned classes are dense in $\mathcal{C}^d_{ar}$ with respect to the unifrom metric:

Corollary 5 ((see [3])) *$\mathcal{C}^d_{ar,dis}$, $\mathcal{C}^d_{ar,abs}$ and $\mathcal{C}^d_{ar,sing}$ are dense in $\mathcal{C}^d_{ar}$ with respect to the uniform metric.*

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References

1. F. Durante and C. Sempi, "Principles of copula theory", Taylor & Francis Group LLC, Boca Raton FL 2016.
2. O. Kallenberg, "Foundations of modern probability", Springer-Verlag, New York 2002.
3. T. Kasper, N. Dietrich and W. Trutschnig, On convergence and mass distributions of multivariate Archimedean copulas and their interplay with the Williamson transform, *arxiv:2210.11868 (2022).*
4. T. Kasper, S. Fuchs, and W. Trutschnig, On weak conditional convergence of bivariate Archimedean and extreme value copulas, and consequences to nonparametric estimation, *Bernoulli* **4(27)** (2021), 2217–2240.
5. A. Klenke, "Wahrscheinlichkeitstheorie", Springer Lehrbuch Masterclass Series, Berlin Heidelberg 2008.
6. K. Lange, Decompositions of Substochastic Transition Functions, *Proceedings of the American Mathematical Society* **37(2)** (1973), 575 – 580.
7. A. McNeil and J. Nešlehová, Multivariate Archimedean copulas d-monotone functions and ℓ_1-norm symmetric distributions, *The Annals of Statistics*, **37** (2009), 3059–3097.
8. R. Schilling, R. Song, and Z. Vondracek, "Bernstein Functions: Theory and Applications", De Gruyter Studies in Mathematics. De Gruyter 2012.

Contagion of deprivations and affluences: a tail dependence story

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1 Introduction

Welfare and related phenomena such as poverty and inequality are multidimensional, that is, they depend not only on income but also on other aspects such as working conditions and material well-being. Hence, to appropriately account for the multivariate nature of these phenomena, it is necessary to measure the potential interdependence between their dimensions. In this paper, we focus on one aspect of multivariate dependence, namely the dependence in the lower and upper tails of the joint distribution. In particular, lower tail dependence captures the risk that individuals who score low in one dimension also score low in the others. Similarly, upper tail dependence captures the risk that individuals who score high in one dimension also score high in the others. In our context, we will refer to these two features as contagion of deprivations and contagion of affluences, respectively. To build up measures of tail dependence, we follow a copula based approach. This approach is particularly appropriate in multidimensional welfare analysis, where we deal with non-Gaussian distributions and possibly non-linear relationships.

Despite its theoretical appeal and its popularity in fields such as finance or environmental sciences, the concept of tail dependence has only recently been applied in welfare economics in D'Agostino et al. (2022).These authors provide a novel application of tail dependence concepts in poverty analysis but limited to a bidimensional setting. Our paper generalises this work by incorporating a multidimensional perspective that includes a pioneering application of multivariate tail dependence concepts to welfare analysis. To do so, we rely on a particular version of the multivariate tail concentration funcion proposed by Di Bernardino and Rullière (2017). This function, which is based on the work of Venter (2002) and Durante et al. (2015) for the bivariate case, has several advantages. First, it allows to represent, in a unit square, the degree of multivariate dependence in both tails of the joint distribution, regardless of the number of dimensions considered. Second, it avoids the cumbersome task of estimating asymptotic tail dependence coefficients. Additionally, we show that this function is closely related to a measure of multivariate overall dependence, namely the Blomqvist's beta.

Our paper illustrates the use of the multivariate TCF by analysing the evolution of tail dependence between the three dimensions of the AROPE rate in the EU-28 over the period 2008-2018. First, we find evidence of lower and upper tail dependence in all EU-28 countries. That is, there is evidence of a risk of contagion of deprivations and affluences. Second, this dependence is time-varying over the period analysed and it is not homogenous over all countries. And third, in most of the EU-28 countries, the risk of contagion of deprivations tends to be higher than the mirrored risk of contagion of affluences.

2 Methodology

The copula approach focuses on the positions of the individuals across the variables, rather than on the values that these variables attain for such individuals. In particular, let the continuous random vector $\mathbf{X} = (X_1, \ldots, X_d)$ represent the relevant d dimensions analysed and let F_i denote the marginal distribution of dimension i, with $i = 1, \ldots, d$. Then, each original variable X_i is transformed by applying the so-called *probability integral transformation*, obtaining a transformed variable $U_i = F_i(X_i)$, with $i = 1, \ldots d$. These transformed variables attach to each individual in the population its relative position in all dimensions.

From probability theory, the transformed variables $U_1, \ldots, U_d$ are standard uniform random variables $U(0,1)$ and the joint distribution of the vector $\mathbf{U} = (U_1, ..., U_d)$ turns out to be the copula function C. Therefore, the copula is a d-dimensional cumulative distribution function, $C : \mathbf{I}^d \to \mathbf{I}$, with $\mathbf{I} = [0,1]$, whose univariate marginals are $U(0,1)$. So, for a given real vector $\mathbf{u} \in \mathbf{I}^d$, the value $C(\mathbf{u})$ represents the proportion of individuals in the population with positions outranked by $\mathbf{u}$, i.e, $C(\mathbf{u}) = p(\mathbf{U} \leq \mathbf{u}) = p(U_1 \leq u_1, \ldots, U_d \leq u_d)$. For instance, $C(0.2, ..., 0.2)$ will represent the probability that a randomly selected individual is simultaneously in the 1^{st} quintile ("low-ranked") in all dimensions.

Another important function, which is not a copula itself but it is related to the copula C, is the survival function, $\bar{C} : \mathbf{I}^d \to \mathbf{I}$, defined as:

$$\bar{C}(\mathbf{u}) = p(\mathbf{U} > \mathbf{u}) = p(U_1 > u_1, \ldots, U_d > u_d),$$

where $\mathbf{U} = (U_1, ..., U_d)$ is a random vector of variables $U(0,1)$ whose joint distribution function is the copula C. For instance, $\bar{C}(0.8, \ldots, 0.8)$ will represent the probability that a randomly selected individual is simultaneously in the 5^{th} quintile in all dimensions.

The copula and survival function defined above play an essential role in determining the so-called lower and upper tail dependence coefficients, respectivelty. Following Joe (2015), in a d-dimensional framework, these coefficients are defined as:

$$\lambda_L^d = \lim_{u \to 0^+} \Pr(U_1 \leq u, \ldots, U_{d-1} \leq u | U_d \leq u) = \lim_{u \to 0^+} \frac{C(u, \ldots, u)}{u}, \quad (1)$$

$$\lambda_U^d = \lim_{u \to 1^-} \Pr(U_1 \geq u, \ldots, U_{d-1} \geq u | U_d \geq u) = \lim_{u \to 1^-} \frac{\bar{C}(u, \ldots, u)}{1-u}. \quad (2)$$

As Durante et al. (2015) point out, “while tail dependence give an asymptotic approximation of the behaviour of the copula in the tail of the distribution, it might be also of interest to consider the case when the tail behaviour is considered at some (finite) points near the corners of the unit square”. To face this goal, we use a particular version of the multivariate tail concentration function (TCF), namely $q_C^d : (0,1) \to \mathbf{I}$, given by:

$$q_C^d(u) = \frac{C(u,\ldots,u)}{u}\mathbf{1}_{(0,0.5]} + \frac{\bar{C}(u,\ldots,u)}{1-u}\mathbf{1}_{(0.5,1)} \qquad (3)$$

where $\mathbf{1}_A$ denotes the indicator function on a set A.

In our framework, if we consider the three dimensions of the AROPE rate (income, work intensity and material well-being), evaluating the TCF at lower (high) values enables analysing the risk of contagion of deprivations (affluences). For instance, $q_C^d(0.2)$ will capture the risk that an individual that is in the 1^{st} quintile in income is also simultaneously in the 1^{st} quintile in both work intensity and material well-being. Similarly, $q_C^d(0.8)$ will capture the risk that an individual that is in the top quintile in income is also simultaneously in the top quintile in both work intensity and material well-being.

The TCF has the advantage of representing, in a unit square, the dependence structure on the tails of a multivariate distribution, regardless of the number of dimensions considered. Moreover, in the limits, this function yields the multivariate tail dependence coefficients defined in (1) and (2), namely $q_C^d(0^+) = \lambda_L^d$ and $q_C^d(1^-) = \lambda_U^d$. Additionally, the multivariate TCF is related to a measure of global dependence, namely the multivariate Blomqvist's beta, $\beta_{d,C}$, proposed by Úbeda-Flores (2005). In particular, the following relathionship holds.

$$\frac{q_C^d(0.5) + q_C^d(0.5^+)}{2} = \beta_{d,C}[1 - 2^{-d+1}] + 2^{-d+1}.$$

In practice, we propose to estimate the multivariate TCF by replacing in (3) both the copula and the survival function with their empirical counterparts. For non-continuous data, the empirical checkerboard copula in Genest et al. (2017) will be used.

3 Results

In this paper, we apply the multivariate TCF to analyse the evolution of multivariate tail dependence between income, work intensity and material well-being in the countries of the EU-28 over the period 2008-2018.

Figure 1 displays, for the EU-28 countries, the estimated trivariate TCF for years 2008 (blue line), 2014 (red line) and 2018 (green line) together with 95% standard bootstrap confidence intervals using 1000 bootstrap replications. As a benchmark, the theoretical trivariate TCF of independence is displayed in black.

Several conclusions emerge from Figure 1. First, there is a risk of contagion of both deprivations and affluences, since the estimated TCFs are above the theoretical TCF of the independence for the three years considered. This means that, in the EU-28, there is a positive probability that a household that scores low (high) in one dimension also scores low (high) in the other two dimensions simultaneously. Second, in most of the EU-28 countries the TCFs are not symmetric, i.e, the risk of contagion of deprivations

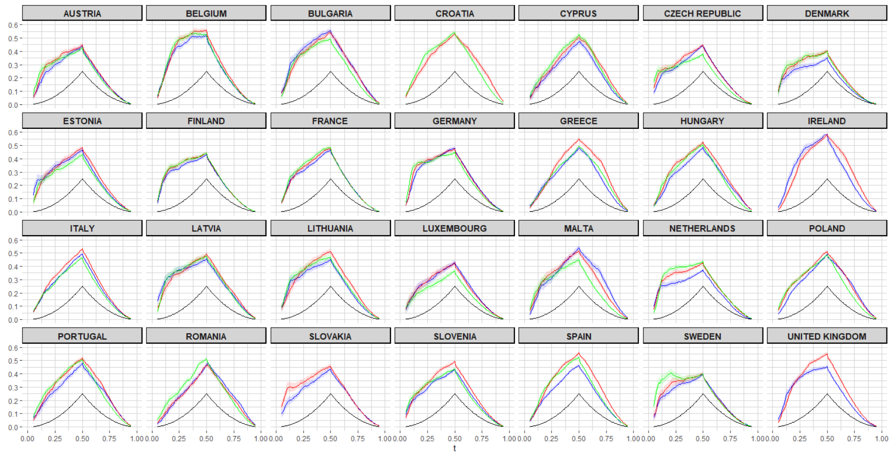

Fig. 1. TCF for the EU-28 countries and years 2008 (blue), 2014 (red) and 2018 (green) with bootstrap confidence intervals and TCF of independence (black).

tends to be higher than the mirrored risk of contagion of affluences. Actually, the latter tends to zero in the limit. Third, multivariate tail dependence changes accross countries and over time. In some countries, the level of dependence between welfare dimensions hardly changed over the period analysed. By contrast, in other countries there was a significant increase of the risk of cumulative deprivation and affluence over the period 2008-2014, but the post-2014 recovery period allowed to reduce that risk to the levels of 2008. However, there are also countries where the Great Recession strengthened tail dependence in such a way that this was still higher in 2018 than in 2008.

References

1. D'Agostino, A., Deluca, G. and Guégan, D., Estimating lower tail dependence between pairs of poverty Dimensions in Europe, *Review of Income and Wealth.* (2022)
2. Di Bernardino, E., Rullière, D., A note on upper-patched generators for Archimedean copulas, *ESAIM: Probability and Statistics* (2017) , 21, 183-200.
3. Durante, F., Fernandez-Sanchez, J. and Pappada, R., Copulas, diagonals, and tail dependence, *Fuzzy Sets Syst.* **264** (2015), 22–41.
4. Genest, C., Nešlehová, J. G. and Rémillard, B., Asymptotic behavior of the empirical multilinear copula process under broad conditions, *Journal of Multivariate Analysis* **159** (2017), 82–110.
5. Joe, H, "Dependence Modeling with Copulas", CRC press 2015.
6. Úbeda-Flores, M., Multivariate versions of Blomqvist's beta and Spearman's footrule, *Annals of the Institute of Statistical Mathematics* **57** (2005), 781–788.
7. Venter, G., Tails of copulas, In: *Proceedings of the Casualty Actuarial Society* (2002), 68–113.

Sparse M-estimators in semi-parametric copula models

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We study the large sample properties of sparse M-estimators in the presence of pseudo-observations. Our framework covers a broad class of semi-parametric copula models, for which the marginal distributions are unknown and replaced by their empirical counterparts. It is well known that the latter modification significantly alters the limiting laws compared to usual M-estimation. We establish the consistency and the asymptotic normality of our sparse penalized M-estimator and we prove the asymptotic oracle property with pseudo-observations, possibly in the case when the number of parameters is diverging. Our framework allows to manage copula-based loss functions that are potentially unbounded. Additionally, we state the weak limit of multivariate rank statistics for an arbitrary dimension and the weak convergence of empirical copula processes indexed by maps. We apply our inference method to Canonical Maximum Likelihood losses with Gaussian copulas, mixtures of copulas or conditional copulas. The theoretical results are illustrated by two numerical experiments.

References

1. Poignard, B., & Fermanian, J. D., Sparse M-estimators in semi-parametric copula models, *arXiv preprint* arXiv:2112.12351.

A model-free and dependence-based forward feature selection for multi-response data and a tool for identifying networks between variables

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Abstract. Building upon the new measure of predictability T^q introduced by Ansari & Fuchs (2023), which quantifies the extent of functional dependence of a response vector $\mathbf{Y} = (Y_1, \ldots, Y_q)$ on a set of explanatory random variables $\mathbf{X} = (X_1, \ldots, X_p)$, we present a model-free and dependence-based feature ranking and forward feature selection of data with multiple response variables, thus facilitating the selection of the most relevant explanatory variables. We further provide a hierarchical clustering method for random variables and devise a visualization tool for the interconnectedness of random variables based on the degree of predictability among them.

Introduction

As a direct and natural extension of Azadkia & Chatterjee's rank correlation T, presented in [2] and given (in its unconditional form) by

$$T(Y|\mathbf{X}) := \frac{\int_{\mathbb{R}} \mathrm{var}(P(Y \geq y \,|\, \mathbf{X}))\, \mathrm{d}P^Y(y)}{\int_{\mathbb{R}} \mathrm{var}(\mathbb{1}_{\{Y \geq y\}})\, \mathrm{d}P^Y(y)}$$

(see also [3]), to a vector $\mathbf{Y} = (Y_1, \ldots, Y_q)$ of $q \geq 1$ response variables, in [1] we introduce the functional T^q defined by

$$T^q(\mathbf{Y}|\mathbf{X}) := \frac{\sum_{i=1}^{q} \left[T(Y_i|(\mathbf{X}, Y_{i-1}, \ldots, Y_1)) - T(Y_i|(Y_{i-1}, \ldots, Y_1))\right]}{\sum_{i=1}^{q} \left[1 - T(Y_i|(Y_{i-1}, \ldots, Y_1))\right]}$$

with $\quad T(Y_1|\emptyset) := 0.$

$T^q(\mathbf{Y}|\mathbf{X})$ quantifies the *degree of predictability* of $\mathbf{Y}$ given $\mathbf{X}$, i.e., the scale-invariant extent of functional dependence of the response vector $\mathbf{Y} = (Y_1, \ldots, Y_q)$ on a number of explanatory variables $\mathbf{X} = (X_1, \ldots, X_p)$, $p \geq 1$, and fulfills the desired characteristics of a measure of predictability, namely

(A1) $0 \leq T^q(\mathbf{Y}|\mathbf{X}) \leq 1$.
(A2) $T^q(\mathbf{Y}|\mathbf{X}) = 0$ if and only if $\mathbf{Y}$ and $\mathbf{X}$ are independent,
(A3) $T^q(\mathbf{Y}|\mathbf{X}) = 1$ if and only if $\mathbf{Y}$ is perfectly dependent on $\mathbf{X}$, i.e., there exists some measurable function $\mathbf{f} : \mathbb{R}^p \to \mathbb{R}^q$ such that $\mathbf{Y} = \mathbf{f}(\mathbf{X})$ almost surely.

In addition, T^q fulfills the so-called information gain inequality:

(P1) $T^q(\mathbf{Y}|\mathbf{X}) \leq T^q(\mathbf{Y}|(\mathbf{X},\mathbf{Z}))$ for all random vectors $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$,

formalizing the idea that additional explanatory variables improve the prediction, and characterizes conditional independence between random vectors:

(P2) $T^q(\mathbf{Y}|\mathbf{X}) = T^q(\mathbf{Y}|(\mathbf{X},\mathbf{Z}))$ if and only if $\mathbf{Y}$ and $\mathbf{Z}$ are conditionally independent given $\mathbf{X}$.

T^q has a model-free, strongly consistent estimator which can be computed in $O(n \log n)$ time and which is given by a simple function of the graph-based estimator T_n for T proposed in [2].

Forward feature selection

Up to our knowledge, there is rather little literature on feature selection methods that are applicable to multivariate response vectors (i.e., for $q > 1$). In the class of linear methods, the lasso allows an extension to multiple output data [6], while the kernel feature ordering by conditional independence in [5] is a general model-free method defined for reproducing kernel Hilbert spaces.

Making use of the above described properties of T^q, we propose a model-free and dependence-based forward feature selection method for multi-response data called MFOCI. We prove that MFOCI is consistent in the sense that the subset $\mathbf{X}_S := (X_j)_{j\in S}$ with $S \subseteq \{1,\dots,p\}$ of selected explanatory variables via MFOCI is sufficient with high probability, i.e., with high probability $\mathbf{Y}$ and $\mathbf{X}_S$ are conditionally independent given $\mathbf{X}_{S^c} := (X_j)_{j\in S^c}$ where $S^c := \{1,\dots,p\} \setminus S$. We further provide several simulations and real-data examples for multi-response data illustrating the superior performance of our method in comparison to existing procedures.

Identifying networks

Since T^q measures the strength of dependence between random vectors, there exist several ways of identifying networks on the level of random variables.

Interconnectedness of banks.
As illustrative example we first consider the interconnectedness of the 3 largest banks in each of the U.S. (US) and Europe (EU), and compare further their connectedness with the 4th largest banks in the US and Europe, which are Citigroup (CG) and Deutsche Bank (DB), respectively.
For revealing the interconnectedness of the banks, we estimate their degree of predictability via T^q from a sample of log-returns of the Banks' stock data. Figure 1 shows the values of T_n^q for the above described interrelations.

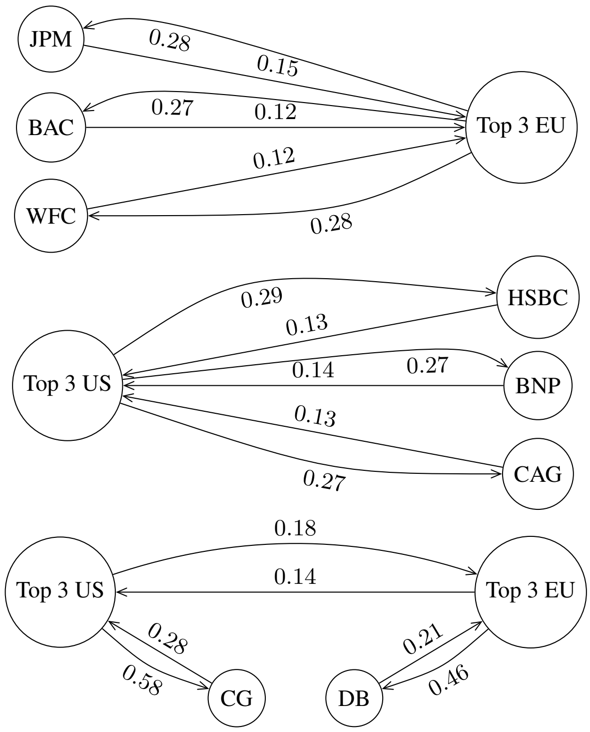

Fig. 1. Interconnectedness of the three largest banks in the US and Europe, as well as connectedness with the banks Citigroup and Deutsche Bank measured by T^q ; for example, the interconnectedness of the three largest US banks with the three largest EU banks was calculated using T_n^q with $p = 3$ explanatory and $q = 3$ response variables.

Clustering.
For T^q we employ the agglomerative hierarchical clustering procedure proposed in [4], where as dissimilarity measure we here propose a suitable function of T^q measuring the maximum degree of predictability between two given disjoint subsets of random variables. Figure 2 illustrates the clustering output of the three largest US and EU banks based on log-return data.

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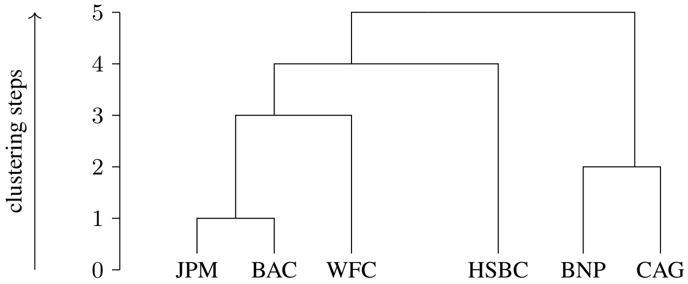

Fig. 2. Hierarchical clustering of the three largest banks in the US (JPM, BAC, WFC) and Europe (HSBC, BNP, CAG) measured by T^q .

References

1. Ansari, J. and S. Fuchs (2023). A simple extension of Azadkia & Chatterjee's rank correlation to a vector of endogenous variables. Available at arxiv.org/abs/2212.01621.
2. Azadkia, M. and S. Chatterjee (2021). A simple measure of conditional dependence. *Ann. Stat. 49*(6), 3070–3102.
3. Dette, H., K. F. Siburg, and P. A. Stoimenov (2013). A copula-based non-parametric measure of regression dependence. *Scand. J. Statist. 40*(1), 21–41.
4. Fuchs, S., F. M. L. Di Lascio, and F. Durante (2021). Dissimilarity functions for rank-based hierarchical clustering of continuous variables. *Comput. Statist. Data Anal. 159*, Article ID 107201, 26 pages.
5. Huang, Z., N. Deb, and B. Sen (2022). Kernel partial correlation coefficient - a measure of conditional dependence. *J. Mach. Learn. Res. 23*(216), 1–58.
6. Tibshirani, R. (1996). Regression shrinkage and selection via the lasso. *J. R. Stat. Soc. Ser. B. Stat. Methodol. 58*(1), 267–288.

Distorted copulas

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Abstract. Transformations of copulas based upon strictly increasing bijections on the real unit interval are discussed. In particular, the problem of determining the subset of bijections which ensure that the transformation of a given copula is still a copula is studied. The real novelty is that several results strongly depend on a fruitful connection between real analysis and group theory.

1 Motivation

Let C be a (bivariate) copula and let $g : [0,1] \to [0,1]$ be a continuous, strictly increasing bijection. The *distorsion* of C by means of g is the function $C_g : [0,1]^2 \to [0,1]$ given by

$$C_g(x,y) = g^{-1}(C(g(x), g(y))). \tag{1}$$

The distortion of a copula as in eq. (1) is also known as *transformation* of C via g. This topic has been considered in many papers both from a theoretical point of view and in applied contexts. See, for instance, [1, 4, 10, 2, 8].

The study of distorted copulas is particularly interesting because they may be used to generate in a very flexible way new families of copulas. Furthermore, it is quite intriguing to analyze how some dependence properties of a copula are modified in a distorted copula, for example the tail dependence coefficients (see [12, 9]). To this purpose, it is essential to have some information about the subset of bijections which ensure that the distortion of a copula is still a copula. More formally, denote by $\mathcal{C}$ the set of bivariate copulas and by Θ the set of strictly increasing and continuous bijections on the real unit interval: given an arbitrary copula C, let $\mathcal{I}(C) = \{g \in \Theta : C_g \in \mathcal{C}\}$. In [4], the authors exactly pose the problem of determining $\mathcal{I}(C)$ for a given $C \in \mathcal{C}$. Note that this problem has been completely solved in the context of triangular norms (see, for instance, [11]) when $C = M$, $C = \Pi$ and $C = W$, where

$$M(x,y) = \min\{x,y\}, \;\; W(x,y) = \max\{0, x+y-1\}, \;\; \Pi(x,y) = xy.$$

In the first case, we have $\mathcal{I}(M) = \Theta$, while in the two remaining cases, when C is either a nilpotent or strict Archimedean copula, $\mathcal{I}(W)$ and $\mathcal{I}(\Pi)$ are given by the concave and the log-concave bijections of Θ, respectively (see also [3]).

To the best of our knowledge, there are no general results related to different types of copulas: moreover, quite surprisingly, only a few investigations appear in the literature with regard to the connection between $\mathcal{I}(C)$ and $\mathcal{I}(C_g)$ for a given copula C and a fixed $g \in \Theta$.

2 Main results

A semi-copula S is a function from $[0,1]^2$ to $[0,1]$ that is increasing in each variable and satisfies $S(x,1) = S(1,x) = x$ for every $x \in [0,1]$, but it may be neither associative nor commutative [6, 5]. It is not difficult to see that the set of continuous semi-copulas is closed under the distortion of any $g \in \Theta$. The same does not occur when S is a copula: indeed a distorted copula is generally only a continuous semi-copula (see, for instance, [7]).

Given any copula C, we call horizontal section of C any mapping of the kind $x \mapsto C(x,v)$ for any fixed $v \in [0,1]$. Analogously, a vertical section of C is any mapping of the kind $x \mapsto C(v,x)$ for any fixed $v \in [0,1]$.

The first results are related to a necessary condition: particularly, $C_g \in \mathcal{C}$ requires that every horizontal and vertical section of C_g is an absolutely continuous function on $[0,1]$. The absolute continuity of both g and its inverse g^{-1} is a sufficient condition for assuring the above requirement, but generally not necessary, as we will show by means of a particular counter-example. However, we prove that the converse is true when C is jointly strictly monotone, i.e.

$$C(x,y) < C(x',y) \text{ and } C(y,x) < C(y,x')$$

for any $x, x', y \in [0,1]$ such that $x < x'$ and $y > 0$.

We can formalize the process of distortion of a semi-copula within the branch of group theory given by the actions. Let X be an arbitrary, non empty set and let G be a group. A (right) *action* of G upon X is a mapping $\Psi : X \times G \to X$ satisfying the following axioms:

- $\Psi(\Psi(x,g),h) = \Psi(x,gh)$ for all $g, h \in G$ and for any $x \in X$;
- $\Psi(x,e) = x$ for all $x \in X$, where e denotes the identity element of G.

We will present some results about the connection between $\mathcal{I}(C)$ and $\mathcal{I}(C_g)$ when we have some information on $\mathcal{I}(C)$ just resorting to the group action, by identifying G with Θ and X with the set of continuous semi-copulas.

Finally, a third group of results is related to some sufficient conditions in order to determine at least a subset of $\mathcal{I}(C)$ for some remarkable examples of copulas.

References

1. E. Alvoni, P.L. Papini and F. Spizzichino. On a class of transformations of copulas and quasi–copulas. *Fuzzy Sets and Systems*, 160(3):334–343, 2009.
2. B. Bassan and F. Spizzichino. Bivariate survival models with Clayton aging functions. *Insurance Math. Econom.*, 37(1):6–12, 2005.
3. B. Bassan and F. Spizzichino. Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. *J. Multivariate Anal.*, 93(2):313–339, 2005.
4. F. Durante and C. Sempi. Copula and semicopula transforms. *Int. J. Math. Math. Sci.*, 4:645–655, 2005.
5. F. Durante and C. Sempi. Semicopulæ. *Kybernetika (Prague)*, 41(3):315–328, 2005.

6. F. Durante, J.J. Quesada-Molina, and C. Sempi. Semicopulas: characterizations and applicability. *Kybernetika (Prague)*, 42(3):287–302, 2006.
7. F. Durante and C. Sempi. *Principles of Copula Theory*. CRC/Chapman & Hall, Boca Raton, FL, 2015.
8. C. Genest and L.P. Rivest. On the multivariate probability integral transformation. *Statist. Probab. Lett.*, 53(4):391–399, 2001.
9. H. Joe. *Dependence Modeling with Copulas*. Chapman & Hall/CRC, London, 2014.
10. E.P. Klement, R. Mesiar and E. Pap. Transformations of copulas. *Kybernetika (Prague)*, 41 (4):425–434, 2005.
11. E.P. Klement, R. Mesiar, and E. Pap. *Triangular norms*, volume 8 of *Trends in Logic—Studia Logica Library*. Kluwer Academic Publishers, Dordrecht, 2000.
12. R.B. Nelsen. *An Introduction to Copulas*. Springer Series in Statistics. Springer, New York, second edition, 2006.

Copula-based dependence measures between random vectors

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In this talk we discuss copula-based dependence quantification between multiple groups of random variables of possibly different sizes. A focus in the talk is on the family of Φ-divergences. An axiomatic framework for this purpose is provided, and we illustrate the divergence measures by means of examples. For statistical inference we focus on the absolutely continuous setting assuming copula densities exist. We consider parametric and semi-parametric frameworks, discuss estimation procedures, and establish asymptotic properties of the proposed estimators. Simulations indicate finite-sample performances, and practical use is discussed.

This talk is based on joint work with Steven De Keyser (KU Leuven).

Chatterjee's rank correlation: what is new?

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This talk will provide an overview of the recent progress made in exploring Sourav Chatterjee's newly introduced rank correlation. The objective is to elaborate on its practical utility and present several new findings pertaining to (a) the asymptotic normality and limiting variance of Chatterjee's rank correlation, (b) its statistical efficiency for testing independence, and (c) the issue of its bootstrap inconsistency. Notably, the presentation will reveal that Chatterjee's rank correlation is root-n consistent, asymptotically normal, but bootstrap inconsistent — an unusual phenomenon in the literature.

References

1. Han, F., On extensions of rank correlation coefficients to multivariate spaces, *Bernoulli News* **28** (2021), 7-–11.
2. Lin, Z. and Han, F., Limit theorems of Chatterjee's rank correlation, (2022) available at arXiv:2204.08031.
3. Lin, Z. and Han, F., On boosting the power of Chatterjee's rank correlation. *Biometrika* (2023) (in press).
4. Lin, Z. and Han, F., On the failure of the bootstrap for Chatterjee's rank correlation, (2023) available at arXiv:2303.14088.
5. Shi, H., Drton, M., and Han, F., On the power of Chatterjee's rank correlation, *Biometrika* **109(2)** (2022), 317-–333.
6. Shi, H., Drton, M., and Han, F., On Azadkia-Chatterjee's conditional dependence coefficient, *Bernoulli* (2023) (in press).

Relations between the shapes of triangular norms and their generators

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Abstract. We study relations between the shape of a strict triangular norm and its generators. In particular, we introduce the class of *balanced* generators.

1 Formulation of the goal

Triangular norms (t-norms for short) are binary operations on the real interval $[0,1]$ which are commutative, associative, nondecreasing, and with neutral element 1. We restrict attention to t-norms T which are *strict*, i.e., continuous and satisfying $T(x,y) < T(x,z)$ whenever $y < z$ and $x > 0$. T-norms are used mainly in fuzzy logics as the interpretation of a conjunction. We refer to [3] for basics on t-norms.

It is known [4, 7] that each strict t-norm T has a (non-unique) additive generator, $t\colon [0,1] \to [0,\infty]$, and a multiplicative generator, $\theta\colon [0,1] \to [0,1]$, which allow to express it as

$$T(x,y) = t^{-1}\big(t(x) + t(y)\big) = \theta^{-1}\big(\theta(x)\,\theta(y)\big)\,.$$

A strict t-norm is a copula iff its (any) additive generator is convex [3]. Conversely, associative copulas are t-norms. Thus many families of binary operations were introduced and studied independently as copulas and t-norms.

Many different proofs of the existence of generators of strict t-norms were published (see the bibliography of [9]). However, we found their interpretability unsatisfactory. It is not known how the "shape" of a generator influences that of the corresponding t-norm and vice versa.

One possible approach is to describe the t-norm by a collection of points which should lie on its graph (or close to it) and fit a t-norm to these data. This can be easily done if we restrict attention to some parametric family of t-norms. If we try to find an optimal approximation among *all* strict or nilpotent t-norms, their associativity makes the task difficult. A feasible solution was proposed by Beliakov in [1]. His solution is not optimal, as shown in [10]. However, all these attempts ignore the local behavior of the t-norm, e.g., its derivatives. We are not aware of any attempt to make an analogy of a Hermite interpolation by a t-norm.

We posed the question of how a (local) change of a t-norm influences its generator or, vice versa, how a change of a generator modifies the t-norm. Particular attention is paid to their first derivatives.

2 Inspiration

There is one result that links the shape of a t-norm and its multiplicative generator. It was first published in [8], here we cite it from [9]:

Theorem 1. *Let θ be a multiplicative generator of a* strict *t-norm T such that $\theta'(0) \in]0, \infty[$. Then*

$$\theta(y) = \frac{\partial}{\partial x} T(x, y)\Big|_{x=0} = \lim_{x \to 0_+} \frac{T(x, y)}{x}$$

for all $y \in [0, 1]$ whenever these expressions are defined.

The latter expression means that, up to the scaling factor $1/x$, the function $y \mapsto T(x, y)$ becomes a very good approximation of a multiplicative generator of T when $x \to 0$, see Fig. 1. Thus a close look at the graph of T near the line segment $\{(0, y) \mid y \in [0, 1]\}$ allows to deduce the "shape" of (one distinguished) multiplicative generator of T. Other multiplicative generators of T are powers of this generator and do not admit such an interpretation. We shall discuss this situation in more detail in Section 3.

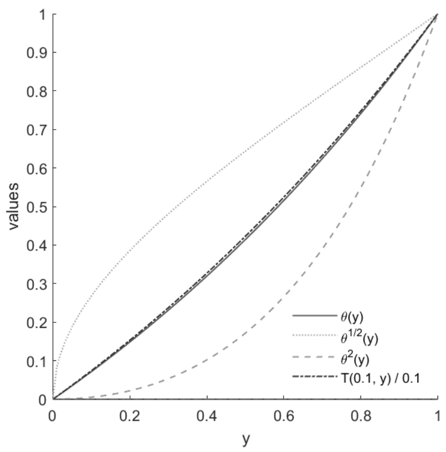

Fig. 1. A balanced generator of a Frank t-norm, two of its powers, and its approximation by Th. 1

3 Derivatives at bounds of the domain

The derivatives at the upper bounds of the domains are linked by the following rules:

Theorem 2. *Let T be a strict t-norm, t its additive generator, and θ its multiplicative generator. From the following conditions, 1 and 2 are equivalent and they imply 3:*

1. *θ has a continuous, nonzero, finite derivative at* 1.
2. *t has a continuous, nonzero, finite derivative at* 1.
3. *T is differentiable at* $(1,1)$ *(i.e., it has there a total differential; explicitly, its linear approximation in the neighborhood of* $(1,1)$ *is* $(x,y) \mapsto x+y-1$*).*

Remark 1. Theorem 2 operates with the existence of derivatives as follows: if one derivative exists and satisfies the assumptions, the other derivative also exists.

Notice that the linear approximation at $(1,1)$ coincides (locally) with the Łukasiewicz t-norm, $T(x,y) = \max(x+y-1,0)$, which is nilpotent, but the theorem applies to strict t-norms, e.g., to all strict Frank t-norms, defined by

$$T(x,y) = \log_\lambda \left(1 + \frac{(\lambda^x - 1)\,(\lambda^y - 1)}{\lambda - 1}\right) \qquad \text{for } \lambda \in (0,\infty) \setminus \{1\}\,,$$

including the product t-norm (as the limit case for $\lambda \to 1$). On the other hand, following [6], any continuous t-norm can be approximated by a strict t-norm with arbitrary precision. As an example, the minimum t-norm is the limit case of Frank t-norms for parameter $\lambda \to 0$, although it is not differentiable at $(1,1)$ (and does not have a generator).

In Theorem 2, if one of the generators satisfies the first two conditions, all generators satisfy it, too. The reason is that additive generators are determined up to a positive multiple and multiplicative generators up to a positive power, and these operations keep the derivatives at 1 finite and non-zero. Now we shall discuss derivatives at 0 and the situation becomes different. Additive generators of strict t-norms are unbounded in a neighborhood of 0, thus speaking of their "shape" is not much useful. We could ask whether they tend to ∞ faster or slower than the negative logarithm, but this does not help our intuition and understanding of the graph. For multiplicative generators the condition similar to Theorem 2 is meaningful.

Definition 1. *[2] A multiplicative generator θ of a strict t-norm is called a* balanced generator *if it has a (right) derivative at* 0 *such that* $0 < \theta'(0) < \infty$.

Example 1. Frank t-norms for parameter $\lambda \in (0,\infty) \setminus \{1\}$ have multiplicative generators $\theta_\lambda = \dfrac{\lambda^x - 1}{\lambda - 1}$ which are balanced. Hamacher product (which belongs to the family of Ali–Mikhail–Haq copulas),

$$T_H(x,y) = \frac{x\,y}{x + y - x\,y} \qquad \text{for } x, y \neq (0,0)\,,$$

has a multiplicative generator

$$\theta_H(x) = \mathrm{e}^{\frac{x-1}{x}}$$

and has no balanced generator.

Theorem 1 reconstructs balanced generators. Every strict t-norm has at most one balanced generator; the derivative of its rth power at 0 is 0 for $r > 1$ and ∞ for $r < 1$, see Fig. 1. A question arises how to find a balanced generator if it exists and we know a multiplicative generator that is not balanced. We have an answer:

Theorem 3. *Let T be a strict t-norm with a multiplicative generator θ. If there is a finite nonzero limit*

$$r = \lim_{x \to 0+} \frac{x\,\theta'(x)}{\theta(x)}\,, \tag{1}$$

then T has a balanced generator

$$\theta_* = \theta^{1/r}\,. \tag{2}$$

If T has a balanced generator θ_*, it is determined by (1) and (2).

If a balanced generator has a continuous derivative at 0, the corresponding t-norm is differentiable at $(0,0)$; its total differential there is zero. We can say more about its similarity to the product near this point:

Theorem 4. *Let T be a strict t-norm with a balanced generator θ such that $\theta'(0) = c \in \,]0,\infty[$. Then*

$$\lim_{(x,y)\to(0,0)} \frac{T(x,y)}{c\,x\,y} = 1\,.$$

4 Diagonals of t-norms

A *diagonal* of a t-norm T is the unary function $\Delta(x) = T(x,x)$. It is known that a diagonal does not determine the t-norm uniquely; see [5] for the characterization of all t-norms with a given diagonal Δ. Its main steps construct a multiplicative generator θ as follows:

1. One point $\big(x,\theta(x)\big) \in \,]0,1[^2$ of the graph of the multiplicative generator can be chosen arbitrarily,
2. Then the diagonal determines θ on a countable infinite set $M \subset \,]0,1[$.
3. The set M is not dense. Choose two subsequent elements $a,b \in M$ such that $a < b$ and $]a,b[\, \cap M = \emptyset$. The restriction $\theta|_{]a,b[}$ can be any function such that $\theta|_{[a,b]}$ is strictly increasing and continuous.
4. The remaining values of θ are uniquely determined by the preceding steps.

A fact that seems to remain unnoticed is that the range of possible values of the multiplicative generator constructed this way is limited by the difference $x - \Delta(x) > 0$. The closer the diagonal is to the identity, the closer must be all t-norms with this diagonal. This restriction may be helpful if the diagonal is close to the identity, which is sometimes the case, e.g., for Frank t-norms with a small parameter.

5 Conclusion

We collected some results which relate the shapes of generators and the corresponding t-norms. Although some of them are vaguely formulated and others are only local properties, referring to derivatives at points, they help to understand the geometry of these objects. These rules can be also demonstrated by graphs of specific aspects of these relations, see [2].

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References

1. G. Beliakov: Fitting triangular norms to empirical data. In: Erich Peter Klement and Radko Mesiar, editors, *Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms*. Elsevier Science B.V., Amsterdam, 2005, 261–272.
2. K. Houšková: *The Interplay of Triangular Norms and Their Generators*. Draft of BSc. Thesis, Czech Technical University in Prague, Czech Republic, April 20, 2023.
3. E. P. Klement, R. Mesiar, E. Pap: *Triangular Norms*, vol. 8 of *Trends in Logic*. Kluwer Academic Publishers, Dordrecht, Netherlands, 2000.
4. C.M. Ling: Representation of associative functions. *Publ. Math. Debrecen*, 12:189–212, 1965.
5. R. Mesiar, M. Navara: Diagonals of continuous triangular norms. *Fuzzy Sets Syst.* **104** (1999), 34–41.
6. V. Marko, R. Mesiar: Continuous Archimedean t-norms and their bounds. *Fuzzy Sets Syst.* **121** (2001), 183–190.
7. P.S. Mostert, A.L. Shields. On the structure of semi-groups on a compact manifold with boundary. *Ann. of Math., II. Ser.*, 65:117–143, 1957.
8. M. Navara, M. Petrík: Two methods of reconstruction of generators of continuous t-norms. In: L. Magdalena, M. Ojeda-Aciego, J.L. Verdegay (eds.), *12th International Conference Information Processing and Management of Uncertainty in Knowledge-Based Systems*, Málaga, Spain, 2008, 1016–1021.
9. M. Navara, M. Petrík: Generators of fuzzy logical operations. In: H.T. Nguyen, V. Kreinovich (eds.), *Algebraic Techniques and Their Use in Describing and Processing Uncertainty*, Studies in Computational Intelligence 878, Springer, 89–112, 2020. DOI 10.1007/978-3-030-38565-1_8
10. A. Popovič: *Approximating by Fuzzy Conjunctions*. BSc. Thesis, Czech Technical University in Prague, Czech Republic, 2020.

Copulas with given values on a given set

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Abstract. We discuss sets of copulas with prescribed values on some fixed subset S of the unit n-cube $\mathbb{I}^n$. The first problem is to find conditions for S and for the values on S, which ensure the existence of at least one copula that attains these values, and the second is to find exact bounds for the set of all such copulas.

1 Introduction

Uncertainty modeling is quickly becoming one of the main topics in applied mathematics and statistics. Modern applications often require statistical estimations to be made even if there is little or no information about the dependence structures involved. To obtain good mathematical models in such situations, it is thus important to take into account any additional information that is available.

In the case of complete dependence uncertainty, the set of all possible distribution functions of a random vector $\mathbf{X}$ is described by the set of all copulas. Having additional information about the distribution of $\mathbf{X}$ manifests as restrictions on the corresponding set of copulas. These restrictions often involve prescribed values for copulas on some subset $S \subseteq \mathbb{I}^n$, such as diagonals, sections, finite sets of points, general compact sets, etc. There are two questions that can be posed in each case. The first question

(Q1) Does there exist at least one copula that attains the prescribed values on S?

leads to finding conditions for the set S and for the prescribed values on S, that will ensure the set of corresponding copulas in nonempty. The second question

(Q2) What are the exact lower and upper bound (constrained Fréchet-Hoeffding bound) for the set of all copulas having the prescribed values on S?

involves finding formulas for the bounds and proving that the bounds are exact, i.e., at each point they are attained by some copula.

We mention a couple of known results on this topic. Existence of bivariate copulas with given diagonal section δ was established by Bertino [2], and Fredricks and Nelsen [5], and the lower and upper bounds for such copulas are given in [11]. The lower bound is known to be exact, while the exactness of the upper bound for general δ is still an open question - it is only solved for special types of δ. Diagonal sections of multivariate copulas were investigated by Jaworski [6]. Copulas with prescribed opposite-diagonal sections were considered by de Amo et al. [3]. Furthermore, a function Q defined on $\mathbb{I}^n$ is a quasi-copula if and only if for any track S in $\mathbb{I}^n$ (i.e. a parametric curve with non-decreasing components) there is a copula C that coincides with Q on S. Some results for a general compact set S can be found in [8].

The first result concerning **(Q1)** for a finite set S (apart from the classical result for $|S| = 1$, which is an integral part of the theory) was given in the bivariate setting in 2010 by Mardani-Fard et al. [9], who showed that if $|S| = 3$ and Q is an arbitrary bivariate quasi-copula, then there exists a bivariate copula C which coincides with Q on S. For $|S| = 4$ this is no longer true, since bivariate copulas have positive volume on any rectangle while proper quasi-copulas do not. Using a linear programming method De Baets et al. [4] proved that an analogous result holds for trivariate copulas when $|S| = 2$, but not when $|S| = 3$. It was an open problem whether the same is true for higher dimensional copulas, see [1]. In the talk we present a positive answer to this question using a constructive induction method.

Very little is know about **(Q2)** for a finite set. In fact, existing results seem to be limited to the case when a single value of a copula is prescribed, i.e. when $|S| = 1$. The bounds for bivariate copulas with a given value at a given point were discovered by Nelsen [10] in 1999. These bounds are given in terms of shuffles of min, so they are again copulas, and thus automatically exact. In 2004 Rodríguez-Lallena and Úbeda-Flores [12] provided the bounds for n-copulas with a given value at a given point $\mathbf{z} \in \mathbb{I}^n$. Whether these bounds are exact when $n \geq 3$ was posed as an open question, with only a partial answer given by the authors, namely in the regions $[\mathbf{0}, \mathbf{z}]$ and $[\mathbf{z}, \mathbf{1}]$. We settle this question by showing that the given bounds are indeed exact for any $n \geq 3$.

2 Results

Let $n \geq 2$ and denote by W and M the lower and upper Fréchet-Hoeffding bound for the set of n-copulas. In this section we provide an answer to **(Q1)** when $|S| = 2$ and an answer to **(Q2)** when $|S| = 1$. It turns out that the former is a consequence of the latter, so we start with the latter.

Let $\mathbf{z} = (z_1, z_2, \dots, z_n)$ be a fixed point in $\mathbb{I}^n$ and choose a real number a such that $W(\mathbf{z}) \leq a \leq M(\mathbf{z})$. It was shown in [12], that if C is a copula (or even a quasi-copula) with $C(\mathbf{z}) = a$, then

$$\max\Big\{W(\mathbf{u}), a - \sum_{i=1}^{n}(z_i - u_i)^+\Big\} \leq C(\mathbf{u}) \leq \min\Big\{M(\mathbf{u}), a + \sum_{i=1}^{n}(u_i - z_i)^+\Big\} \quad (1)$$

for all $\mathbf{u} = (u_1, u_2, \dots, u_n) \in \mathbb{I}^n$, where $r^+ = \max\{0, r\}$ denotes the positive part of a real number r. Our first theorem shows that these bounds, which are n-quasi-copulas, are exact (i.e. best possible).

Theorem 1 **([7]).** *Fix a point* $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{I}^n$ *and a real number* a *such that* $W(\mathbf{z}) \le a \le M(\mathbf{z})$. *Then for every point* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{I}^n$ *there exists an* n*-copula* C_u *satisfying the conditions*

$$C_u(\mathbf{z}) = a \qquad \textit{and} \qquad C_u(\mathbf{x}) = \min\Big\{M(\mathbf{x}), a + \sum_{i=1}^{n}(x_i - z_i)^+\Big\},$$

and an n*-copula* C_l *satisfying the conditions*

$$C_l(\mathbf{z}) = a \qquad \textit{and} \qquad C_l(\mathbf{x}) = \max\Big\{W(\mathbf{x}), a - \sum_{i=1}^{n}(z_i - x_i)^+\Big\}.$$

So n-copula C_u attains the prescribed value at $\mathbf{z}$, and the upper bound at $\mathbf{x}$, while n-copula C_l attains the prescribed value at $\mathbf{z}$, and the lower bound at $\mathbf{x}$. This implies that the given bounds are exact.

Now suppose two points $\mathbf{z}$ and $\mathbf{w}$ in $\mathbb{I}^n$ are fixed, and we prescribe the values at $\mathbf{z}$ and $\mathbf{w}$ to be two values coming from a single n-quasi-copula Q. In this case, question **(Q2)** asks whether there is a n-copula, that coincides with Q at points $\mathbf{z}$ and $\mathbf{w}$. Since the bounds given in (1) hold also for n-quasi-copulas, Theorem 1 can be used to construct such an n-copula by taking an appropriate convex combination of copulas C_u and C_l from Theorem 1, with $\mathbf{x}$ replaced by $\mathbf{w}$. We thus have the following result.

Theorem 2 **([7]).** *Let* $\mathbf{z}$ *and* $\mathbf{w}$ *be two points in* $\mathbb{I}^n$ *and let* Q *be an* n*-quasi-copula. Then there exists an* n*-copula* C *such that*

$$C(\mathbf{z}) = Q(\mathbf{z}) \quad \textit{and} \quad C(\mathbf{w}) = Q(\mathbf{w}).$$

The essence of our construction method thus lies in the proof of Theorem 1. The core idea of the proof is the notion of an $\mathbf{F}$-copula, where $\mathbf{F}$ is an n-tuple of increasing 1-Lipschitz functions defined on $\mathbb{I}$, with value 0 at 0 and a common value $T \in \mathbb{I}$ at 1. An $\mathbf{F}$-copula is a grounded n-increasing function with marginals $\mathbf{F}$, and it is meant to model a slice of a higher dimensional copula, namely, a slice of a slice of a slice ... of a copula is an $\mathbf{F}$-copula for appropriate $\mathbf{F}$.

The proof of Theorem 1, say for the upper bound, can be summarized as follows (see also Figure 1).

1. Formulate Theorem 1 for $\mathbf{F}$-copulas.
2. Reorder the coordinates if necessary.
3. Take a slice through $\mathbf{x}$ in the x_1 coordinate, namely $\{x_1\} \times \mathbb{I}^{n-1}$, and define an appropriate $(n-1)$-tuple $\mathbf{F}'$ that will serve as the marginals on the slice, and an appropriate value a' at the projection $\mathbf{z}'$ of $\mathbf{z}$ to the slice.
4. Use induction to find an $\mathbf{F}'$-copula C' that will serve as a slice of the eventual solution $\mathbf{F}$-copula.
5. Extend C' defined on the slice, together with the value a at $\mathbf{z}$, to an $\mathbf{F}$-copula C defined on $\mathbb{I}^n$, which will automatically satisfy the necessary conditions.
6. Apply the result to the n-tuple $\mathbf{F}$ of uniform marginals.

Additionally, the technical aspects of the proof require generalizations of certain classical results from the theory of copulas to the setting of $\mathbf{F}$-copulas, including the Fréchet-Hoeffding bounds and the answer to **(Q1)** for $|S| = 1$. Furthermore, a byproduct of step 5. is also an answer to **(Q1)** when S is a union of an $(n-1)$-dimensional slice of $\mathbb{I}^n$ and an arbitrary point in $\mathbb{I}^n$.

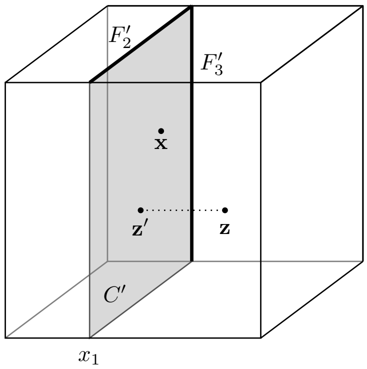

Fig. 1. A visualization of the procedure used in the proof of Theorem 1 (figure adapted from [7]).

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References

1. J. J. Arias-García, R. Mesiar, B. De Baets, A hitchhiker's guide to quasi-copulas, *Fuzzy Sets and Systems*, **393** (2020), 1–28.
2. S. Bertino, Sulla dissomiglianza tra mutabili cicliche, *Metron*, **35** (1977), 53–88.
3. E. de Amo, H. De Meyer, M. Díaz Carrillo, J. Fernández Sánchez, Characterization of copulas with given diagonal and opposite diagonal sections, *Fuzzy Sets and Systems*, **284** (2016), 63–77.
4. B. De Baets, H. De Meyer, J. Fernández Sánchez, M. Úbeda-Flores, On the existence of a trivariate copula with given values of a trivariate quasi-copula at several points, *Fuzzy Sets and Systems*, **228** (2013), 3–14.
5. G. A. Fredricks, R. B. Nelsen, Copulas constructed from diagonal sections, *Distributions with given marginals and moment problems* (Prague, 1996), Kluwer Acad. Publ., Dordrecht, 1997, 129–136.
6. P. Jaworski, On copulas and their diagonals, *Inf. Sci.*, **179** (2009), 2863–2871.
7. E. P. Klement, D. Kokol Bukovšek, M. Omladič, S. Saminger-Platz, N. Stopar, Multivariate copulas with given values at two arbitrary points, *Stat Papers* (2022). https://doi.org/10.1007/s00362-022-01362-4
8. T. Lux, A. Papapantoleon, Improved Fréchet-Hoeffding bounds on d-copulas and applications in model-free finance, *Ann. Appl. Probab.*, **27** (2017), 3633–3671.
9. H. A. Mardani-Fard, S. M. Sadooghi-Alvandi, Z. Shishebor, Bounds on bivariate distribution functions with given margins and known values at several points, *Comm. Statist. Theory Methods*, **39** (2010), 3596–3621.
10. R. B. Nelsen, *An Introduction to Copulas*, Springer, New York, 1999.
11. R. B. Nelsen, J. J. Quesada-Molina, J. A. Rodríguez-Lallena, M. Úbeda-Flores, On the construction of copulas and quasi-copulas with given diagonal sections, *Insurance Math. Econom.*, **42** (2008), 473–483.
12. J. A. Rodríguez-Lallena, M. Úbeda-Flores, Best-possible bounds on sets of multivariate distribution functions, *Comm. Statist. Theory Methods*, **33** (2004), 805–820.

On extreme generators of shock–induced copulas

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Abstract. We extend the notion of extreme points in the Krein–Milman sense of the class of semilinear copulas introduced by Durante et al. to the class of all bivariate shock–induced copulas. This class properly contains semilinear copulas for which the results of our procedure are consistent with the existing notion. We show that our extreme copulas are dense in every class to which they belong (including the class of semilinear copulas) in a stronger sense than in the Krein–Milman approach; in fact, they are dense in a similar way that shuffles of M are dense in the set of all copulas. We also provide a stochastic interpretation of our extension of extremality. Roughly speaking, a shock–induced copula is extreme whenever the inducing shocks have pairwise disjoint supports.

1 Introduction

Let U and V, respectively, denote the two lifetimes of the components of a system. The independent times of occurrence of three types of shocks are denoted by X, Y and Z, respectively. The first two are idiosyncratic shocks that are fatal to only one of the two components at a time. The third one is an exogenous shock that affects both components. Note that

$$U = \min\{X, Z\} \quad \text{and} \quad V = \min\{Y, Z\}. \tag{1}$$

In this case, the random vector (U, V) obeys the Marshall copula $C_{\varphi,\psi}$, and the two functions φ, ψ can be interpreted in terms of distribution functions of the previously introduced random variables as $F_X(x) = \varphi(F_U(x))$ for all x with $F_U(x) > 0$, and $F_Y(y) = \psi(F_V(y))$ for all y with $F_V(y) > 0$.

In various applications, random events may have a different meaning than shocks, e.g., lifetime, in which case the two minima above may become maxima. In finance, random events can have opposite effects on long and short investments. Therefore, we often cannot consider these events as either shocks or lifetimes. Nevertheless, we use the common terminology "shock–induced copulas" when studying these copulas and their properties, regardless of their actual application.

In the seminal paper by Durante, Girard, and Mazo [2], which introduces a broad family of copulas, including those induced by shocks, the operators such as “min” or “max” in the Expression (1) above are called the *linking functions*. While symmetric linking functions lead to the Marshall copula, the asymmetric choice $U = \max\{X, Z\}$ and $V = \min\{Y, Z\}$ yields the maxmin copula. Although this copula is obtained by a reflection in the linkage from the Marshall case, another reflection in a component does not bring us back to the Marshall copula – it brings us to RMM.

Let us briefly recall the classes of shock–induced copulas that form the main background of our investigation.

(A) Marshall copulas: Maps $C_{\varphi,\psi}(u, v) = \min\{v\varphi(u), u\psi(v)\}$, where

1. φ, ψ are two increasing real valued maps on $[0, 1]$;
2. $\varphi(0) = \psi(0) = 0$ and $\varphi(1) = \psi(1) = 1$;
3. $\varphi^*(u) = \dfrac{\varphi(u)}{u}$ and $\psi^*(v) = \dfrac{\psi(v)}{v}$ are decreasing,

are called *Marshall copulas* [6]. We denote this class of copulas with $\mathcal{M}$.

(B) Maxmin copulas: Maps $C_{\varphi,\psi}(u, v) = uv + \min\{u(1 - v), (\varphi(u) - u)(v - \psi(v))\}$, where

1. φ, ψ are two increasing real valued maps on $[0, 1]$;
2. $\varphi(0) = \psi(0) = 0$ and $\varphi(1) = \psi(1) = 1$;
3. $\varphi^*(u) = \dfrac{\varphi(u)}{u}$ and $\psi_*(v) = \dfrac{1 - \psi(v)}{v - \psi(v)}$ are decreasing,

are called *maxmin copulas* [7]. We denote this class of copulas with $\mathcal{M}^m$.

(C) Reflected maxmin copulas: Maps $C_{f,g}(u, v) = \max\{0, uv - f(u)g(v)\}$, where

1. f, g are two real valued maps on $[0, 1]$;
2. the functions $f^*(u) = \dfrac{f(u)}{u}$ and $g^*(v) = \dfrac{g(v)}{v}$ are decreasing;
3. the functions $\widehat{f}(u) = f(u) + u$ and $\widehat{g}(v) = g(v) + v$ are increasing;
4. $f(0) = g(0) = 0$, $f(1) = g(1) = 0$, $f^*(1) = g^*(1) = 0$,

are called *reflected maxmin copulas* (*RMM* for short) [5]. We denote this class of copulas with $\mathcal{M}^{m\sigma}$. Every such copula can be obtained from a maxmin copula after applying the reflection on the second variable, i.e., $v \mapsto 1 - v$. The functions φ and ψ transform into f and g with the following rules: $f(u) = \varphi(u) - u$ and $g(u) = 1 - u - \psi(1 - u)$.

(D) Semilinear copulas: The family of *semilinear copulas* [3] may be considered as a special case of Marshall copulas. Although they deserve to be defined on their own, they are precisely symmetric Marshall copulas, i.e., we have $\psi = \varphi$.

2 Results

In this section we present the results of [4]. A set of functions $\mathcal{F} = \{f : \mathbb{I} \longrightarrow \mathbb{I} : f(0) = f(1) = 0,\ u + f(u) \nearrow,\ f(u)/u \searrow\}$, where $\mathbb{I} = [0, 1]$, will be used as a set of *generators* of certain families of copulas. Observe that this set is equivalently defined as $\widehat{\mathcal{F}} = \{\varphi : \varphi(0) = 0, \varphi(1) = 1, \varphi(u) \nearrow, \varphi(u)/u \searrow\}$. Indeed, the bijective correspondence $\mathcal{F} \to \widehat{\mathcal{F}}$ is given by $f \mapsto \varphi(u) = f(u) + u$. Functions of these sets are continuous on $(0, 1]$ and may have a jump at 0. If we redefine their values at 0 as their right-hand side limits, these sets are compact and convex.

We also introduce two sets of points

$$\begin{aligned} A &= \left\{u \in \mathbb{I} \,:\, f'(u) = \frac{f(u)}{u}\right\} \text{ and } B = \{u \in \mathbb{I} \,:\, f'(u) = -1\}, \text{ for } f \in \mathcal{F}; \\ A &= \left\{u \in \mathbb{I} : \varphi'(u) = \frac{\varphi(u)}{u}\right\} \text{ and } B = \{u \in \mathbb{I} : \varphi'(u) = 0\}, \text{ for } \varphi \in \widehat{\mathcal{F}}. \end{aligned} \tag{2}$$

With μ we denote the Lebesgue measure on $\mathbb{I}$. We now present a characterization of the extreme points of $\mathcal{F}$ (respectively $\widehat{\mathcal{F}}$).

Theorem 1. *A function f (respectively φ) is an extreme point of $\mathcal{F}$ (respectively $\widehat{\mathcal{F}}$) if and only if $\mu(A \cup B) = 1$.*

For X, Y, Z continuously distributed random variables we obtain the following characterization of extreme generators.

Theorem 2. *If X (respectively Y) and Z are continuously distributed, then f (respectively g) is an extreme point of $\mathcal{F}$ if and only if the supports of X (respectively Y) and Z are disjoint.*

A rough definition of *the support of* X, denoted $\operatorname{supp} X \subseteq \mathbb{R}$, is that $x \in \operatorname{supp} X$ if either *(1)* the probability of the set $\{X = x\}$ is nonzero (the set of these points is called *the discrete support*, denoted $\operatorname{supp}_d X$), or *(2)* F_X is differentiable with nonzero derivative at x and belongs to an interval in which the points with this property are dense (the set of these points is called *the absolutely continuous support*, denoted $\operatorname{supp}_a X$). The support also contains accumulation points of the points of types *(1)* and *(2)*. However, there are shocks whose supports cannot be obtained only in this way.

1. Using the Cantor function, we can generate an example with shocks whose supports are neither of type *(1)* nor *(2)*, nor their combination.
2. The Cantor function can provide generators that are not extreme.
3. Two shocks that are both singular can provide an example of a generating function that is extreme.

Introduce x_0 and z_0 as the infimum of the support of X and Z, respectively, and let $w_0 = \max\{x_0, z_0\}$. Similarly, introduce y_1 and z_1 respectively as the supremum of the support of Y and Z respectively, and let $w_1 = \min\{y_1, z_1\}$. If we assume that the random variables X, Y, Z are combinations of discrete and continuous type, we obtain the following theorem.

Theorem 3. *Assume that each of the three shocks is distributed discretely, continuously, or as a combination of the two. Then*

1. *generator φ is extreme if and only if*

$$(\operatorname{supp}_d X \cap \operatorname{supp}_d Z) \setminus \{w_0\} = \emptyset \quad \textit{and} \quad \mu\left(\operatorname{supp}_a X \cap \operatorname{supp}_a Z\right) = 0;$$

2. *generator ψ is extreme if and only if*

$$(\mathrm{supp}_d Y \cap \mathrm{supp}_d Z) \setminus \{w_1\} = \emptyset \quad \textit{and} \quad \mu\,(\mathrm{supp}_a Y \cap \mathrm{supp}_a Z) = 0.$$

The following three theorems show the density of extreme generators among all generators and the density of extreme copulas, i.e. copulas generated by extreme generators, among shock–induced copulas of their respective types. It follows that extreme copulas are dense in the class of semilinear copulas.

Theorem 4. *Extreme generators are dense in the set of generators $\mathcal{F}$.*

Theorem 5. *In either of the three families of copulas, Marshall, maxmin, and RMM, the extreme copulas given by the formulas above form a dense subset.*

Remark 1. In the class of semilinear copulas, where the extreme copulas of Durante et al. [1] are defined, their definition coincides with our notion given above.

Corollary 1. *In the class of semilinear copulas extreme copulas form a dense subset in sup norm topology.*

The next two theorems give the stochastic interpretation of continuously distributed shocks and combinations of discrete in continuously distributed shocks, respectively.

Theorem 6. *Let X, Y and Z be continuously distributed independent shocks generating a copula that belongs to either of the three families $\mathcal{M}$, $\mathcal{M}^m$, or $\mathcal{M}^{m\sigma}$. Then the so obtained copula is an extreme one if and only if $\mu([\mathrm{supp}\, X \cup \mathrm{supp}\, Y] \cap \mathrm{supp}\, Z) = 0$.*

Theorem 7. *Let X, Y, and Z be independent shocks. Assume that each of the three shocks is distributed discretely, continuously, or as a combination of the two, and suppose that they generate a copula C that belongs to either of the three families $\mathcal{M}$, $\mathcal{M}^m$, or $\mathcal{M}^{m\sigma}$. Then*

1. *In the case of $\mathcal{M}$ copula C is extreme if and only if*

$$(\mathrm{supp}_d X \cap \mathrm{supp}_d Z) \setminus \{\inf \mathrm{supp}\, U\} = \emptyset, \quad (\mathrm{supp}_d Y \cap \mathrm{supp}_d Z) \setminus \{\inf \mathrm{supp}\, V\} = \emptyset,$$

$$\mu\,(\mathrm{supp}_a X \cap \mathrm{supp}_a Z) = 0, \quad \mu\,(\mathrm{supp}_a Y \cap \mathrm{supp}_a Z) = 0.$$

2. *In the case of $\mathcal{M}^m$ or $\mathcal{M}^{m\sigma}$ copula C is extreme if and only if*

$$(\mathrm{supp}_d X \cap \mathrm{supp}_d Z) \setminus \{\inf \mathrm{supp}\, U\} = \emptyset, (\mathrm{supp}_d Y \cap \mathrm{supp}_d Z) \setminus \{\sup \mathrm{supp}\, V\} = \emptyset,$$

$$\mu\,(\mathrm{supp}_a X \cap \mathrm{supp}_a Z) = 0, \mu\,(\mathrm{supp}_a Y \cap \mathrm{supp}_a Z) = 0.$$

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References

1. F. Durante, J. Fernández Sánchez, M. Úbeda Flores, Extreme semilinear copulas, *Fuzzy Sets and Systems*, doi: 10.1016/j.fss.2020.12.009.
2. F. Durante, S. Girard, G. Mazo, Copulas Based on Marshall-Olkin Machinery, Chapter 2 in: U. Cherubini, F. Durante, S. Mulinacci (Eds.), Marshall-Olkin Distributions – Advances in Theory and Practice, in: *Springer Proceedings in Mathematics & Statistics*, Springer, 2015, pp. 15–31.
3. F. Durante, A. Kolesarovà, R. Mesiar, C. Sempi, Semilinear copulas, *Fuzzy Sets and Systems*, **159** (2008), 63–76.
4. D. Kokol Bukovšek, T. Košir, B. Mojškerc, M. Omladič, Matjaž, Extreme generators of shock induced copulas, *Appl. Math. Comput.*, **429** (2022), Paper No. 127214, 16 pp.
5. T. Košir, M. Omladič, Reflected maxmin copulas and modeling quadrant subindependence, *Fuzzy Sets and Systems*, **378** (2020), 125–143.
6. A. W. Marshall, Copulas, marginals, and joint distributions, in: L. Rüschendorf, B. Schweitzer, M. D. Taylor (eds.), Distributions with Fixed Marginals and Related Topics in LMS, *Lecture Notes – Monograph Series*, vol. **28**, 1996, 213–222.
7. M. Omladič, N. Ružić, Shock models with recovery option via the maxmin copulas, *Fuzzy Sets and Systems*, **284** (2016), 113–128.

On the exact regions determined by pairs of various concordance measures

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Abstract. We consider the bounds for possible values of one concordance measure of a bivariate copula given the value of some other concordance measure. Most of the obtained bounds are also attained.

1 Introduction

A concordance measure, or its generalization, a weak concordance measure, is often a better way to model dependence than Pearson's correlation coefficient since it is invariant with respect to monotone increasing transformations of the random variables. Because of this invariance, the concordance of a random vector (X, Y) is uniquely determined by its copula.

The most commonly used concordance measures are Spearman's rho, Kendall's tau, Gini's gamma and Blomqvist's beta. Additionally, Spearman's footrule is probably the most used weak concordance measure. They are defined as follows:

Spearman's rho $$\rho(C) = 12 \int_0^1 \int_0^1 C(x,y)dxdy - 3,$$

Kendall's tau $$\tau(C) = 4 \int_0^1 \int_0^1 C(x,y)dC(x,y) - 1,$$

Gini's gamma $$\gamma(C) = 4 \int_0^1 C(x,x)dx + 4 \int_0^1 C(x,1-x)dx - 2,$$

Spearman's footrule $$\phi(C) = 6 \int_0^1 C(x,x)dx - 2,$$

Blomqvist's beta $$\beta(C) = 4\, C\left(\tfrac{1}{2}, \tfrac{1}{2}\right) - 1.$$

The range of every concordance measure is the interval $[-1, 1]$, while the range of Spearman's footrule is $[-\frac{1}{2}, 1]$.

Given their widespread use in a variety of practical applications, it is natural to compare different concordance measures in terms of the values that they can attain. In particular, if a value of one measure is known, we may ask what are the possible values of the other measures.

The investigation of the above question was started in 1950 by Daniels [1], who compared Spearman's rho and Kendall's tau and gave some estimates for the values of the two measures. The exact region of all possible pairs of values $(\tau(C), \rho(C))$, $C \in \mathcal{C}$, was determined in 2017 by Schreyer, Paulin, and Trutschnig [7]. The regions determined by Blomqvist's beta and the other three concordance measures (Spearman's rho, Kendall's tau, and Gini's gamma) are given in [6] as an exercise for the reader, while the region determined by Blomqvist's beta and Spearman's footrule is given in [2].

In the talk we will describe the exact regions determined by Spearman's footrule, Gini's gamma and Kendall's tau. We will also tightly estimate the exact region determined by Spearman's footrule and Spearman's rho. The exact region determined by Spearman's rho and Gini's gamma remains open.

2 Results

The exact regions determined by Spearman's footrule, Gini's gamma and Kendall's tau are polygons, more precisely triangular or quadrilateral regions.

Theorem 1 (KB, Mojškerc, [3]). *The exact region determined by Spearman's footrule and Gini's gamma of all points* $\{(\phi(C), \gamma(C)) \in [-\frac{1}{2}, 1] \times [-1, 1];\ C \in \mathcal{C}\}$ *is given by*

$$\frac{4}{3}\phi(C) - \frac{1}{3} \le \gamma(C) \le \min\left\{\frac{4}{3}\phi(C) + \frac{1}{6}, \frac{2}{3}\phi(C) + \frac{1}{3}\right\}.$$

Theorem 2 (KB, Stopar, [5]). *The exact region determined by Spearman's footrule and Kendall's tau of all points* $\{(\phi(C), \tau(C)) \in [-\frac{1}{2}, 1] \times [-1, 1];\ C \in \mathcal{C}\}$ *is a triangular region given by*

$$\frac{4}{3}\phi(C) - \frac{1}{3} \le \tau(C) \le \frac{2}{3}\phi(C) + \frac{1}{3}.$$

Theorem 3 (KB, Stopar, [5]). *The exact region determined by Gini's gamma and Kendall's tau of all points* $\{(\gamma(C), \tau(C)) \in [-1, 1] \times [-1, 1];\ C \in \mathcal{C}\}$ *is given by*

$$\max\left\{\frac{2}{3}\gamma(C) - \frac{1}{3}, 2\gamma(C) - 1\right\} \le \tau(C) \le \min\left\{\frac{2}{3}\gamma(C) + \frac{1}{3}, 2\gamma(C) + 1\right\}.$$

All the boundary points of these regions are attained by shuffles of min.

The exact region determined by Spearman's rho and Spearman's footrule is more complicated.

Theorem 4 (KB, Stopar, [4]). *The exact region determined by Spearman's footrule and Spearman's rho of all points* $\{(\phi(C), \rho(C)) \in [-\frac{1}{2}, 1] \times [-1, 1]; C \in \mathcal{C}\}$ *is given by*

$$\frac{2}{9}\sqrt{3}(1 + 2\phi(C))^{3/2} - 1 \le \rho(C) \le s(\phi(C))$$

where $s : [-\frac{1}{2}, 1] \to [-1, 1]$ *is a concave function satisfying*

$$r(x) \le s(x) \le 1 - \frac{2}{3}(1 - x)^2$$

and r *is the function defined by*

$$r(x) = \begin{cases} 2x + \frac{1}{2} - \frac{\sqrt{3}}{9}(1 + 2x)^{3/2}; & x \in [-\frac{1}{2}, -\frac{1}{8}], \\ \frac{4}{3}x + \frac{7}{24}; & x \in [-\frac{1}{8}, \frac{1}{4}], \\ \dfrac{2n+1}{n^2+n}x + \dfrac{2n^2 - 2n + 1}{2(n^2+n)}; & x \in [1 - \frac{3}{2n}, 1 - \frac{3}{2(n+1)}] \text{ for } n = 2, 3, \dots, \\ 1; & x = 1. \end{cases}$$

The exact region determined by Spearman's footrule and Spearman's rho is similar in shape to the exact region determined by Sperman's rho and Kendall's tau, i.e., the upper bound seems to be a piecewise function with finer and finer pieces. However, the exact region determined by Sperman's rho and Kendall's tau is not convex while in our case the region is convex.

The obtained regions are shown in Figure 1.

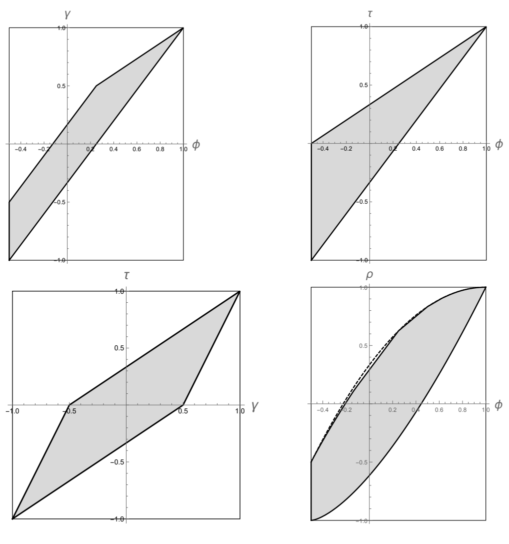

Fig. 1. The exact regions determined by pairs of various (weak) concordance measures, [3–5].

3 Concordance similarity measure

We introduce the (κ_1, κ_2)-*similarity measure* between (weak) concordance measures κ_1 and κ_2 as

$$\kappa sm(\kappa_1, \kappa_2) = 1 - \frac{A(\kappa_1, \kappa_2)}{(1 - \kappa_1(W))(1 - \kappa_2(W))},$$

where $A(\kappa_1, \kappa_2)$ is the area of the exact region determined by κ_1 and κ_2 and W is the lower Fréchet-Hoeffding bound, [3]. It describes how closely related two (weak) concordance measures are.

In Table 1 we give (κ_1, κ_2)-similarity measures between all pairs of (weak) concordance measures for which the exact region determined by them is known.

κsm	ρ	τ	γ	ϕ	β
ρ	1	0.7114		0.65??	0.3750
τ	0.7114	1	0.7500	0.7500	0.3333
γ		0.7500	1	0.8125	0.5000
ϕ	0.65??	0.7500	0.8125	1	0.5000
β	0.3750	0.3333	0.5000	0.5000	1

Table 1. (κ_1, κ_2)-similarity measure between pairs of (weak) concordance measures

It turns out that Spearman's footrule and Gini's gamma are most closely related. On the other hand, the value of Blomqvist's beta gives us little information about the values of other (weak) concordance measures.

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References

1. H. E. Daniels, Rank correlation and population models, *J. Roy. Statist. Soc. Ser. B* **12** (1950), 171–181.
2. D. Kokol Bukovšek, T. Košir, B. Mojškerc, and M. Omladič, Spearman's footrule and Gini's gamma: local bounds for bivariate copulas and the exact region with respect to Blomqvist's beta, *J. Comput. Appl. Math.* **390** (2021), Paper No. 113385.
3. D. Kokol Bukovšek and B. Mojškerc, On the exact region determined by Spearman's footrule and Gini's gamma, *J. Comput. Appl. Math.* **410** (2022), Paper No. 114212.
4. D. Kokol Bukovšek and N. Stopar, On the exact region determined by Spearman's rho and Spearman's footrule, arXiv:2212.02443 [math.ST].
5. D. Kokol Bukovšek and N. Stopar, On the exact regions determined by Kendall's tau and other concordance measures, *Mediterr. J. Math.* **20**, 147 (2023).
6. R. B. Nelsen, *An introduction to copulas*, second ed., Springer Series in Statistics, Springer, New York, 2006.
7. M. Schreyer, R. Paulin, and W. Trutschnig, On the exact region determined by Kendall's τ and Spearman's ρ, *J. R. Stat. Soc. Ser. B. Stat. Methodol.* **79** (2017), no. 2, 613–633.

Approximation of copulas using Cramér-von Mises statistics in the case of missing data

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1 General aspects

When considering multivariate data from practice, it often happens that data values are missing. The causes of missing values can be different: defective machines, failed measurements, plants dying, etc. Imputation and various maximum likelihood techniques are well-established strategies for dealing with missing data. The classical theory can be found in several textbooks, see [3] for example.

We introduce a new approach and consider parametric estimation of the copula of dimension d, taking into account the concrete data structure at hand. No missing data values are replaced. We just use the available data. Moreover, we do not assume special probabilistic mechanisms for missing data. The data items with the same pattern of complete and missing data are combined into a subset. For each of these patterns, the corresponding marginal copula of non-missing components is introduced. Using these marginal copulas and the copula itself, an adapted version of the Cramér-von Mises divergence is constructed. The minimization of the estimated Cramér-von Mises divergence results in the minimum distance estimator for the parameters. As an example, we fit product copulas (see [1]) to ecological data.

2 Data structure

Let $X = (X^{(1)}, \ldots, X^{(d)})^T$ be a d-dimensional random vector representing the data without missing values. In the case of a complete observation vector, we denote the joint distribution function by H and the (continuous) marginal distribution functions of $X^{(j)}$ by $F_1, \ldots, F_d$. According to Sklar's theorem, we have

$$H(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)) \quad \text{for } x_i \in \mathbb{R},$$

where $C : [0,1]^d \to [0,1]$ is the d-dimensional copula. The detailed theory of copulas can be found in the popular monograph by [4].

Next we describe the structure of the data, including missing values. The sample breaks down into m subsets of data items with the same pattern of missing data. The number m does not depend on n. Every pattern is modeled as a binary vector $\mathbf{b} = (b_1, \ldots, b_d)^T \in \{0,1\}^d$ which has at least two components equal to 1:

$$b_j = \begin{cases} 1 \text{ if the } j\text{-th component is observed,} \\ 0 \text{ if the } j\text{-th component is not observed.} \end{cases}$$

Here $\mathbf{b}^{(1)}, \ldots, \mathbf{b}^{(m)} \in \{0,1\}^d$ are the pattern vectors of the data subsets. We give an example: the pattern $\mathbf{b}^{(k)} = (0,1,0,1)^T$ of data subset k means that the data items of this subset have a non-missing second component and a non-missing fourth one, whereas components 1 and 3 are missing. We assume that for all data subsets the distribution function of the data items coincides with the corresponding multivariate marginal distribution functions resulting from H. The complete data items have distribution function H. The data structure and the subset distribution functions are given in Table 1.

Table 1: Structure of the data including subset distribution functions ($\tilde{x}_\mu = (x_j, j \in J_\mu)$)

subset	data	pattern	distribution fcn.
1	$Y_{11}, \ldots, Y_{1n_1} \in \mathbb{R}^d$ complete data	$\mathbf{b}^{(1)} = \mathbf{1}$, $J_1 = \{1, \ldots, d\}$	H
2	$Y_{21}, \ldots, Y_{2n_2} \in \mathbb{R}^{d_2}$	$\mathbf{b}^{(2)}$, $J_2 = \{l : b_l^{(2)} = 1\}$	$\tilde{x}_2 \mapsto C_2(F_j(x_j), j \in J_2)$
…			
m	$Y_{m1}, \ldots, Y_{mn_m} \in \mathbb{R}^{d_m}$	$\mathbf{b}^{(m)}$, $J_m = \{l : b_l^{(m)} = 1\}$	$\tilde{x}_m \mapsto C_m(F_j(x_j), j \in J_m)$

The copulas of the subsets are determined by

$$C_\mu(u_j, j \in J_\mu) = C(\mathbf{u} \odot \mathbf{b}^{(\mu)} + \mathbf{1} - \mathbf{b}^{(\mu)}) \quad (\mu = 1 \ldots m),$$

where $\mathbf{u} = (u_1, \ldots, u_d)^T$, $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^d$, $\mathbf{a} \odot \mathbf{b} = \operatorname{diag}(\mathbf{a})\, \mathbf{b}$ is the Hadamard product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$.

3 Divergence and Estimation

Let $\mathcal{F} = \{\mathcal{C}(. \mid \theta)\}_{\theta \in \Theta}$ be a parametric family of copulas. $\Theta \subset \mathbb{R}^q$ is the parameter space. Due to the high complexity of the multivariate data, it cannot be expected that a copula model class can be found to which the copula of the data belongs. Therefore, model fitting aims at the best possible approximation of the underlying copula C by the family $\mathcal{F}$. For this purpose, we consider the *Cramér-von-Mises divergence* as a measure for the discrepancy between the copula C and $\mathcal{F}$. Define the model copula for subset μ:

$$\mathcal{C}_\mu(u_j, j \in J_\mu \mid \theta) = \mathcal{C}(\mathbf{u} \odot \mathbf{b}^{(\mu)} + \mathbf{1} - \mathbf{b}^{(\mu)} \mid \theta)$$

for $u \in [0,1]^d, \theta \in \Theta, \mu = 1 \dots m$. We introduce the population version of the divergence as

$$\mathcal{D}(C, \mathcal{C}(. \mid \theta)) = \sum_{\mu=1}^{m} \int_{[0,1]^{d_\mu}} \left(C_\mu(\mathbf{u}) - \mathcal{C}_\mu(\mathbf{u} \mid \theta)\right)^2 w_\mu(\mathbf{u}) \, \mathrm{d}C_\mu(\mathbf{u}),$$

where $w_\mu : [0,1]^{d_\mu} \to [0, +\infty)$ ($\mu \in \{1 \dots m\}$) is the Lipschitz-continuous weight function for the data subset μ. This quantity $\mathcal{D}(C, \mathcal{C}(. \mid \theta))$ describes the deviation of the sample copula from the model copula.

Let $\hat{F}_{nj}$ be the empirical marginal distribution function of the j-th component using all available non-missing data. Define $\check{F}^*_{n\mu}(y_j, j \in J_\mu) = (\hat{F}_{nj}(y_j))_{j \in J_\mu}$. Further $\hat{H}_{n\nu}$ denotes the empirical distribution function of the subset ν:

$$\hat{H}_{n\nu}(\mathbf{y}) = \frac{1}{\bar{n}_\nu} \sum_{\mu : 1 \le \mu \le m,\ b_j^{(\mu)} \ge b_j^{(\nu)} \forall j} \ \sum_{i=1}^{n_\mu} \mathbf{1}\left\{\psi_{\nu\mu}(Y_{\mu i}) \le \mathbf{y}\right\}$$

for $\mathbf{y} \in \mathbb{R}^d$, where $\bar{n}_\nu$ is the number of data items for subsets μ with $b_j^{(\mu)} \ge b_j^{(\nu)} \forall j$. Here function $\psi_{\nu\mu}$ selects the components of subset μ which are also present in subset ν: $\psi_{\nu\mu}(y_j, j \in J_\mu) = (y_j, j \in J_\nu)$. Next we consider the following estimator for $\mathcal{D}(C, \mathcal{C}(. \mid \theta))$.

$$\widehat{\mathcal{D}}_n(\theta) = \sum_{\mu=1}^{m} \frac{1}{n_\mu} \sum_{i=1}^{n_\mu} \left(\hat{H}_{n\mu}(Y_{\mu i}) - \mathcal{C}_\mu(\check{F}^*_{n\mu}(Y_{\mu i}) \mid \theta)\right)^2 w_\mu(\check{F}^*_{n\mu}(Y_{\mu i})).$$

In our approach, we deal with approximate minimum distance estimators $\hat{\theta}$ defined as approximate minimizers of $\widehat{\mathcal{D}}_n(.)$. Compared to the maximum likelihood method, the Cramér-von Mises statistic has the advantage that no copula densities have to be calculated and the partial statistics of the patterns are comparable.

4 Results

Let $\lim_{n\to\infty} \frac{n_\mu(n)}{n} = \gamma_\mu$ be fulfilled for positive constants $\gamma_\mu, \mu = 1 \dots m$. Suppose that the Hessian of $\theta \mapsto \mathcal{D}(C, \mathcal{C}(. \mid \theta))$ is positive definite at $\theta = \theta_0$. Under these assumptions and some additional assumptions on partial derivatives of $(\mathbf{u}, \theta) \mapsto \mathcal{C}(\mathbf{u} \mid \theta)$ and w_μ, we have proved consistency and the asymptotic normality of the estimator $\hat{\theta}$. These results allow us to write down formulas for confidence intervals. The theorems generalise former results in the case of complete data ($m = 1$), see Liebscher (2009). Moreover, a result on asymptotic normality of $\widehat{\mathcal{D}}_n(\hat{\theta})$ has been derived.

References

1. Liebscher, E. Construction of asymmetric multivariate copulas. *J. Multivariate Anal.* **99** (2008), 2234-2250.
2. Liebscher, E. Semiparametric Estimation of the Parameters of Multivariate Copulas. *Kybernetika* **6** (2009), 972–991.
3. Little, R.J.A. Rubin, D.B. “Statistical Analysis with Missing Data”. third Edition. J. Wiley&Sons 2019.
4. Nelsen, R.B. *An Introduction to Copulas*. Lecture Notes in Statistics 139. second edition, Springer 2006.

Uninorms, n-uninorms and pseudo-uninorms with continuous underlying functions

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The introduction of statistical metric spaces (probabilistic metric spaces as they are called today) by Menger in [3] and exploration of related concepts has initiated a deep study of triangular norms (see Schweizer and Sklar [4]) and related operations on the unit interval. In particular, commutative and associative binary functions on the unit interval were considered and applied in many theoretical and applied fields, for example in probability, statistic, many-valued logic, decision theory, artificial intelligence, neural networks, image processing, data fusion, however, also in economics, social sciences, and many others.

Generalization of the position of the neutral element or the annihilator of a t-norm yielded the definition of uninorms and nullnorms [1, 5]. Since these operations behave differently below and above the neutral element (annihilator) it was soon observed that they can be used in bipolar aggregation, or bipolar many-valued logic. In fact, uninorms and nullnorms can be taken as bipolar t-norms and t-conorms. Observe that similarly as in the case of t-norms and t-conorms we can construct uninorms using continuous additive generators which yields the class of representable uninorms. From an algebraic point of view, proper uninorms are the only binary operations $*$ on the unit interval which make the structures $([0, 1], \max, *)$ and $([0, 1], \min, *)$ distributive commutative semi-rings [2].

The theoretical study of structural properties of uninorms has attracted many researchers. Nevertheless, similarly as in the case of t-norms, the class of all uninorms is very complex and therefore their characterization results always rely on some additional properties. In our case this additional property will be the continuity of the underlying functions.

First, we will focus on the characterization of uninorms with continuous underlying functions. We will discuss the ordinal sum construction for uninorms, their characterizing set-valued functions and decomposition of uninorms with continuous underlying functions into representable and trivial semigroups, and the cardinality of the related ordinal sum. Then we will focus on n-uninorms which generalize uninorms and nullnorms and are related to reference levels of interest for scores. We will again discuss characterizing set-valued functions of n-uninorms and decomposition of n-uninorms with continuous underlying functions into z-ordinal sum of trivial and representable semigroups. Finally, we will focus on further generalizations: pseudo-uninorms and pseudo-n-uninorms with continuous underlying functions, where the axiom of commutativity is relaxed. We will identify all pairs of points of non-commutativity and give similar characterizations as in the case of their commutative counterparts.

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References

1. T. Calvo, B. De Baets, J. Fodor, The functional equations of Frank and Alsina for uninorms and nullnorms, *Fuzzy Sets Syst.* **120(3)** (2001), 385–394.
2. J. Golan, "The theory of semirings with applications in mathematics and theoretical computer science", Pitman Monographs and Surveys in Pure and Applied Mathematics 54, Longman Scientific and Technical, 1992.
3. K. Menger, Statistical metrics, *Proc. of the National Academy of Sciences of U.S.A.* **8** (1942), 535–537.
4. B. Schweizer, A. Sklar, Espaces métriques aléatoires, *Comptes Rendus Hebdomadaires des Scéances de l'Académie des Sciences, Paris* **247** (1958), 2092–2094.
5. R. R. Yager, A. Rybalov, Uninorm aggregation operators, *Fuzzy Sets Syst.* **80** (1996), 111—120.

Pseudo lack-of-memory properties and related copula functions

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Abstract. Survival analysis, also known as time-to-event analysis, is a branch of statistics which studies the expected duration of time until one event happens: however, it is well known that the standard lack-of-memory property is generally not satisfied by the distributions used in this kind of analysis. For this reason, in this talk, we propose an extension of the standard strong and weak lack-of-memory properties: after investigating the main statistical and probabilistical aspects of these distributions, we present applications to insurance and risk management.

1 Pseudo Lack-of-Memory Properties

Let us consider, in place of the standard product, a strict Archimedean t-norm given by

$$x \otimes_h y = h(h^{-1}(x)\, h^{-1}(y)), \quad x, y \in [0,1]$$

where h is a strictly increasing bijection of $[0,1]$, see, among the others, B. Schweizer and A. Sklar (1961). In this setup, we introduce the function

$$\exp_h(x) = h(e^x)$$

which represents a generalization of the exponential function since it solves the functional equation

$$f(x+y) = f(x) \otimes_h f(y).$$

Now, we say that the distribution of a non-negative random vector (X_1, X_2) with joint survival function $\bar{F}$ satisfies the pseudo lack-of-memory in its strong form if

$$\bar{F}(t_1 + s_1, t_2 + s_2) = \bar{F}(t_1, t_2) \otimes_h \bar{F}(s_1, s_2) \quad (1)$$

and in its weak version if

$$\bar{F}(s_1 + t, s_2 + t) = \bar{F}(s_1, s_2) \otimes_h \bar{F}(t, t), \quad (2)$$

with $t_1, s_1, t_2, s_2, t \geq 0$. The standard lack-of-memory property, analyzed in Marshall and Olkin (1967), is recovered when $h = id$.

We prove that the solution of (1) is given by

$$\bar{F}(x, y) = \exp_h(-\lambda_1 x - \lambda_2 y),\ \lambda_1, \lambda_2 > 0, \tag{3}$$

which is a bivariate survival function when h is convex and, in this case, the resulting bivariate distribution is absolutely continuous with Archimedean dependence.
As for (2), we find quite general sufficient conditions on h, λ and on the marginal distributions $\bar{F}_1$,$\bar{F}_2$ of the components X_1 and X_2 such that its solution, given by,

$$\bar{F}_{X_1,X_2}(x, y) = \begin{cases} \exp_h(-\lambda y) \otimes_h \bar{F}_1(x - y), x \geq y \\ \exp_h(-\lambda x) \otimes_h \bar{F}_2(y - x), x < y \end{cases}, \tag{4}$$

is a bivariate survival function with $\mathbb{P}(X_1 = X_2) > 0$.

The copula and the dependence structure are studied both in the most general form and for specific choices of the marginals and of h. Moreover, a comparison between survival probabilities obtained with the standard lack-of-memory property and the pseudo one is made for different super-additive and sub-additive generators both in the strong as well as in the weak cases.

2 Distributions of Residual Lifetimes

We also show that the survival distribution of the excess-of-loss random vector $(X_1 - t_1, X_2 - t_2)$ in the strong lack of memory property case is again of type (3) with $h_{t_1,t_2}(x) = \frac{h\left(xe^{-\lambda_1 t_1 - \lambda_2 t_2}\right)}{h\left(e^{-\lambda_1 t_1 - \lambda_2 t_2}\right)}$ in place of h. Similarly, the survival distribution of the excess-of-loss random vector $(X_1 - t, X_2 - t)$ in the weak case is of type (4) with $h_t(x) = \frac{h\left(xe^{-\lambda t}\right)}{h\left(e^{-\lambda t}\right)}$ in place of h. Moreover, we analyze the evolution of their dependence structure as a function of the elapsed time.

References

1. B. Schweizer, A. Sklar. "Associative functions and statistical triangle inequalities". In: Publicationes Mathematicae, Debrecen **8**, pp. 169-186 (1961).
2. A. W. Marshall, I. Olkin. "A multivariate exponential distribution". In: Journal of American Statistical Association **62**, pp. 30–44 (1967).
3. Muliere, P., Scarsini. M. "Characterization of a Marshall-Olkin type class of distributions". In: Ann Inst Stat Math **39**, 429–441 (1987).
4. F. Durante, R. Foschi, P. Sarkoci. " Distorted Copula: Construction and Tail Dependence ". In: Communication in Statistics-Theory and Methods " **39**", pp 2288-2301 (2010).
5. A. W. Marshall, I. Olkin: "A Bivariate Gompertz-Makeham Life Distribution". In: Journal of Multivariate Analysis, **139**, 219-226 (2015).

Decisions under uncertainty: sufficient conditions for multivariate almost stochastic dominance

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Decision making under risk involves a ranking of distributions, which is typically based on a method for assigning a real number to a distribution using a risk measure, a premium principle or a context of expected utility. As it is typically difficult to assess a concrete risk measure or utility function it is a well established idea to use stochastic dominance rules in form of stochastic orders to compare distributions. However, it is often equally difficult to completely specify a distribution, in particular in a multivariate setting. Therefore it is an interesting question whether one can derive unambiguous decisions under partial knowledge of the distributions.

In this talk we address this question under the condition that we only know the mean and variance of the involved distributions or that we know the marginal distributions but not the copulas in a multivariate context. Under such assumptions we derive sufficient conditions for concepts of almost stochastic dominance that are based on restrictions on marginal utilities.

The talk is based on joint work with Marco Scarsini, Ilia Tsetlin, and Robert L. Winkler.

Non-metric unfolding via copula

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Abstract. In this work an effective procedure to avoid degeneracies in multidimensional unfolding for preference rank data is proposed. We adopt the strategy of augmenting the data matrix, trying to build a complete dissimilarity matrix, by using copula-based association measures among rankings (individuals), and between rankings and objects (namely, a rank-order representation of the objects through tied rankings). Our proposal is able to both recover the order of the preferences and reproduce the position of both rankings and objects in a geometrical space. Application on real datasets show that our procedure returns non-degenerate unfolding solutions.

1 Copula

Copula are functions that join multivariate distribution functions to their marginal distribution functions [8]. They describe the dependence structure existing across pairwise marginal random variables. In this way we can consider bivariate distributions with dependency structures different from the linear one that characterizes the multivariate normal distribution.

Each copula is related to the most important measures of dependency: the Pearson correlation coefficient and the Spearman ρ correlation coefficient. The Spearman's ρ coefficient (see [8] pp. 169-170 for the definition of the ρ correlation coefficient for continuous random variables) measures the association between two variables and can be expressed as a function of the copula. More precisely, if two random variables are continuous and have copula C with parameter θ, then the Spearman ρ correlation is

$$\rho_s(C) = 12 \int_{I^2} C_\theta(u_1, u_2) du_1 du_2 - 3. \tag{1}$$

For continuous random variables it is invariant with respect to the two marginal distributions, i.e. it can be expressed as a function of its copula. This property is also known as 'scale invariance'. Note that not all measures of association satisfy this property, e.g. Pearson's linear correlation coefficient [5].

2 Unfolding as a special case of multidimensional scaling on copula-based association between rankings

Unfolding, originally formulated by Coombs [3] for the analysis of the two-mode preference choice data, is a technique that allows the estimation of two configurations usually representing the coordinates for a set of m individuals and a set of n objects on the basis of proximity values between them, typically expressing preferences of each individual over each object.

Therefore unfolding applies multidimensional scaling [4] to an off-diagonal $n \times m$ matrix, usually representing the scores (or the rank) assigned to a set of m items by n individuals or judges [1]. Using of either scores or rankings traditionally discriminates between metric and non-metric unfolding.

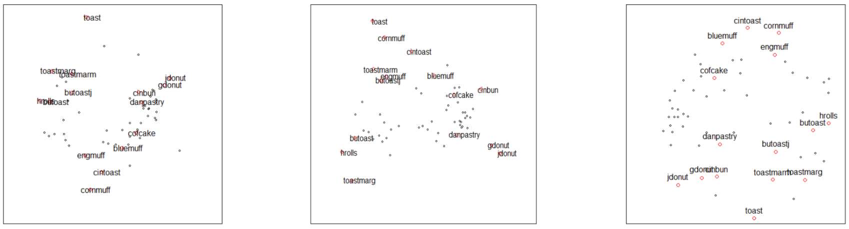
The goal is to obtain two configuration of points representing the position of the judges (X) and the items (Y) in a reduced geometrical space. Each point representing the individuals is considered as an ideal point so that its distances to the object points correspond to the preference scores [3].

Unfolding can be seen as a special case of multidimensional scaling because the off-diagonal matrix is considered as a block of an ideal distance matrix in which both the within judges and the within items dissimilarities are missing. The presence of blocks of missing data causes the phenomenon of the so-called degenerate solutions, i.e., solutions that return excellent badness of fit measures but not graphically interpretable at all.
To tackle the problem of degenerate solutions, several proposals have been presented in the literature [1]. By following the approach introduced by [9], we adopt the strategy of augmenting the data matrix, trying to build a complete dissimilarity matrix, and then applying any MDS algorithms.
Let $\boldsymbol{\Gamma}$ be the original $m \times n$ original preference data matrix. In order to augment the data matrix we add to this n additional rows, one for each of the n objects, that correspond to tied rankings representing the jth item, $j = 1, \ldots, n$. As a result, a new $N \times n$ $\boldsymbol{\Gamma}^*$ matrix is obtained, with $N = n + m$. Then we use copula-based association measures among rankings (the individuals), and between rankings and objects (namely, a rank-order representation of the objects through tied rankings), obtaining in fact a $N \times N$ dissimilarity matrix to be analyzed with any MDS algorithm.

3 An application on a real data set

Fig. 1 shows a comparison between the Unfolding solutions of PRESCAL [2], which actually is the most used algorithm for Unfolding analysis, and our proposal by using the Spearman ρ correlation coefficient via copula on the breakfast data set. Green and Rao [6] collected 42 rankings of 15 objects by asking 21 students and their wives to order 15 breakfast items in terms of their preference.

PREFSCAL works by setting two penalties on a modified loss function in such a way to guarantee non degenerate solutions. A possible drawback of this algorithm is that it is not always clear how set the penalty terms. In fact the user must make attempts in order to find the right solution.
The figure emphasizes that the solution of our procedure is not degenerate and it is comparable with the one of PREFSCAL, especially with its unconditional output. It is normal that our output looks like the unconditional PREFSCAL solution because we propose a solution that, depending on how we defined the dissimilarity matrix, is unconditional as well.



(a) Row-conditional Prefscal solution (b) Unconditional Prefscal solution (c) Spearman augmented unfolding

Fig. 1: Unfolding solutions for breakfast data. Breakfast items are labeled as follow: Toast pop-up; Buttered toast; English muffin and margarine; Jelly donut; Cinnamon toast; Blueberry muffin and margarine; Hard rolls and butter; Toast and marmalade; Buttered toast and jelly; Toast and margarine; Cinnamon bun; Danish pastry; Glazed donut; Coffee cake; Corn muffin and butter.

4 Concluding remarks

We propose an unfolding algorithm based on the augmentation of the data matrix and a copula-based association between rankings. The shown result highlights that our proposal produces non-degenerate unfolding solutions that are comparable with the ones obtained with PREFSCAL. With respect to PREFSCAL, any parameter must be a priori chosen by the user. On the other hand PREFSCAL always guarantees non degenerate solutions. A robust simulation study will be discussed.

References

1. Borg, I., Groenen, P.: Modern multidimensional scaling. Theory and applications. Springer-Verlag, New York (1997)
2. Busing, Frank M. T. A. and Groenen, Patrick J. K and Heiser, Willem J. Avoiding degeneracy in multidimensional unfolding by penalizing on the coefficient of variation. Psychometrika, **70(1)** 71–98, (2005).
3. Coombs,C.H.: Psychological scaling without a unit measurement. Psychological review, **57** 145–158 (1998)
4. Cox, T.F., Cox, M. A. A.: Multidimensional scaling. Chapman & Hall, London (1994)
5. Embrechts, P., McNeil, A. J., Straumann, D.: Correlation and dependence in risk management: properties and pitfalls. Risk management: value at risk and beyond, 176–223 (1998)
6. Green, P E and Rao, V R: Applied Multidimensional Scaling: A Comparison of Approaches and Algorithms. Holt. Rinheart and Winston, New York (1972).
7. Kaufman, L., Rousseeuw, P. J.: Finding groups in data. An introduction to cluster analysis. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. John Wiley & Sons, Inc., New York (1990)

8. Nelsen, R.B.: An introduction to copulas. Springer Series in Statistics. Springer, New York (2013)
9. Van Deun, K., Heiser W., Delbeke, L.: Multidimensional unfolding by nonmetric multidimensional scaling of spearman distances in the extended permutation polytope. Multivariate Behavioral Research (2007) doi: 10.1080/00273170701341167

Relations between copula-based dependence and ageing. A new approach based on partial orderings for copulas

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Abstract. In the case where dependence concept is based on notions of dependence orderings for copulas, we propose a new approach to investigate the inter-relations among dependence, univariate and bivariate ageing concepts for exchangeable random variables. Such an approach allows to extend some previously known inter-relations to weak dependence concepts and to propose two indeces of univariate a bivariate ageing.

The inter-relations among dependence, univariate and bivariate ageing concepts have been analyzed in the past literature (see, e.g., Averous and Dortet-Bernadet (2004) [1], Bassan and Spizzichino (2005) [3], and more recently Nappo and Spizzichino (2020) [4]). Such studies essentially hint on copula-based concepts of stochastic dependence.

As far as the univariate ageing of a random lifetime X is concerned, the main role is played by dependence properties of the Archimeadean function generated by the marginal survival function $\overline{G}$ of X, i.e.,

$$S_G(u,v) := \overline{G}\big(\overline{G}^{-1}(u) + \overline{G}^{-1}(v)\big). \tag{1}$$

The function S_G is not always a copula, actually it is a semicopula when $\overline{G}$ is not convex. Therefore the dependence concepts of interest need to be extended to Archimedean semicopulas.

In the bivariate exchangeable case the main role is played by the dependence properties of the ageing functions $B(u,v)$ (see (4) for the definition of B). The ageing function B is a transformation of the survival copula $\widehat{C}(u,v)$ and had been introduced and applied in Bassan and Spizzichino (2001) [2] (see also [3] and [4]). We recall that also the ageing functions B are not always copulas, indeed they generally belong to the class of semicopulas. Then this approach necessarily requires the extension of the dependence concept to such a class, which is actually larger than the subclass of the Archimedean semicopulas.

For the case where the dependence concept is based on notions of dependence orderings we propose a different approach. Such an approach aims to extend the analysis of the inter-relations among dependence, univariate and bivariate ageing concepts also to weak notions of dependence A further advantage of our approach dwells in the possibility of replacing dependence properties (of the ageing function B) by a comparison between the survival copula of the model and S_G. In such a framework the dependence concepts and the related dependence orderings only need to be extended to the class of the Archimedean semicopulas.

Furthermore the extension to weak dependence allows us to propose some theoretical indeces of ageing, somehow inspired to the well known Spearman correlation index. Though this idea seems rather natural, to our knowledge no index of ageing has ever been defined, neither for univariate nor for bivariate ageing.

More in details a brief description of our approach goes as follows.

We start by recalling one of the simplest property of Positive / Negative Univariate Ageing (**1-AG+ /1-AG−**), i.e., the New Better than Used / New Worse than Used (NBU/NWU), namely a positive random variable X with survival function $\overline{G}$ has the NBU **1-AG+** (NWU **1-AG−**) property when

$$\mathbb{P}(X > x) \underset{(\leq)}{\geq} \mathbb{P}(X > x + y | X > y), \; \forall\, x, y > 0 \quad \Leftrightarrow \quad \overline{G}(x)\overline{G}(y) \underset{(\leq)}{\geq} \overline{G}(x+y).$$

Setting $u := \overline{G}(x)$ and $v := \overline{G}(y)$, the NBU **1-AG+** property becomes

$$S_G(u,v) := \overline{G}\big(\overline{G}^{-1}(u) + \overline{G}^{-1}(v)\big) \leq uv,$$

or equivalently

$$S_G(u,v) \leq \Pi(u,v) \quad \Leftrightarrow \quad S_G \preceq_{PQD} \Pi \tag{2}$$

where $\Pi(u,v) = uv$ is the independent copula, and $\succeq_{PQD}$ is the PQD stochastic order. The condition at the r.h.s. of (2) can be seen as Negaitive Quadrant Dependence (NQD) of the Archimedean semicopula S_G.

As far as the bivariate ageing property we restrict to the exchangeable case, i.e., to a couple of random variables (X,Y) with symmetric joint survival function $\overline{F}(x,y)$, with common marginal survival function $\overline{G}$, and with symmetric survival copula $\widehat{C}(u,v)$. We recall the NBU positive bivariate ageing property (**2-AG+**) related to the previous univariate ageing property as defined in [2]:

$$B(u,v) \geq \Pi(u,v) \quad \Leftrightarrow \quad B \succeq_{PQD} \Pi \tag{3}$$

where, setting $\gamma(u) = \exp\{-\overline{G}^{-1}(u)\}$ and $\gamma^{-1}(z) = \overline{G}(-\log(z))$,

$$B(u,v) := \exp\{-\overline{G}^{-1}\left(\overline{F}\left(-\log u, -\log v\right)\right)\} = \gamma\big(\widehat{C}(\gamma^{-1}(u), \gamma^{-1}(v))\big). \tag{4}$$

In this case it is easily seen that (3) is equivalent to $\widehat{C} \succeq_{PQD} S_G$.

The idea is to extend this procedure to other dependence concepts. We will restrict to weak / strong dependence concepts defined as a comparison between the copula and the independent copula Π by means of a weak / strong Partial Dependence Order $\succeq_{(w/s)PDO}$:

$$C \quad \text{is} \quad \begin{matrix}\textbf{(w/s)Dep+} \\ \textbf{(w/s)Dep}-\end{matrix} \quad \Leftrightarrow \quad \begin{matrix} C \succeq_{(w/s)PDO} \Pi \\ C \preceq_{(w/s)PDO} \Pi \end{matrix}$$

As shown in the above cited papers ([1, 3]) many univariate ageing concepts (starting from the NBU) can be expressed as

$$S_G \quad \text{is} \quad \begin{matrix}\textbf{1-AG+} \\ \textbf{1-AG}-\end{matrix} \quad \Leftrightarrow \quad S_G \text{ is } \begin{matrix}\textbf{Dep}- \\ \textbf{Dep+}\end{matrix}$$

Then, in our framework, the related concepts of univariate e ageing are defined as a comparison of the Archimedean semicopula S_G with the Independent copula Π:

$$\overline{G} \text{ is } \begin{matrix}\textbf{1-AG+} \\ \textbf{1-AG}-\end{matrix} \;\Leftrightarrow\; \begin{matrix} S_G(u,v) \preceq_{(w/s)PDO} \Pi \\ S_G(u,v) \succeq_{(w/s)PDO} \Pi \end{matrix} \;\Leftrightarrow\; S_G(u,v) \text{ is } \begin{matrix}\textbf{(w/s)Dep}- \\ \textbf{(w/s)Dep+}\end{matrix}$$

Similarly (see [2, 3]) for exchangeable models (X, Y), with survival copula $\widehat{C}$ and common marginal distributions $\overline{G}$, many bivariate ageing concepts can be expressed as

$$(\widehat{C}, \overline{G}) \quad \text{is} \quad \begin{matrix}\textbf{2-AG+} \\ \textbf{2-AG}-\end{matrix} \quad \Leftrightarrow \quad B \text{ is } \begin{matrix}\textbf{Dep+} \\ \textbf{Dep}-\end{matrix}$$

In many examples (starting from the PQD dependence) the above condition is equivalent to a comparison of the Archimedean semicopula S_G with the survival copula $\widehat{C}$. Then in our framework the bivariate ageing property is

$$(\widehat{C}, \overline{G}) \quad \text{is} \quad \begin{matrix}\textbf{2-AG+} \\ \textbf{2-AG}-\end{matrix} \;\Leftrightarrow\; \begin{matrix} S_G(u,v) \preceq_{(w/s)PDO} \widehat{C} \\ S_G(u,v) \succeq_{(w/s)PDO} \widehat{C} \end{matrix}$$

Note that, when S_G is a true copula, the pair $(S_G, \overline{G})$ characterizes the Schur constant model with joint survival fucntion $\overline{G}(x+y)$. Such a model may be seen as representing the "noageing model" for two exchangeable random variables with common marginal survival function $\overline{G}$. Similarly, when S_G is not a true copula, the pair $(S_G, \overline{G})$ may be interpreted as a "noageing (pseudo) model".

As already recalled, in [3] (see also [4]) some inter-relations among (**1-AG**$\pm$, **2-AG**$\pm$, **Dep**$\pm$) have been exploited by using the ageing function: for example

1. **Dep+ 1-AG+** $\Rightarrow$ **2-AG+**
2. **2-AG+ 1-AG**$-$ $\Rightarrow$ **Dep+**
3. **2-AG**$-$ **Dep+** $\Rightarrow$ **1-AG**$-$

In our framework the bivariate the above inter-relations emerge in a direct and natural way, and their proof becomes elementary.

We end by explaining briefly the proposed indeces of ageing related to Spearman correlation index ρ. As it is well known ρ can be expressed in terms either of the joint copula C or the joint survival copula $\widehat{C}$. Furthermore $\rho = \rho_C = \rho_{\widehat{C}}$ can be seen as a measure of average quadrant dependence. Indeed

$$\begin{aligned}\rho_{\widehat{C}} &= 12\int_0^1\int_0^1 \widehat{C}(u,v)\,du\,dv - 3 = 12\int_0^1\int_0^1 \big(\widehat{C}(u,v) - \Pi(u,v)\big)\,du\,dv \\ &= \mathbb{E}\big(\overline{G}(X_0)\overline{G}(Y_0)\big) - \mathbb{E}\big(\overline{G}(X_0)\big)\mathbb{E}\big(\overline{G}(Y_0)\big)\end{aligned}$$

where X_0 and Y_0 are independent random variables with common survival function $\overline{G}$. Furthermore we stress that the average quadrant dependence can be seen as a weak dependence concept (see Navarro and Pellerey (2021) [5] for a complete analysis of weak notion of weak dependence concepts), which in its turn is associated to the following weak Partial Order

$$C_1(u,v) \preceq_\rho C_2(u,v) \quad \Leftrightarrow \quad \int_0^1\int_0^1 C_1(u,v)\,du\,dv \leq \int_0^1\int_0^1 C_2(u,v)\,du\,dv.$$

and therefore $\rho \geq 0$ is equivalent to $\Pi \preceq_\rho \widehat{C}$. Given two exchangeable random variables X and Y with joint survival function $\overline{F} = \widehat{C}\big(\overline{G}(x), \overline{G}(y)\big)$, we propose the following indeces ν_G^ρ and s_F^ρ of univariate and bivariate ageing:

$$\begin{aligned}\nu_G^\rho := -\rho_{S_G} &= 12\int_0^1\int_0^1 \big(\Pi(u,v) - S_G(u,v)\big)\,du\,dv \\ &= \mathbb{E}\big(\overline{G}(X_0)\big)\mathbb{E}\big(\overline{G}(Y_0)\big) - \mathbb{E}\big(\overline{G}(X_0+Y_0)\big) \\ s_F^\rho := \rho_{\widehat{C}} + \nu_G^\rho &= 12\int_0^1\int_0^1 \big(\widehat{C}(u,v) - S_G(u,v)\big)\,du\,dv \\ &= \mathbb{E}\big(\widehat{C}(\overline{G}(X_0), \overline{G}(Y_0))\big) - \mathbb{E}\big(\overline{G}(X_0)\big)\mathbb{E}\big(\overline{G}(Y_0)\big).\end{aligned}$$

Then positive univariate ageing is equivalent to $\nu_G^\rho \geq 0$ and positive bivariate ageing is equivalent to $s_F^\rho \geq 0$.

References

1. Avérous, J. and J.-L. Dortet-Bernadet (2004). Dependence for Archimedean copulas and aging properties of their generating functions. *Sankhya A 66*(4), 607–620.
2. Bassan, B. and F. Spizzichino (2001). Dependence and multivariate aging: the role of level sets of the survival function. In *System and Bayesian reliability*, Volume 5 of *Ser. Qual. Reliab. Eng. Stat.*, pp. 229–242. World Sci. Publ., River Edge, NJ.
3. Bassan, B. and F. Spizzichino (2005). Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. *J. Multivariate Anal. 93*(2), 313–339.
4. Nappo, G. and F. Spizzichino (2020). Relations between ageing and dependence for exchangeable lifetimes with an extension for the IFRA/DFRA property. *Dependence Modeling 8*(1), 1–33.
5. Navarro, J. and F. Pellerey (2021). Weak Dependence Notions and Their Mutual Relationships. *Mathematics 9*(1), 1–27. https://www.mdpi.com/2227-7390/9/1/81

New classes of multivariate quasi-copulas emerging from the Dedekind-MacNeille completion of multivariate copulas

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Abstract. There is a growing interest in the studies of subclasses of multivariate quasi-copulas [1, 3, 2]. The question about the Dedekind-MacNeille completion of multivariate copulas has recently been solved [14, 15]. Its solution has brought into light some subclasses of interest in imprecise probability. This presentation should encourage open discussion among the participants of the Linz Seminar 2023 on problems in d-variate quasi-copulas for $d \geqslant 3$.

1 Three motivations

We start by some motivation for the method presented in this contribution.

(1) In 2005 B. Nelsen and M. Úbeda-Flores showed in a historical paper [9] that the Dedekind-MacNeille completion of the poset of bivariate copulas is just the class of bivariate quasi-copulas. The class of copulas is a poset (partially ordered set) for pointwise order. This raises the question, how to get the abstract Dedekind-MacNeille completion, of the poset of copulas in the multivariate setting. In [5] the two authors together and Fernández-Sánchez demonstrate that for dimension $d \geqslant 3$ a simple extension of this result to d-variate (quasi-)copulas does not work any more.

To every distribution function H there exists a copula C such that $H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n))$, where F_i are the marginal distributions of H (the Sklar's theorem). The definitions of copula and quasi-copula will be given in Section 2. The respective families will be denoted by $\mathcal{C}_d$ and $\mathcal{Q}_d$.

(2) Another motivation, and as a matter of fact chronologically our first one, arises from the imprecise probability. Applications of copulas in the imprecise settings are a relatively new area. In 2015 I. Montes et al. published in [8] a possible approach to bivariate imprecise copulas. If $\mathcal{C}$ is a nonempty set of copulas, then $\underline{C} = \inf\{C\}_{C\in\mathcal{C}}$ and $\overline{C} = \sup\{C\}_{C\in\mathcal{C}}$ are quasi-copulas and the ordered interval $(\underline{C}, \overline{C})$, i.e., the set of all intermediate quasi-copulas, may be called *an imprecise copula*. Following the ideas of p-boxes it would be nice if the order ideal defined by these quasi-copulas contained a "true" copula. However, we show in [10] that there exists an imprecise copula (A, B) in this sense such that there is no copula C with $A \leqslant C \leqslant B$.

(3) A recent motivation emerged from the Linz-Ljubljana project. Part of it considers applications of semicopulas (capacities) in imprecise models [6]. Such a model is typically required to satisfy some reasonable consistency conditions, such as avoidance of sure loss and coherence. The method we now present extends to this situation as well.

2 The method

Our approach upgrades the ideas of M. Dibala et al. [4], where possibly negative volumes of quasi-copulas are studied as defects from being copulas. The attained technique is called Algebraic Obstacles in the Geometry of Negative Volumes (ALGEN for short). This method has been developed to solve some questions related to imprecise probabilities.

Let us start by the promised definition of a quasi-copula. Let $\mathbb{I} = [0,1]$. For points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we write $\mathbf{x} \leqslant \mathbf{y}$ if $x_i \leqslant y_i$ for every $i \in [d]$ where $[d] = \{1, \ldots, d\}$. If $\mathbf{x} \leqslant \mathbf{y}$, a *d-box* (rectangle) is the set $[\mathbf{x}, \mathbf{y}] = [x_1, y_1] \times \cdots \times [x_d, y_d]$. Denote also by $\operatorname{ver}[\mathbf{x}, \mathbf{y}] = \{x_1, y_1\} \times \cdots \times \{x_d, y_d\}$ the *set of vertices* of this rectangle. For a real valued function A and $R = [\mathbf{x}, \mathbf{y}]$ we define the A-*volume* (or simply the *volume* if A is understood) *of the rectangle* R by $V_A(R) = \sum_{\mathbf{v} \in \operatorname{ver} R} \operatorname{sign}_R(\mathbf{v}) A(\mathbf{v})$, where $\operatorname{sign}_R(\mathbf{v})$ equals 1 if $v_j = x_j$ for an even number of indices, and -1 otherwise. A d-variate *quasi-copula* is a function $A\colon \mathbb{I}^d \to \mathbb{I}$ that satisfies the following three conditions:

(i) for every $j \in [d]$ we have $A(1, \ldots, 1, u_j, 1, \ldots, 1) = u_j$;
(ii) A is increasing in each of its variables;
(iii) A satisfies the 1-Lipschitz condition in each of its variables.

It is well-known and not hard to see that A then satisfies also

(iv) for every $j \in [d]$ and for every point $(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_d) \in \mathbb{I}^{d-1}$ we have $A(u_1, \ldots, u_{j-1}, 0, u_{j+1}, \ldots, u_d) = 0$.

Function A is a *copula* (equivalently) if it satisfies Conditions *(i)*, *(iv)*, and

(ii') for every rectangle $R \subseteq \mathbb{I}^d$ its A-volume is positive.

For any point $\mathbf{x} \in \mathbb{I}^d$ we define its *multiplicity* with respect to R by $m_R(\mathbf{x}) = \operatorname{sign}_R(\mathbf{x})$, if $\mathbf{x} \in \operatorname{ver} R$, and 0 otherwise.

Now, we denote by $\mathfrak{R}$ the family of all objects of the form $R = \bigsqcup_{i=1}^k R_i$, where $\{R_i\}_{i=1}^k$ is an arbitrary finite family of d-boxes with vertices in $\mathbb{I}^d$, and $\bigsqcup$ denotes the disjoint union. Let us extend the definition of *multiplicity* to any $R \in \mathfrak{R}$ by $m_R(\mathbf{x}) = \sum_{i=1}^k m_{R_i}(\mathbf{x})$. Let A and B be a pair of real valued functions on $\mathbb{I}^d$ and let

$$L^{(A,B)}(R) = \sum_{\substack{\mathbf{y} \in \mathbb{I}^n \\ m_R(\mathbf{y})>0}} B(\mathbf{y}) m_R(\mathbf{y}) + \sum_{\substack{\mathbf{y} \in \mathbb{I}^n \\ m_R(\mathbf{y})<0}} A(\mathbf{y}) m_R(\mathbf{y})$$

$$P_M^{(A,B)}(\mathbf{x}) = \inf_{\substack{R \in \mathfrak{R} \\ m_R(\mathbf{x})>0}} \frac{L^{(A,B)}(R)}{|m_R(\mathbf{x})|} \quad \text{and} \quad P_O^{(A,B)}(\mathbf{x}) = \inf_{\substack{R \in \mathfrak{R} \\ m_R(\mathbf{x})<0}} \frac{L^{(A,B)}(R)}{|m_R(\mathbf{x})|}.$$

Denote $\mathcal{C}(A,B) = \{C \in \mathcal{C}_d \mid A \leqslant C \leqslant B\}$, for d-variate quasi-copulas $A \leqslant B$.

Theorem 1 (The first law of ALGEN method). *Let $A \leqslant B$ be quasi-copulas, then $\mathcal{C}(A,B)$ is nonempty if and only if $L^{(A,B)}(R) \geqslant 0$ for all $R \in \mathfrak{R}$.*

Theorem 2 (The second law of ALGEN method). *Let $A \leqslant B$ be quasi-copulas with $\mathcal{C}(A,B) \neq \emptyset$. Then* *(a)* $B = \sup \mathcal{C}(A, B)$ *if and only if* $B(\mathbf{x}) - A(\mathbf{x}) \leqslant P_O^{(A,B)}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{I}^d$.

(b) $A = \inf \mathcal{C}(A, B)$ *if and only if* $B(\mathbf{x}) - A(\mathbf{x}) \leqslant P_M^{(A,B)}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{I}^d$.

Our first law relates to "avoiding sure loss" principle and the second one to "coherence" principle. We can show that there are "good-looking" quasi-copulas $A \leqslant B$ with empty $\mathcal{C}(A, B)$, i.e., not avoiding sure loss (cf. [10]). The roles of the ALGEN test functions L, P_M, and P_O are self-explanatory from these laws.

3 The Dedekind-MacNeille completion

The Dedekind-MacNeille completion of a poset, is the "smallest possible" complete lattice containing this poset. Due to applications our solution should not be too abstract – it should sit in the poset of, say, d-variate quasi-copulas. Let us give here a shortcut to our solution. Denote by $\mathcal{C}_d$, respectively $\mathcal{Q}_d$, the d-variate poset of copulas, respectively quasi-copulas. We first present the *upper Dedekind-MacNeille completion* $\overline{\mathcal{C}}_d$ of copulas within the set of quasi-copulas.

Theorem 3 (The upper Dedekind-MacNeille completion). *The upper completion of* $\mathcal{C}_d$ *is given by* $\overline{\mathcal{C}}_d = \left\{ Q \in \mathcal{Q}_d \mid Q = M_d - P_M^{(\max\{W_d + P_O^{(W_d,Q)}, W_d\}, M_d)} \right\}$ *with meet and join operations given by*

$$\overline{\bigwedge} S = M_d - P_M^{(\max\{W_d + P_O^{(W_d, \bigwedge S)}, W_d\}, M_d)} \quad \text{and} \quad \overline{\bigvee} S = M_d - P_M^{(\bigvee S, M_d)}.$$

The lower completion is expressed slightly differently than the upper one.

Theorem 4 (The lower Dedekind-MacNeille completion). *The lower completion of* $\mathcal{C}_d$ *is given by* $\underline{\mathcal{C}}_d = \left\{ Q \in \mathcal{Q}_d \mid Q = \max\{W_d + P_O^{(W_d, M_d - P_M^{(Q,M_d)})}, W_d\} \right\}$ *with meet and join operations given by*

$$\underline{\bigwedge} S = \max\{W_d + P_O^{(W_d, \bigwedge S)}, W_d\} \text{ and } \underline{\bigvee} S = \max\{W_d + P_O^{(W_d, M_d - P_M^{(\bigvee S, M_d)})}, W_d\}.$$

Our completions are "small", $\overline{\mathcal{C}}_d$, say, is not even convex. There are other completions, all lattice isomorphic to each other but made of some other quasi-copulas.

4 Is the class of d-variate quasi-copulas too big?

Our completions are small subclasses of d-variate quasi-copulas. So, is there a "general" well-defined smaller subclass that would serve our purposes? This brings us to the pretty unexplored field of d-variate quasi-copulas for $d \geqslant 3$. A recent "Hitchhiker's" problem list was given in [2]. Problem 6, say, is our Motivation **(2)** of Section 1. Our negative answer was given in [10] as explained in Section 2. Problem 2 of [2] (or at least an important part of it) was solved as a part of Linz-Ljubljana project and published in

[7]. In our oral presentation we intend to explain how this solution can be used in the proof of the following theorem, which answers in the negative the question above. We will also explain some other relations among the notions presented to encourage open discussion among the participants of the Linz Seminar 2023

Theorem 5 (cf. [15]). *Every d-variate quasi-copula can be obtained as a pointwise supremum of a set of pointwise infima of sets of d-variate copulas. It can also be obtained as a pointwise infimum of a set of pointwise suprema of sets of d-variate copulas.*

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References

1. J.J. Arias-García, R. Mesiar, B. De Baets, The unwalked path between quasi-copulas and copulas: Stepping stones in higher dimensions, *Intern. J. Appr. Reasoning* (2017), 89–99.
2. J. J. Arias-Garcia, R. Mesiar, and B. De Baets, A hitchhiker's guide to quasi-copulas, *Fuzzy Sets Syst.* **393** (2020), 1–28.
3. J.J. Arias-García, R. Mesiar, E.P. Klement, S. Saminger-Platz, B. De Baets, Extremal Lipschitz continuous aggregation functions with a given diagonal section, *Fuzzy Sets Syst.* **346** (2018), 147–167.
4. M. Dibala, S. Saminger-Platz, R. Mesiar, and E. P. Klement, Defects and transformations of quasi-copulas, *Kybernetika* **52** (2016), 848–865.
5. J. Fernández-Sánchez, R. Nelsen, M. Úbeda-Flores, Multivariate copulas, quasi-copulas and lattices, *Stat. Probab. Lett.* **81** (2011), 1365–1369.
6. E.P. Klement, D. Kokol Bukovšek, B. Mojškerc, M. Omladič, S. Saminger-Platz, N. Stopar, Relating semicopulas to a k-increasing semicopula by the ALGEN method, *work in progress*.
7. E.P. Klement, D. Kokol Bukovšek, M. Omladič, S. Saminger-Platz, N. Stopar, Multivariate copulas with given values at two arbitrary points, *Statistical Papers* 41 pages, DOI 10.1007/s00362-022-01362-4.
8. I. Montes, E. Miranda, R. Pelessoni, P. Vicig, Sklar's theorem in an imprecise setting, *Fuzzy Sets and Syst.* **278** (2015), 48–66.
9. R.B. Nelsen, M. Úbeda-Flores, The lattice-theoretic structure of sets of bivariate copulas and quasi-copulas, *C. R. Acad. Sci. Paris, Sér. I, Math.* **341** (2005), 583–586.
10. M. Omladič, N. Stopar, Final solution to the problem of relating a true copula to an imprecise copula, *Fuzzy Sets and Syst.* **393** (2020), 96–112.
11. M. Omladič, N. Stopar, A full scale Sklar's theorem in the imprecise setting, *Fuzzy Sets and Syst.* **393** (2020), 113–125.
12. M. Omladič, N. Stopar, On a new partial order on bivariate distributions and on constrained bounds of their copulas, *Fuzzy Sets and Syst.* **415** (2021), 118–135.
13. M. Omladič, N. Stopar, Multivariate imprecise Sklar type theorems, *Fuzzy Sets and Syst.* **428** (2022), 80–101.
14. M. Omladič, N. Stopar, *Dedekind-MacNeille completion of multivariate copulas via ALGEN method*, *Fuzzy Sets and Syst.* **441** (2022), 321–334.
15. M. Omladič, N. Stopar, Novel methods and results in multivariate quasi-copulas, *work in progress*.

On perturbation and truncation involving the Fréchet-Hoeffding bounds

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1 Introduction

It might be well known to the audience of this years's Linz seminar that the three basic copulas $W(x,y) = \max\{x+y-1, 0\}$, $\Pi(x,y) = x \cdot y$, and $M(x,y) = \min\{x,y\}$ are, respectively, characteristic for the counter-monotonicity, independence, and comonotonicity of a pair of two continuous random variables on the same probability space. Moreover, W and M serve as the lower resp. upper bound of the set of all bivariate copulas $\mathcal{C}_2$. It might also be known that the approach to "perturb" copulas may be seen as an attempt to modify an underlying copula in order to obtain different, but somehow similar ones. More precisely, a continuous function $f\colon [0,1]^2 \to \mathbb{R}$ (usually called the perturbation factor) is added to a given copula D, leading to the natural question whether or not or under which conditions $D + f$ as a perturbation of D is again a copula (see, e.g., [1, 3, 17, 19]).

A most prominent example of a perturbation of Π is the family of Eyraud-Farlie-Gumbel-Morgenstern (or EFGM) copulas (see also [16, 15, 8, 5]) given by $\mathbf{EFGM}_\theta\colon [0,1]^2 \to [0,1]$,

$$\mathbf{EFGM}_\theta(x,y) = xy + \theta xy(1-x)(1-y), \tag{1}$$

where a parameterized family of perturbation factors is added to Π (for more details about EFGM copulas and for other families of copulas which can be considered as perturbations see e.g. [2, 4, 6, 11, 12, 14, 16, 18]). $\mathbf{EFGM}_\theta$ is a copula if and only if $\theta \in [-1,1]$ (see, e.g., [16, Example 3.12]).

It has been Hürlimann [9, Theorem 3.1] who extended the family of EFGM copulas by allowing the parameter θ to take any value in $[-\infty, \infty]$ together with a truncation by the Fréchet-Hoeffding bounds M resp. W and thus obtained a comprehensive parametric family of copulas $\mathrm{H}_\theta\colon [0,1]^2 \to [0,1]$, being defined by

$$\mathrm{H}_\theta = \begin{cases} W & \text{if } \theta = -\infty, \\ \Pi_{(\theta)} \vee W & \text{if } \theta \in \left]-\infty, 0\right[, \\ \Pi_{(\theta)} \wedge M & \text{if } \theta \in \left[0, \infty\right[, \\ M & \text{if } \theta = \infty. \end{cases} \tag{2}$$

2 Transformations and truncation of copulas

If $C \in \mathcal{C}_2$ is an arbitrary copula, then also the x-flipping $C^{x\text{flip}}$, the y-flipping $C^{y\text{flip}}$ of C, and the *survival copula* C^{surv} of C, given by

$$\begin{aligned} C^{x\text{flip}}(x,y) &= y - C(1-x,y), \\ C^{y\text{flip}}(x,y) &= x - C(x,1-y), \\ C^{\text{surv}}(x,y) &= x + y - 1 + C(1-x,1-y), \end{aligned}$$

are copulas. Moreover they are closely related to some symmetries of the random vector (X,Y) associated to the copula $C_{X,Y}$ by means of

$$(C_{X,Y})^{x\text{flip}} = C_{-X,Y}, \qquad (C_{X,Y})^{y\text{flip}} = C_{X,-Y}, \qquad (C_{X,Y})^{\text{surv}} = C_{-X,-Y}$$

(compare also [7, 10]). Since for Π all these symmetries coincide, i.e., $\Pi = \Pi^{x\text{flip}} = \Pi^{y\text{flip}} = \Pi^{\text{surv}}$, we may formulate the following equivalent expressions for EFGM copulas

$$\begin{aligned} \mathbf{EFGM}_\theta(x,y) &= \Pi(x,y) + \theta\Pi(x,y)(1-x)(1-y) \\ &= \Pi(x,y) + \theta\Pi(x,y)\cdot\Pi^{\text{surv}}(1-x,1-y) \\ &= \Pi(x,y) + \theta\Pi^{x\text{flip}}(1-x,y)\cdot\Pi^{y\text{flip}}(x,1-y). \end{aligned}$$

Based on these observations we introduce the following two types of perturbations of a copula:

Definition 1. *Let $C\colon [0,1]^2 \to [0,1]$ be a bivariate copula and $\theta \in \mathbb{R}$. Then the (θ)-*transform $C_{(\theta)}\colon [0,1]^2 \to \mathbb{R}$ *of C and the $[\theta]$-*transform $C_{[\theta]}\colon [0,1]^2 \to \mathbb{R}$ *of C are defined by, respectively,*

$$\begin{aligned} C_{(\theta)}(x,y) &= C(x,y) + \theta C(x,y)\cdot C^{\text{surv}}(1-x,1-y) \\ &= C(x,y) + \theta C(x,y)(C(x,y) - x - y + 1), \end{aligned} \tag{3}$$

$$\begin{aligned} C_{[\theta]}(x,y) &= C(x,y) + \theta C^{x\text{flip}}(1-x,y)\cdot C^{y\text{flip}}(x,1-y) \\ &= C(x,y) + \theta(x - C(x,y))(y - C(x,y)). \end{aligned} \tag{4}$$

As is obvious from above, we obtain $\Pi_{(\theta)} = \Pi_{[\theta]}$ for the independence copula Π, even more, Π is the only copula for which the two transformations coincide for all $\theta \in \mathbb{R}$; for any other copula C and parameter $\theta \neq 0$, $C_{(\theta)}$ is different from $C_{[\theta]}$.

Note that for any copula C and any $(\hat{x}, \hat{y}) \in [0,1]^2$ with $C(\hat{x}, \hat{y}) \notin \{0, \hat{x} + \hat{y} - 1\}$, $C_{(\theta)}(\hat{x}, \hat{y})$ can take any value in $\mathbb{R}$ for an appropriate choice of $\theta \in \mathbb{R}$. A similar statement holds for $C_{[\theta]}(\hat{x}, \hat{y})$ whenever $C(\hat{x}, \hat{y}) \notin \{\hat{x}, \hat{y}\}$, i.e. though $C_{(\theta)}$ and $C_{[\theta]}$ fulfill the boundary conditions of a copula, they need not map into $[0, 1]$.

Truncating the functions $C_{(\theta)}$ and $C_{[\theta]}$ by the Fréchet-Hoeffding bounds M and W following the approach of Hürlimann as outlined above (compare also [13]), on the one hand, guarantees that the range of the functions stays within $[0, 1]$ and, on the other hand, might also enlarge the set of admissible parameters for obtaining not only a function but also a copula.

In our presentation we will discuss some of the properties of sets (Θ_C) and $[\Theta_C]$ of parameters θ ensuring that $(C_{(\theta)})_{\theta \in (\Theta_C)}$ resp. $(C_{[\theta]})_{\theta \in [\Theta_C]}$ are families of copulas. And we will discuss for some special copulas C how these sets change and enlarge when employing truncation. In particular, we look at $C = M$ resp. $C = W$, i.e. the copulas modeling co- and counter-monotonicity, and show that their truncated transformations interestingly lead to families of copulas with non-convex sets of admissible parameters.

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References

1. C. Amblard and S. Girard. A new extension of bivariate FGM copulas. *Metrika*, 70:1–17, 2009.
2. D. A. Conway. Farlie-Gumbel-Morgenstern distributions. In S. Kotz and N. L. Johnson, editors, *Encyclopedia of Statistical Sciences*, volume 3, pages 28–31. Wiley, New York, 1983.
3. F. Durante, J. Fernández-Sánchez, and M. Úbeda-Flores. Bivariate copulas generated by perturbations. *Fuzzy Sets and Systems*, 228:137–144, 2013.
4. F. Durante and C. Sempi. *Principles of Copula Theory*. CRC Press, Boca Raton, 2015.
5. D. J. G. Farlie. The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, 47:307–323, 1960.
6. J. Fernández-Sánchez and M. Úbeda-Flores. Solution to two open problems on perturbations of the product copula. *Fuzzy Sets and Systems*, 354:116–122, 2019.
7. S. Fuchs and K. D. Schmidt. Bivariate copulas: transformations, asymmetry and measures of concordance. *Kybernetika*, 50:109–125, 2014.
8. E. J. Gumbel. Distributions à plusieurs variables dont les marges sont données. *C. R. Acad. Sci. Paris Sér. A*, 246:2717–2719, 1958.
9. W. Hürlimann. A comprehensive extension of the FGM copula. *Statist. Papers*, 58:373–392, 2017.
10. H. Joe. *Dependence Modeling with Copulas*. CRC Press, Boca Raton, 2015.
11. N. L. Johnson and S. Kotz. On some generalized Farlie-Gumbel-Morgenstern distributions. *Comm. Statist.*, 4:415–427, 1975.
12. N. L. Johnson and S. Kotz. On some generalized Farlie-Gumbel-Morgenstern distributions. II. Regression, correlation and further generalizations. *Comm. Statist. Theory Methods*, 6:485–496, 1977.
13. E. P. Klement and A. Kolesárová. Extension to copulas and quasi-copulas as special 1-Lipschitz aggregation operators. *Kybernetika*, 41:329–348, 2005.

14. S. Kotz and N. L. Johnson. Propriétés de dépendance des distributions itérées, généralisées à deux variables Farlie-Gumbel-Morgenstern. *C. R. Acad. Sci. Paris Sér. A-B*, 285:A277–A280, 1977.
15. D. Morgenstern. Einfache Beispiele zweidimensionaler Verteilungen. *Mitteilungsblatt für Mathematische Statistik*, 8:234–235, 1956.
16. R. B. Nelsen. *An Introduction to Copulas*, volume 139 of *Lecture Notes in Statistics*. Springer, New York, second edition, 2006.
17. J. A. Rodríguez-Lallena and M. Úbeda-Flores. A new class of bivariate copulas. *Statist. Probab. Lett.*, 66:315–325, 2004.
18. S. Saminger-Platz, A. Kolesárová, A. Šeliga, R. Mesiar, and E. P. Klement. The impact on the properties of the EFGM copulas when extending this family. *Fuzzy Sets and Systems*, 415:1–26, 2021.
19. S. Saminger-Platz, A. Kolesárová, A. Šeliga, R. Mesiar, and E. P. Klement. New results on perturbation-based copulas. *Depend. Model.*, 9:347–373, 2021.

Copula-based change point detection in non-stationary multivariate time series with mixture marginals

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Abstract. Non-stationarity is one of the most challenging issues in time series analysis and prediction. The nature of non-stationary time series may follow mixture distributions, such as bimodal or multimodal. Also, in multivariate time series, this non-stationarity problem may lead to a change in the associated joint distribution, and therefore to a different type of copula used to calculate this joint distribution. In this work we consider non-stationary multivariate time series in presence of bimodal marginals, and we use copulas to carry out predictions. We use two methods including dynamic copulas and mixture copulas to calculate forecasts for real financial multivariate time series. Finally, we compare the performance of these two methods.

1 Instruction for authors

As the use of large amount of data has grown in many disciplines, the structure of data has become more and more complex. This complexity is severe in longitudinal and time series data, especially in multivariate cases [1, 2]. Such cases often show stationarity violation as well as bimodality or multi-modality, see e.g., DeYoreo and Kottas (2017) [3]. Dealing with the non-stationarity of multivariate time series when the marginal distribution is a mixture of distributions is the main aim of this work.

For a d-dimensional random vector $\boldsymbol{X} = (X_1, X_2, ..., X_d)$, let us assume that $\boldsymbol{X}$ is generated from a (finite) mixture of K-components mixture model and its density at $\boldsymbol{x} = (x_1, ..., x_d)$ can be written as

$$f(\boldsymbol{x}; \boldsymbol{\eta}) = \sum_{k=1}^{K} \pi_k g_k(\boldsymbol{x}; \boldsymbol{\eta}_k),$$

where $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_K)^T$ is the parameter vector that contains all the mixture model parameters with $\boldsymbol{\eta}_k = (\boldsymbol{\theta}_k, \pi_k)$ in which $\boldsymbol{\theta}_k$ is the parameter vector of all the parameters of the k-th component. Also, $g_k(\cdot; \boldsymbol{\eta}_k)$ and π_k are respectively the density of the k-th component and its weight and satisfy $\pi_k \geq 0$, $\sum_{k=1}^{K} \pi_k = 1$, for $k = 1, \ldots, K$. We refer to McLachlan and Lee (2019) [4] for more details. Similarly, the corresponding joint distribution can be presented as

$$F(\boldsymbol{x}; \boldsymbol{\eta}) = \sum_{k=1}^{K} \pi_k G_k(\boldsymbol{x}; \boldsymbol{\eta}_k),$$

where $G_k(.; \boldsymbol{\eta}_k)$ is the distribution of the k-th component.

It is well known that copulas, which have been introduced by Sklar (1959) [5], are sophisticated tools to capture the non linear as well as the linear relationships between the variables in the multivariate case [6, 15]. If the continuous random vector $\boldsymbol{X} = (X_1, X_2, ..., X_d)$ follows the joint multivariate distribution $F_{\boldsymbol{X}} : \mathbb{R}^d \to [0, 1]$ and $F_i : \mathbb{R} \to [0, 1]$, $i = 1, 2, ..., d$, are the related marginal distribution functions of X_i, $i = 1, 2, ..., d$, then, there exist a copula $C : [0, 1]^d \to [0, 1]$ in which

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F_1(x_1), F_2(x_2), ..., F_d(x_d)). \quad (1)$$

Also, under some smoothness assumptions of $\boldsymbol{X}$, from (1), one can obtain the joint density f of $\boldsymbol{X}$ as

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = c(F_1(x_1), \ldots, F_d(x_d)) \prod_{k=1}^{d} f_k(x_k) \quad (2)$$

where c is the density of the copula C and $f_k, k = 1, .., d$, are the respective densities of the random variables $X_1, .., X_d$. One may refer to Nelsen (2006) [8] and Durante and Sempi (2016) [7] for more information.

For marginal modelling, many popular approaches such as ARMA, GARCH, and their mixture (ARMA-GARCH) have been often applied to time series. The majority of these models are constructed based on the stationarity assumption (see e.g., [9], p. 19) of the underlying time series, which is not realistic in real-world data sets (see, e.g., Kleibergen et al., (1993) [12] and Mikosch and Stărică, (2004) [13]). We also refer to [10] and [11] for more details about mixture of univariates in time series. Considering non-stationary multivariate time series, some techniques specifically deal with non-stationarity, such as change point detection methods [21].

In this work, we use copulas as connection links between variables. Then, in order to carry out multivariate change point detection, we use dynamic copulas. Following the approach of Bücher et al. (2014) [16] and their proposed test statistics for stationarity, we address the stationarity of time series and find the relevant change point(s), see also Hofert et al. (2018) [15] for more explanations.

Since non stationarity in a multivariate time series affects marginals as well, we assume that the distributions of univariate time series follow mixture distributions. Therefore, we follow three scenarios to determine the best method to handle change point detection in non-stationary multivariate time series with a mixture of marginal models.

In the first scenario, we determine the best-fitting marginal for each univariate variable. By estimating the copula of the marginal residuals, we simulate M realizations from a copula. Hence, from each simulation, we calculate the predicted values using the inverse cumulative distribution function (cdf) and the relevant fitted marginal models (see e.g., Ansell and Dalla Valle (2022) [18]).

In the second scenario, similarly to Sheikhi et. al (2023) [19], after finding the change point(s), we split the time series into segments based on the obtained change points. In each segment, we fit the best-fitting marginals for each univariate variable. By estimating the copula of the marginal residuals, we simulate M realizations from the copula. Hence, we calculate the predicted values for each simulation, using the inverse cdf and the relevant fitted marginal models.

In the third scenario, we estimate the best-fitting mixture marginal for each univariate variable. By estimating the copula of the marginal residuals, we simulate M realizations from the copula. Again, we calculate the predicted values for each simulation, using the inverse cdf and the relevant fitted marginal models, see, for instance, Sahin and Czado (2022) [20] for a similar approach in clustering.

In this work we consider data split in two groups: market data and social media data. The market data are the stock prices of Tesla and the social media data are the corresponding sentiment scores obtained from Twitter in a period of 6 years from Jan. 1, 2013, to Sep. 31, 2018. The market data contain daily open-high-low-close (aka as OHLC) information on the Tesla stock; while the social media data are gathered from Twitter and consist of relevant tweets, along with their sentiment with daily resolution, see Mendoza et al. (2022) [17].

In order to compare the performances of the considered approaches we calculate in-sample predictions for the last 10 percent observations of the time series in the three scenarios. In this regard, the difference between true observations and our in-sample prediction is our criterion for comparison. More specifically, we use the continuous ranked probability score (CRPS) of Matheson and Winkler (1979) [23].

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References

1. Ponce-Flores, M., Frausto-Solís, J., Santamaría-Bonfil, G., Pérez-Ortega, J. & González-Barbosa, J. Time series complexities and their relationship to forecasting performance. *Entropy*. **22**, 89 (2020)
2. Sardá-Espinosa, A. Comparing time-series clustering algorithms in r using the dtwclust package. *R Package Vignette*. **12** pp. 41 (2017)
3. DeYoreo, M. & Kottas, A. A Bayesian nonparametric Markovian model for non-stationary time series. *Statistics And Computing*. **27** pp. 1525-1538 (2017)
4. McLachlan, G., Lee, S. & Rathnayake, S. Finite mixture models. *Annual Review Of Statistics And Its Application*. **6** pp. 355-378 (2019)
5. Sklar M. Fonctions de repartition a n dimensions et leurs marges *Publ. Inst. Statist. Univ. Paris* , 229–231, (1959)
6. Czado, C. Analyzing dependent data with vine copulas. *Lecture Notes In Statistics, Springer*. **222** (2019)
7. F. Durante & C. Sempi: Principles of Copula Theory. CRC Chapman and Hall, Boca Raton 2016.

8. R. B. Nelsen: An introduction to copulas. Second edition. Springer Series in Statistics, Springer-Verlag, New York 2006.
9. R. Shumway, D. Stoffer and D. Stoffer: Time series analysis and its applications, Springer, 2000
10. C. Vasconcelos Modeling, clustering, and segmenting video with mixtures of dynamic textures. *IEEE transactions on pattern analysis and machine intelligence*. **30(5)**, 909-926 (2008)
11. X. Yimin and D. Yeung. Time series clustering with ARMA mixture. *Pattern Recognition* **37(8)**, 1675-1689 (2004)
12. Kleibergen, F. & Van Dijk, H. Non-stationarity in garch models: A bayesian analysis. *Journal Of Applied Econometrics*. **8**, S41-S61 (1993)
13. Mikosch, T. and Stărică, C. Nonstationarities in financial time series, the long-range dependence, and the IGARCH effects. *Review Of Economics And Statistics*. **86**, 378-390 (2004)
14. Aue, A. & Horváth, L. Structural breaks in time series. *Journal Of Time Series Analysis*. **34**, 1-16 (2013)
15. Hofert, M., Kojadinovic, I., Mächler, M. & Yan, J. Elements of copula modeling with R. (Springer,2018)
16. Bücher, A., Kojadinovic, I., Rohmer, T. & Segers, J. Detecting changes in cross-sectional dependence in multivariate time series. *Journal Of Multivariate Analysis*. **132** pp. 111-128 (2014)
17. Mendoza-Urdiales, R., Núñez-Mora, J., & Santillán-Salgado, R. and Valencia-Herrera, H. Twitter Sentiment Analysis and Influence on Stock Performance Using Transfer Entropy and EGARCH Methods. *Entropy*. **24**, 874 (2022)
18. Ansell, L. & Dalla Valle, L. Social media integration of flood data: A vine copula-based approach. *Journal of Environmental Informatics*. **39**, 97-110 (2022)
19. Sheikhi, A., Dalla Valle, L. & Mesiar, R. (2023) On the use of time-varying vine copulas in multivariate time series analysis, *The 7th Seminar on Copula Theory and its Applications*, Mashhad, Iran, pp. 24.
20. Sahin, Ö. & Czado, C. Vine copula mixture models and clustering for non-Gaussian data. *Econometrics And Statistics*. **22** pp. 136-158 (2022)
21. Aue, A. & Horváth, L. Structural breaks in time series. *Journal Of Time Series Analysis*. **34**, 1-16 (2013)
22. Carlos-Sandberg, L. & Clack, C. Incorporation of causality structures to complex network analysis of time-varying behaviour of multivariate time series. *Scientific Reports*. **11**, 1-16 (2021).
23. Matheson, J. & Winkler, R. Scoring rules for continuous probability distributions. *Management Science*. **22**, 1087-1096 (1976).

Dependence property for comparison of bivariate measures of concordance for copulas

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1 Motivation

A popular method for measuring the dependence between random variables is using measures of concordance such as Kendall's τ, Spearman's ρ and Gini's γ, which are margin-free so that their value only depends on the random variables connecting copula. Since there exist various measures of concordance the question arises how different measures relate to each other. For instance, in [9] it was shown that if the copula C fulfills certain dependence properties (C has to be *left tail decreasing* and *right tail increasing*) then $\rho(C) \geq \tau(C) \geq 0$ holds. Hutchinson and Lai (see [7]) conjectured that

$$-1 + \sqrt{1 + 3\tau(C)} \leq \rho(C) \leq \min\left\{\frac{3\tau(C)}{2}, 2\tau(C) - \tau(C)^2\right\}$$

holds for *stochastically increasing* copulas C. However, the counterexample given in [9] disproves the upper bound. Apparently, it depends on the properties of the particular copula C whether certain relations hold.

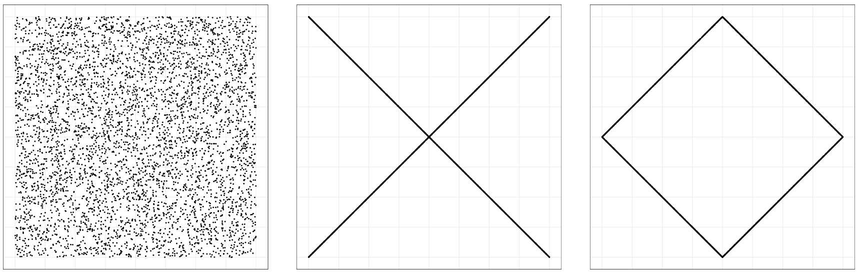
In this paper we examine a well-known class of measures of concordance induced by so-called *invariant* (with respect to permutations and reflections) copulas and derive a dependence property that allows the elements of this class to be related to each other.

2 Invariant copulas and measures of concordance

Among the best-known invariant copulas are, for example, the

- independence copula Π,
- arithmetic mean of the lower and upper Fréchet-Hoeffding bound W anf M defined by $M_\Sigma := (M + W)/2$ and
- copula V defined by

$$V(u,v) := \begin{cases} M(u,v), & \text{if } |u-v| > \frac{1}{2}, \\ W(u,v), & \text{if } |u+v-1| > \frac{1}{2}, \\ \frac{u}{2} + \frac{v}{2} - \frac{1}{4}, & \text{otherwise.} \end{cases}$$



(a) Sample of Π (b) Sample of M_Σ (c) Sample of V

Fig. 1: Examples of supports of reflection and permutation invariant copulas for sample size $N = 5.000$.

It can be easily seen that those copulas are invariant by considering their supports (see Figure 1).

[2] introduced an approach to generate invariant copulas via a transformation $\vartheta : \mathcal{C} \to \mathcal{C}$ as illustrated in Figure 2.

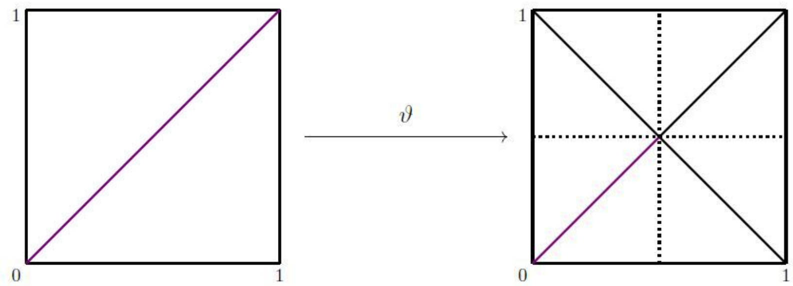


Fig. 2: Illustration of the map ϑ using the example of M.

Let $\mathcal{C}$ denote the class of all bivariate copulas and consider the map $[.,.] : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ given by

$$[C, D] := \int_{[0,1]^2} C(u, v) dQ^D(u, v)$$

where Q^D denotes the probability measure associated with the copula D; see [5]. The map $[.,.]$ is in either argument linear with respect to convex combinations and is therefore called a *biconvex form*. Moreover, the map $[.,.]$ is symmetric.

Now, consider a fixed copula $A \in \mathcal{C}$. Then we have $[M, A] > [\Pi, A]$ such that the map $\kappa_A : \mathcal{C} \to \mathbb{R}$ given by

$$\kappa_A(C) := \frac{[C, A] - [\Pi, A]}{[M, A] - [\Pi, A]}$$

is well-defined; see [6]. We have the following results (see [1, 3, 4, 6]).

1. The map κ_A is a measure of concordance if and only if A is invariant.
2. If A is invariant, then κ_A is convex, order preserving and continuous.
3. If A is invariant, then $[\Pi, A] = 1/4$ and the identity

$$\kappa_A(C) = \frac{[C, A] - 1/4}{[M, A] - 1/4}$$

   holds for all $C \in \mathcal{C}$.

Next we give some examples and see which well-known measures of concordance can be obtained using this approach.

1. **Spearman's ρ:** The copula Π is invariant and κ_Π satisfies

$$\kappa_\Pi(C) = 12[C, \Pi] - 3$$

   which means that κ_Π is Spearman's ρ; see [4, 9].
2. **Gini's γ:** The copula $M_\Sigma = (M + W)/2$ is invariant and κ_{M_Σ} satisfies

$$\kappa_{M_\Sigma}(C) = 8[C, M_\Sigma] - 2$$

   which means that κ_{M_Σ} is Gini's γ; see [4, 9].
3. The copula V is invariant and therefore we obtain the measure of concordance

$$\kappa_V(C) := 16[C, V] - 4$$

Note that Kendall's τ can not be constructed using invariant copulas.

3 An ordering of measures of concordance

Introducing an ordering for invariant copulas in the sense that $\vartheta(A) \preceq \vartheta(B)$ whenever $A \leq B$ pointwise, the question arises how the ordering of invariant copulas transfers to the induced measures of concordance, i.e., does $\vartheta(A) \preceq \vartheta(B)$ imply $\kappa_{\vartheta(A)}(C) \leq \kappa_{\vartheta(B)}(C)$ for every copula $C \in \mathcal{C}$? Apparently, this is not the case, in general. However, we were able to derive a dependence property for copulas C such that

$$\alpha(B) \cdot \kappa_{\vartheta(B)}(C) - \alpha(A) \cdot \kappa_{\vartheta(A)}(C) \geq 0$$

holds whenever $\vartheta(A) \preceq \vartheta(B)$ for certain values $\alpha(A), \alpha(B) > 0$ only depending on A and B. Among others, the following families of copulas satisfy the obtained dependence property:

– **FGM-copula:** For $\theta \in [-1,1]$, the mapping $C_\theta : [0,1]^2 \to [0,1]$ given by

$$C_\theta(u,v) := uv + \theta uv(1-u)(1-v)$$

is a copula and called Farlie-Gumbel-Morgenstern (FGM) copula.

– **Gaussian copula:** For $\rho \in (-1,0)\cup(0,1)$, the mapping $C_\rho : [0,1]^2 \to [0,1]$ given by

$$C_\rho(u,v) = \int\limits_{(-\infty,\phi^{-1}(u)]\times(-\infty,\phi^{-1}(v)]} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}\right) d\lambda_2(s,t)$$

is a copula and called Gaussian copula, where ϕ^{-1} denotes the inverse of the standard Gaussian distribution function.

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References

1. J. Behboodian, A. Dolati, M. Úbeda-Flores, Measures of association based on average quadrant dependence, *J. Probab. Statist. Sci 3* (2005) 161–173.
2. F. Durante, S. Fuchs, Reflection invariant copulas, *Fuzzy Sets and Systems* **354** (2019) 63–73
3. H. H. Edwards, P. Mikusiński, M. D. Taylor, Measures of concordance determined by d_4-invariant copulas, *Int. J. Math. Math. Sci. 2004* **(70)** (2004) 3867–3875.
4. S. Fuchs, K. D. Schmidt, Bivariate copulas: transformations, asymmetry and measures of concordance, *Kybernetika 50* **(1)** (2014) 109–125
5. S. Fuchs, A biconvex form for copulas, *Depend. Model. 4* **(1)** (2016).
6. S. Fuchs, Copula–induced measures of concordance, *Depend. Model. 4* **(1)** (2016).
7. T.P. Hutchinson, C.D. Lai, *Continuous Bivariate Distributions. Emphasizing Applications, Rumsby Scientific*, 1990
8. W. Hürlimann, Hutchinson-Lai's conjecture for bivariate extreme value copulas, *Statist. Probab. Lett. 61* **2** (2003) 191–198.
9. R. Nelson, An Introduction to Copulas, Springer-Verlag, Berlin Heidelberg, 2006. Kluwer, Dordrecht 2002.
10. W. Trutschnig, T. Mroz, A sharp inequality for Kendall's τ and Spearman's ρ of Extreme-Value Copulas, *Depend. Model. 6* **11** (2018) 369-376.

A relational algebraic approach to universal integrals

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1 Introduction

In a series of papers [3–6] the authors investigated discrete integrals such as the Choquet and Sugeno integral and their axiomatization. As part of their study they showed that universal integrals are based on semicopulas, and they provided lower and upper bounds of the integral operations based on a given semicopula. These real-valued resp. unit interval valued integrals can be considered as proper aggregation tool in the context of fuzzy sets. The aim of the current paper is to generalize this approach to so-called L-fuzzy sets and relations, i.e., fuzzy sets and relations that use an arbitrary Heyting algebra L as membership degree instead of the unit interval. Furthermore, we present the theory within arrow categories, i.e., we abstract from concrete sets and relations and work within a suitable algebraic framework. The results of this work also show that the results of [3] can be proven without referring to the real numbers and specific measures such as the Lebesque measure and the induced measurable spaces.

2 Arrow categories

In this section we want to recall some basic notions from categories, allegories and arrow categories [2, 10, 11].

We will write $R : A \to B$ to indicate that a morphism R of a category $\mathcal{C}$ has source A and target B. Composition and the identity morphism are denoted by ; and $\mathbb{I}_A$.

First, we are going to introduce Dedekind categories. These categories are called locally complete division allegories in [2].

Definition 1. *A Dedekind category $\mathcal{R}$ is a category satisfying the following:*

1. *For all objects A, B the collection $\mathcal{R}[A, B]$ is a complete Heyting algebra. Meet, join, the order, the least and greatest element are denoted by $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$.*
2. *There is a monotone operation $^\smile$ (called converse) mapping a relation $Q : A \to B$ to $Q^\smile : B \to A$ such that for all relations $Q : A \to B$ and $R : B \to C$ the following holds: $(Q;R)^\smile = R^\smile;Q^\smile$ and $(Q^\smile)^\smile = Q$.*
3. *For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law $(Q;R) \sqcap S \sqsubseteq Q;(R \sqcap (Q^\smile;S))$ holds.*
4. *For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) such that for all $X : A \to B$ the following holds: $X;R \sqsubseteq S \iff X \sqsubseteq S/R$.*

It is easy to verify that the collection of all L-fuzzy relations, i.e., binary relations between A and B that are given by a characteristic function $A \times B \to L$, forms a Dedekind category.

Using the converse operation a second residual can be defined by $Q \backslash R := (R^{\smile}/Q^{\smile})^{\smile}$. Furthermore, we define the symmetric quotient by $\mathrm{syQ}(Q,R) := Q \backslash R \sqcap Q^{\smile}/R^{\smile}$.

In a Dedekind category we are already able to identify the membership values, i.e., the Heyting algbera L, that are used by the relations in an abstract manner. For this we define a so-called scalar relation $\alpha : A \to A$ on an object A by $\alpha \sqsubseteq \mathbb{I}_A$ and $\top\!\!\!\top_{AA}; \alpha = \alpha; \top\!\!\!\top_{AA}$. Concretely, an L-fuzzy relation is a scalar iff every element of A is related to itself with a fixed degree a from L. As a consequence, there is a one-one correspondence between the scalar relations on A and the lattice L.

A crisp L-fuzzy relation R satisfies $R(x,y) = 0$ or $R(x,y) = 1$ for all pairs (x,y) where 0 and 1 are the smallest resp. greatest element of L. These relations can be identified with Boolean valued relations, i.e., regular set-theoretic relations. The notion of crispness cannot be defined abstractly in the theory of Dedekind categories [9, 10]. Because of this arrow categories were introduced [10, 11]. These categories add two operations $(.)^{\downarrow}$ and $(.)^{\uparrow}$ to Dedekind categories. Intuitively, the down-arrow (or kernel) operation maps an L-fuzzy relation R to the greatest crisp relation included in R and the up-arrow (or support) operation maps R to the least crisp relation that includes R.

Definition 2. *An arrow category $\mathcal{A}$ is a Dedekind category with $\top\!\!\!\top_{AB} \neq \perp\!\!\!\perp_{AB}$ for all objects A and B together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:*

1. $R^{\uparrow}, R^{\downarrow} : A \to B$ *for all* $R : A \to B$.
2. $(^{\uparrow}, ^{\downarrow})$ *is a Galois correspondence, i.e.,* $Q^{\uparrow} \sqsubseteq R$ *iff* $Q \sqsubseteq R^{\downarrow}$ *for all* $Q, R : A \to B$.
3. $(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ *for all* $R : B \to A$ *and* $S : B \to C$.
4. *If* $\alpha \neq \perp\!\!\!\perp_{AA}$ *is a non-zero scalar then* $\alpha^{\uparrow} = \mathbb{I}_A$.
5. $(Q \sqcap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$ *for all* $Q, R : A \to B$.

A relation $R : A \to B$ of an arrow category $\mathcal{A}$ is called crisp iff $R^{\uparrow} = R$ (or equivalently $R^{\downarrow} = R$). The collection of crisp relations is closed under all operations of a Dedekind category, and, hence, forms a sub-Dedekind category of $\mathcal{A}$ [10, 11].

An abstract version of a singleton set is given by the notion of a unit. A unit $\mathbf{1}$ is an object so that $\top\!\!\!\top_{\mathbf{11}} = \mathbb{I}_{\mathbf{1}}$ and $\top\!\!\!\top_{A\mathbf{1}}$ is total for every object A, i.e., $\top\!\!\!\top_{A\mathbf{1}}; \top\!\!\!\top_{\mathbf{1}A} = \top\!\!\!\top_{AA}$. A relation $v : \mathbf{1} \to A$ is called a vector on A and we denote the set of all vectors on A by $\mathrm{Vec}(A) = \mathcal{A}[\mathbf{1}, A]$.

An operation $* : \mathrm{Sc}(\mathbf{1}) \times \mathrm{Sc}(\mathbf{1}) \to \mathrm{Sc}(\mathbf{1})$ is called a pseudo multiplication if $*$ is monotone in both parameters and we have $\alpha * \mathbb{I}_{\mathbf{1}} = \mathbb{I}_{\mathbf{1}} * \alpha = \alpha$. Notice that pseudo multiplications satisfy $\alpha * \perp\!\!\!\perp_{\mathbf{11}} = \perp\!\!\!\perp_{\mathbf{11}} * \alpha = \perp\!\!\!\perp_{\mathbf{11}}$. In the context of the unit interval a pseudo multiplication is called a semicopula [1]. Please note that $*_m$ defined by $\alpha *_m \beta = \alpha$ iff $\beta = \mathbb{I}_{\mathbf{1}}$, $\alpha *_m \beta = \beta$ iff $\alpha = \mathbb{I}_{\mathbf{1}}$, and $\alpha *_m \beta = \perp\!\!\!\perp_{\mathbf{11}}$ otherwise is the smallest pseudo multiplication, and $*_g$ defined by $\alpha *_g \beta = \alpha \sqcap \beta$ is the greatest pseudo multiplication.

The disjoint union of two sets can be defined abstractly as a relation sum. A relation sum $A + B$ of two objects A and B is an object together with two crisp relations

$\iota : A \to A + B$ and $\kappa : B \to A + B$ satisfying $\iota^{\smile};\iota \sqsubseteq \mathbb{I}_A$, $\kappa^{\smile};\kappa \sqsubseteq \mathbb{I}_A$, $\iota;\kappa^{\smile} = \perp\!\!\!\perp_{AB}$, and $\iota;\iota^{\smile} \sqcup \kappa;\kappa^{\smile} = \mathbb{I}_{A+B}$.

An abstract version of power sets is given by the notion of a relational (or direct) power [7]. Please note that in the context of arrow categories we are interested in the set of all L-fuzzy subsets of A, i.e., in the set of all functions $f : A \to L$. An object L^A together with a relation $\varepsilon : A \to L^A$ is called a relational power iff $\mathrm{syQ}(\varepsilon, \varepsilon)^{\downarrow} = \mathbb{I}_{L^A}$ and $\mathrm{syQ}(R, \varepsilon)^{\downarrow}$ is total for every $R : A \to B$. Notice that the relation $\mathrm{syQ}(R^{\smile}, \varepsilon)^{\downarrow} : B \to L^A$ is a crisp map for every relation $R : B \to A$. For concrete relation this construction yields the direct image of R. We denote this relation by $\Lambda(R)$.

In this paper we are also interested in the set of all crisp subsets of a set. We call this the crisp (relational) power and it is given by an object $\mathcal{P}(A)$ and a crisp relation $\in : A \to \mathcal{P}(A)$ satisfying $\mathrm{syQ}(\in, \in) = \mathbb{I}_{\mathcal{P}(A)}$ and $\mathrm{syQ}(R^{\downarrow}, \in)$ is total for every $R : A \to B$. Similar to above the relation $\mathrm{syQ}(R^{\smile}, \in)$ is a map for every crisp relation $R : B \to A$ and we denote this by $\Lambda_c(R)$. The order relation and the empty and the full set within a crisp power are given by the relations $\Omega = \in \setminus \in$, $e = \Lambda_c(\perp\!\!\!\perp_{\mathbf{1}A})$ and $f = \Lambda_c(\top\!\!\!\top_{\mathbf{1}A})$, respectively.

3 Relation measures and universal integrals

A concrete L-fuzzy relation $R : A \to \mathbf{1}$ is given by its characteristic function from $A \times \mathbf{1}$ to L. Since $A \times \mathbf{1}$ is isomorphic to A this function corresponds to a function that assigns to every element of A a value of L. This observation leads to the following definition of a relational measure.

Definition 3. *A relation $\mu_A : \mathcal{P}(A) \to \mathbf{1}$ is called a measure (or capacity) on A iff $\mu_A^{\smile};\Omega \sqsubseteq \mu_A^{\smile}$, $e;\mu_A = \perp\!\!\!\perp_{\mathbf{11}}$, and $f;\mu_A = \mathbb{I}_{\mathbf{1}}$.*

As usual we will regularly omit the index A, and we will denote the set of measures on A by $\mathrm{M}(A) \subseteq \mathcal{A}[\mathcal{P}(A), \mathbf{1}]$.

Definition 4. *A family of operations $I_A : \mathrm{M}(A) \times \mathrm{Vec}(A) \to \mathrm{Sc}(\mathbf{1})$ with $A \in \mathrm{Obj}_{\mathcal{A}}$ is called a (discrete) universal integral iff the following holds:*

(A1) *For each object A the operation I_A is monotone in both arguments, i.e., $\mu_1 \sqsubseteq \mu_2$ and $v_1 \sqsubseteq v_2$ implies $I_A(\mu_1, v_1) \sqsubseteq I_A(\mu_2, v_2)$.*
(A2) *$I_A(\mu, v) = \Lambda_c(v);\mu$ if v is crisp.*
(A3) *$I_A(\mu, \alpha;\top\!\!\!\top_{1A}) = \alpha$ for all scalars $\alpha \in \mathrm{Sc}(\mathbf{1})$.*
(A4) *For all measures $\mu_1 \in \mathrm{M}(A), \mu_2 \in \mathrm{M}(B)$ and $v_1 \in \mathrm{Vec}(A), v_2 \in \mathrm{Vec}(B)$ the inclusion $\Lambda_c((\alpha \backslash v_1)^{\downarrow});\mu_1 \sqsubseteq \Lambda_c((\alpha \backslash v_2)^{\downarrow});\mu_2$ for all scalars $\alpha \in \mathrm{Sc}(\mathbf{1})$ implies $I_A(\mu_1, v_1) \sqsubseteq I_B(\mu_2, v_2)$.*

Furthermore, we say that I is based on the pseudo multiplication $*$ iff $I_A(\mu, \alpha;v) = \alpha * \Lambda_c(v);\mu$ for all measures $\mu \in \mathrm{M}(A)$ and crisp $v \in \mathrm{Vec}(A)$.

Please note that [3, 4] only requires a weaker version of (A4) where the inclusion $\sqsubseteq$ on both sides of the implication is replaced by $=$. However, Proposition 2.7 of [3] implies our (stronger) version in the context of the (extended) real numbers resp. the unit interval. We leave the investigation whether (A4) can be replaced by this weaker version in the more general context of arbitrary Heyting algebras for future work.

Given a scalar $\beta \in \mathrm{Sc}(\mathbf{1})$ the relation $\mu^{\beta}_{\mathbf{1+1}} : \mathcal{P}(\mathbf{1}+\mathbf{1}) \to \mathbf{1}$ defined by $\mu^{\beta}_{\mathbf{1+1}} = \in^{\smile}; \mathbb{T}_{A\mathbf{1}}; \beta \sqcup f^{\smile}$ is a measure on $\mathbf{1}+\mathbf{1}$. For concrete L-fuzzy relations and $\mathbf{1}+\mathbf{1} = \{a,b\}$ the relation $\mu^{\beta}_{\mathbf{1+1}}$ turns out to be the map $\mu^{\beta}_{\mathbf{1+1}}(M) = \perp\!\!\!\perp_{\mathbf{11}}$ iff $M = \emptyset$, $\mu^{\beta}_{\mathbf{1+1}}(M) = \beta$ iff $M = \{a\}$ or $M = \{b\}$, and $\mu^{\beta}_{\mathbf{1+1}}(M) = \mathbb{I}_{\mathbf{1}}$ iff $M = \{a,b\}$. Given an universal integral I we may use this measure and define $\alpha \hat{*} \beta = I_{\mathbf{1+1}}(\mu^{\beta}_{\mathbf{1+1}}, \alpha; \iota)$ where $\iota : \mathbf{1} \to \mathbf{1}+\mathbf{1}$ is the first injection of the relational sum.

Lemma 1. *The operation $\hat{*}$ is a pseudo multiplication and I is based on $\hat{*}$.*

If we define ${}_*I_A(\mu, v) = \bigsqcup_{\alpha \in \mathrm{Sc}(\mathbf{1})} \alpha * \Lambda_c((\alpha \backslash v)^{\downarrow}); \mu$, then we obtain the following theorem.

Theorem 1. *${}_*I$ is the smallest universal integral based on $*$.*

In order to show that there is also a greatest universal integral based on a pseudo multiplication we first define $\delta^{\mu}_{v} \in \mathrm{Sc}(\mathbf{1})$ by $\delta^{\mu}_{v} = \bigsqcup_{\alpha \in \mathrm{Sc}(\mathbf{1})} \alpha; \Lambda_c((\alpha \backslash v)^{\downarrow}); \mu^{\uparrow}$. Please note that δ^{μ}_{v} was denoted by $\mathrm{essup}_{\mu} v$ in [3, 4]. Using δ^{μ}_{v} and following [3, 4] we now define ${}^*I_A(\mu, v) = \delta^{\mu}_{v} * (\bigsqcup_{\perp\!\!\!\perp_{\mathbf{11}} \neq \alpha \in \mathrm{Sc}(\mathbf{1})} \Lambda_c((\alpha \backslash v)^{\downarrow}); \mu)$.

Theorem 2. *${}^*I$ is the greatest universal integral based on $*$.*

Contrarily to [3] the proof of the previous theorem does not use the Lebesque measure and Borel measurable functions. Instead it use the relational measure $\eta : \mathcal{P}(L^{\mathbf{1}}) \to \mathbf{1}$ defined by $\eta = \in^{\smile}; \varepsilon^{\smile}$.

From the fact that $*_m$ is the smallest and $*_g$ is the greatest pseudo multiplication we obtain the following corollary.

Corollary 1. *${}_{*_m}I$ is the smallest and ${}^{*_g}I$ is the greatest universal integral.*

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References

1. Bassan B., Spizzichino F.: Relations among univariate aging, bivariate aging and dependence for exchangeable lifetimes. Journal of Multivariate Analysis 93(2), 313-339 (2005).
2. Freyd P., Scedrov A.: Categories, Allegories. North-Holland Mathematical Library Vol. 39, North-Holland, Amsterdam (1990).
3. Klement E.P., Mesiar R., Pap E.: A Universal Integral as Common Frame for Choquet and Sugeno Integral. IEEE Transactions on Fuzzy Systems 18(1), 178-187 (2010).
4. Klement E.P., Mesiar R.: Discrete Integrals and Axiomatically Defined Functionals. Axioms 1(1), 9-20 (2012).

5. Klement E.P., Mesair R.: On the axiomatization of some classes of discrete universal integrals. Knowledge-based Systems 28, 13-18 (2012).
6. Klement E.P., Mesiar R., Spizzichino F., Stupñanová: Universal integrals based on copulas. Fuzzy Optimization and Decision Making 13, 273-286 (2014).
7. Schmidt G.: Relational Mathematics. Cambridge University Press, Cambridge (2011).
8. Schmidt G., Winter M.: Relational Topology. LNM 2208 (2018).
9. Winter M.: A new Algebraic Approach to L-Fuzzy Relations Convenient to Study Crispness. INS Information Science 139, 233-252 (2001).
10. Winter M.: Goguen categories – A Categorical Approach to L-fuzzy Relations. Springer, Berlin (2007).
11. Winter M.: Arrow Categories. Fuzzy Sets and Systems 160, 2893-2909 (2009).

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